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Research Article

Uniformly Random Attractor for the Three-Dimensional Stochastic Nonautonomous Camassa-Holm Equations

Zhehao Huang and Zhengrong Liu

Department of Mathematics, South China University of Technology, Guangzhou 510640, China

Correspondence should be addressed to Zhengrong Liu; liuzhr@scut.edu.cn

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We consider the uniformly random attractor for the three-dimensional stochastic nonautonomous Camassa-Holm equations in the periodic box $[0,l]^3$ in this paper. We associate with the concepts of uniform attractor and random attractor and produce the concept of uniformly random attractor for a process. Then we establish the existence of the uniformly random attractor in $D(A^{1/2})$ and $D(A)$ for the equations.

1. Introduction

The Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \tag{1}$$

models the unidirectional propagation of shallow water waves over a flat bottom [1–4]. It has been paid a large number of attentions due to its rich nonlinear phenomenology. It is completely integrable [1], and it has stable solutions [5]. It possesses the peakons $u = e^{|x-ct|}$ which has been proved stable [6, 7]. It has been shown that (1) is locally well-posed for initial data $u_0 \in H^s(\mathbb{R})$ $(s > 3/2)$ [8, 9]. There are a rich variety of global solutions and blow-up solutions obtained in [8–10]. The global existence of weak solutions, conservative solutions, and disispative solutions was investigated in [6, 11, 12].

Following the Camassa-Holm (1), some generalized types of the equation have been deeply considered by many authors, for instance [13–18]. The authors in [19] considered the three-dimensional Camassa-Holm equations subject to periodic boundary conditions:

$$\begin{gathered}\frac{\partial}{\partial t}\left(\alpha_0^2 u - \alpha_1^2 \Delta u\right) - \nu\Delta\left(\alpha_0^2 u - \alpha_1^2 \Delta u\right) - u \\ \times\left(\nabla\times\left(\alpha_0^2 u - \alpha_1^2 \Delta u\right)\right) + \frac{1}{\rho_0}\nabla p = f(x), \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x).\end{gathered} \tag{2}$$

They established the global regularity of solutions of the equation and provided the estimates for the Hausdorff and fractal dimensions of the global attractor.

In [20] the authors analyzed the effects produced by stochastic perturbations in the deterministic version of the three-dimensional Lagrangian averaged Navier-Stokes-α model:

$$\begin{gathered}\frac{\partial}{\partial t}(u - \alpha\Delta u) + \nu(Au - \alpha\Delta(Au)) \\ + (u\cdot\nabla)(u - \alpha\Delta u) - \alpha\nabla u^* \cdot \Delta u + \nabla p \\ = F(t,u) + G(t,u)\dot{W}(t), \\ \nabla\cdot u = 0, \quad \text{in } D\times(0,+\infty), \\ u = 0, \qquad Au = 0, \quad \text{on } \partial D\times(0,+\infty), \\ u(0) = u_0,\end{gathered} \tag{3}$$

that is, the persistence of exponential stability as well as possible stabilization effects produced by the noise.

In [21] the authors proved the existence of the pullback and forward attractors for three-dimensional Lagrangian averaged Navier-Stokes-α model with delay:

$$\begin{gathered}\frac{\partial}{\partial t}(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))\\ +(u\cdot\nabla)(u-\alpha\Delta u)-\alpha\nabla u^{*}\cdot\Delta u+\nabla p\\ =f(t)+F(t,u_t),\\ \nabla\cdot u=0,\quad \text{in } D\times(\tau,+\infty),\\ u=0,\qquad Au=0,\quad \text{on } \partial D\times(\tau,+\infty),\\ u(t)=\phi(t-\tau),\quad \text{in } (\tau-h,\tau),\\ u(\tau)=u_0.\end{gathered}\tag{4}$$

In [22] the author investigated the existence of finite dimensional uniform attractor for three-dimensional nonautonomous Camassa-Holm equations with periodic boundary conditions:

$$\begin{gathered}\frac{\partial}{\partial t}\left(\alpha_0^2u-\alpha_1^2\Delta u\right)-\nu\Delta\left(\alpha_0^2u-\alpha_1^2\Delta u\right)\\ -u\times\left(\nabla\times\left(\alpha_0^2u-\alpha_1^2\Delta u\right)\right)+\frac{1}{\rho_0}\nabla p=f(x,t),\\ \nabla\cdot u=0,\\ u(x,\tau)=u_\tau(x).\end{gathered}\tag{5}$$

In [23, 24] the author studied the existence of uniform attractor and convergence of the attractor as $\varepsilon\to0^+$ for a nonautonomous three-dimensional Lagrangian averaged Navier-Stokes-α model with singularly oscillating external force:

$$\begin{gathered}\frac{\partial}{\partial t}(u-\alpha\Delta u)+\nu(Au-\alpha\Delta(Au))\\ +(u\cdot\nabla)(u-\alpha\Delta u)-\alpha\nabla u^{*}\cdot\Delta u+\nabla p=g^{\varepsilon},\\ \nabla\cdot u=0,\quad \text{in } D\times(\tau,+\infty),\\ u=0,\qquad Au=0,\quad \text{on } \partial D\times(\tau,+\infty),\\ u(\tau)=u_\tau,\end{gathered}\tag{6}$$

where

$$g^{\varepsilon}(x,y,z,t)=\begin{cases}g_0(x,y,z,t)+\dfrac{1}{\varepsilon^{\rho}}g_1(x,y,z,t), & \varepsilon\in(0,1],\ \rho\in[0,1),\\ g_0(x,y,z,t), & \varepsilon=0.\end{cases}\tag{7}$$

Motivated by all their works, we initial our work to investigate the equations perturbed by an additive noise. We consider the following viscous version of three-dimensional stochastic nonautonomous Camassa-Holm equation in the periodic box $[0,l]^3$:

$$\begin{gathered}\frac{\partial}{\partial t}\left(\alpha_0^2u-\alpha_1^2\Delta u\right)-\nu\Delta\left(\alpha_0^2u-\alpha_1^2\Delta u\right)\\ -u\times\left(\nabla\times\left(\alpha_0^2u-\alpha_1^2\Delta u\right)\right)+\frac{1}{\rho_0}\nabla p\\ =f(x,t)+QW_t,\\ \nabla\cdot u=0,\\ u(x,\tau)=u_\tau(x),\end{gathered}\tag{8}$$

where $p/\rho_0=\pi/\rho_0+\alpha_0^2|u|^2-\alpha_1^2(u\cdot\Delta u)$ is the modified pressure, while π is the pressure, $\nu>0$ is the constant viscosity, and $\rho_0>0$ is a constant density. The function f is a given body forcing, and $\alpha_0>0$, $\alpha_1\geq0$ are scale parameters. $W(t)$ are two-sided real-valued Wiener processes on a probability space which will be specified later. $Q:\mathbb{R}^n\to L^2([0,l]^3)^3$ is a bounded linear operator. Also observing that at the limiting case $\alpha_0=1$, $\alpha_1=0$, we obtain the three-dimensional stochastic Navier-Stokes with periodic boundary conditions.

Attractor is an important concept to describe the long-time behavior of solutions for a system in mathematical physics [25–27]. The notion of uniform attractor parallelling to that of the global autonomous systems has been systematically considered in [26]. In the approach presented in [27], to construct the uniform attractor, instead of the associated process $\{\mathscr{U}_\sigma(t,\tau),t\geq\tau,\tau\in\mathbb{R}\}$, one should consider a family of processes $\{\mathscr{U}_\sigma(t,s)\}$, $\sigma\in\Sigma$, in some Banach space E where the functional parameter $\sigma_0(s)$, $s\in\mathbb{R}$, is called the symbol and Σ is the symbol space including $\sigma_0(s)$. The approach implies that the structure of uniform attractor is described by the representation as a union of sections of all kernels of the family of processes. The kernel is the set of all complete trajectories of a process.

While in the real world, a system is usually uncertain due to some external noise, which is random. The random effects are considered not only as compensations for the defects in some deterministic models but rather essential phenomena [28–32]. In order to capture the essential dynamics of random dynamical systems with wide fluctuations, the concept of pullback random attractor was introduced in [29, 33, 34], as an extension to stochastic systems of the theory of attractors for deterministic systems in [25, 27, 35–43]. A pullback random attractor $\mathscr{A}(\omega)$ which can be constructed by a closed random absorbing set $\mathscr{K}(\omega)$ for an asymptotically compact stochastic mapping $\mathscr{S}(t,s,\omega)$ is given by

$$\mathscr{A}(\omega)=\bigcap_{s\geq\tau}\overline{\bigcup_{t\geq s}\mathscr{S}(t,\tau,\theta_{\tau-t}\omega,\mathscr{K}(\theta_{\tau-t}\omega))},\tag{9}$$

where θ_t is the metric process on probability space. The existence of random attractors for stochastic dynamical systems has been investigated extensively by many authors [29, 33, 34, 44–49]. In our paper, we associate with the concepts of uniform attractor and random attractor together and give the concept of uniformly random attractor. Then we consider (8) in an appropriate space and show that there is a uniformly (with respect to f) random attractor $\mathscr{A}(\omega)$ which all solutions approach as $t \to \infty$. To our best knowledge, the long-time dynamical behavior of the three-dimensional stochastic nonautonomous Camassa-Holm equations has not been discussed, and we believe that it is a significant work to obtain a uniformly (with respect to f) random attractor for the system.

The paper is organized as follows. In Section 2, we present the abstract results describing the uniformly random attractor and some relevant definitions. In Section 3, we give some functional settings which are foundations for us to obtain the existence of uniformly (with respect to f) random attractor of (8). In Section 4, we convert (8) with an additive noise to deterministic equations with random parameters and define a process corresponding to the equations. In Section 5, we obtain the existence of uniformly (with respect to f) random attractors for (8) on the basis of the above preparations.

2. Abstract Results

In this section, we associate with the concepts of uniform attractor and random attractor and obtain the notion of uniformly random attractor. Let $(E, \|\cdot\|_E)$ be a separable Hilbert space with the Borel σ-algebra $\mathscr{B}(E)$, and let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 1. $(\theta_t)_{t\in\mathbb{R}}$ is called a measurable flow on probability space $(\Omega, \mathscr{F})$ if $\theta : \mathbb{R}\times\Omega \to \Omega$ is $(\mathbb{R}\times\mathscr{F}, \mathscr{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$, and $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2. A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of E, is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for $\mathbb{P}$ a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} d\left(B\left(\theta_{-t}\omega\right)\right) = 0 \quad \forall\beta > 0, \tag{10}$$

where $d(B) = \sup_{x\in E}\|x\|_E$.

Definition 3. A random set $\{\mathscr{K}(\omega)\}_{\omega\in\Omega}$ is called an absorbing set of a stochastic mapping $\mathcal{S}(t, s, \omega)$ in E if for every random bounded set B and $\mathbb{P}$ a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\mathcal{S}\left(t, s, \theta_{s-t}\omega\right) B\left(\theta_{s-t}(\omega)\right) \subseteq \mathscr{K}(\omega) \quad \forall t \geq t_B(\omega). \tag{11}$$

Let $\{\mathscr{U}(t, \tau, \omega)\} = \{\mathscr{U}(t, \tau, \omega), t \geq \tau, t, \tau \in \mathbb{R}, \omega \in \Omega\}$ be a three-parameter family of mappings acting on E:

$$\mathscr{U}(t, \tau, \omega) : E \longrightarrow E, \quad t \geq \tau,\ t, \tau \in \mathbb{R},\ \omega \in \Omega. \tag{12}$$

Definition 4. A three-parameter family of random mappings $\{\mathscr{U}(t, \tau, \omega)\}$ is said to be a process in E if it is $(\mathscr{B}(\mathbb{R}^+) \times \mathscr{B}(\mathbb{R}^+) \times \mathscr{F} \times \mathscr{B}(E), \mathscr{B}(E))$-measurable and for $\mathbb{P}$ a.e. $\omega \in \Omega$ it satisfies

$$\begin{gathered}\mathscr{U}\left(t, s, \theta_{s-\tau}\omega\right) \circ \mathscr{U}(s, \tau, \omega) = \mathscr{U}(t, \tau, \omega), \\ \forall t \geq s \geq \tau,\ \tau \in \mathbb{R}, \\ \mathscr{U}(\tau, \tau, \omega) = Id, \quad \tau \in \mathbb{R}.\end{gathered} \tag{13}$$

Definition 5. A family of processes $\mathscr{U}_\sigma(t, \tau, \omega)$, $\sigma \in \Sigma$, acting in E is said to be $(E\times\Sigma, E)$-continuous, if, for $\mathbb{P}$ a.e. $\omega \in \Omega$ and fixed $t, \tau, t \geq \tau$, the random mapping $(u, \sigma) \mapsto \mathscr{U}_\sigma(t, \tau, \omega)u$ is continuous from $E \times \Sigma$ into E.

Definition 6. A random curve $u(s, \omega)$, $s \in \mathbb{R}$, is said to be a complete trajectory of the process $\{\mathscr{U}(t, \tau, \omega)\}$ if for $\mathbb{P}$ a.e. $\omega \in \Omega$

$$\{\mathscr{U}(t, \tau, \omega)\} u(\tau, \omega) = u(t, \omega), \quad \forall t \geq \tau,\ t, \tau \in \mathbb{R}. \tag{14}$$

Definition 7. The random kernel $\mathscr{N}(\omega)$ of the process $\{\mathscr{U}(t, \tau, \omega)\}$ consists of all bounded complete trajectories of the process $\{\mathscr{U}(t, \tau, \omega)\}$:

$$\begin{aligned}\mathscr{N}(\omega) = \{&u(\cdot, \omega) \mid u(\cdot, \omega) \text{ satisfies Definition 6} \\ &\text{and } \|u(s, \omega)\|_E \leq M_u(\omega) \text{ for } s \in \mathbb{R}\}.\end{aligned} \tag{15}$$

Definition 8. The random set

$$\mathscr{N}(s, \omega) = \{u(s, \omega) \mid u(\cdot, \omega) \in \mathscr{N}(\omega)\} \subseteq E \tag{16}$$

is said to be the random kernel section at time $t = s$, $s \in \mathbb{R}$.

Let $B_t(\omega) = \bigcup_{\sigma\in\Sigma}\bigcup_{s\geq t}\mathscr{U}_\sigma(s, t, \theta_{t-s}\omega)B(\theta_{t-s}\omega)$, where $B(\omega)$ is a random set. Denote that $\overline{B(\omega)}$ is closure of the set $B(\omega)$ and $\mathbb{R}_\tau = \{t \in \mathbb{R} \mid t \geq \tau\}$.

Definition 9. A random set $\mathscr{W}_{\tau,\Sigma}(B)(\omega) = \bigcap_{t\geq\tau}\overline{B_t(\omega)}$ is called the uniformly (with respect to $\sigma \in \Sigma$) random (pullback) omega-limit set of $B(\omega)$ which can be characterized as follows, analogously to that for semigroups,

$$\begin{gathered} y \in \mathscr{W}_{\tau,\Sigma}(B)(\omega) \Longleftrightarrow \text{there are sequences } \{x_n\} \subset B\left(\theta_{\tau-t_n}\omega\right), \\ \{\sigma_n\} \subset \Sigma,\ \{t_n\} \subset \mathbb{R}_\tau, \\ \text{such that } t_n \longrightarrow +\infty \text{ and} \\ \mathscr{U}_{\sigma_n}\left(t_n, \tau, \theta_{\tau-t_n}\omega\right) x_n \longrightarrow y \quad (n \longrightarrow \infty). \end{gathered} \tag{17}$$

Let $B(\omega) \in \mathscr{B}(E)$, and its Kuratowski measure of noncompactness $\kappa(B)$ is defined by

$$\begin{aligned}\kappa(B) = \inf\{&\delta > 0 \mid B \text{ admits a finit} \\ &\text{covering by sets of diameter} \leq \delta\}.\end{aligned} \tag{18}$$

Definition 10. A family of processes $\{\mathscr{U}_\sigma(t, \tau, \omega)\}$, $\sigma \in \Sigma$, is said to be uniformly (with respect to $\sigma \in \Sigma$) random (pullback) omega-limit compact if, for $\mathbb{P}$ a.e. $\omega \in \Omega$ and any $\tau \in \mathbb{R}$, the set $B_t(\omega)$ is bounded for every t and $\lim_{t\to\infty}\kappa(B_t(\omega)) = 0$.

We now present a method to verify the uniformly (with respect to $\sigma \in \Sigma$) random (pullback) omega-limit compactness.

Definition 11. A family of processes $\{\mathscr{U}_\sigma(t, \tau, \omega)\}$, $\sigma \in \Sigma$, is said to satisfy uniformly (with respect to $\sigma \in \Sigma$) condition

(C) if, for $\mathbb{P}$ a.e. $\omega \in \Omega$ and any fixed $\tau \in \mathbb{R}$, tempered set $B(\omega) \in \mathscr{B}(E)$, $\varepsilon > 0$, there exist $t_0 = t(\tau, B, \varepsilon, \omega) \geq \tau$ and a finite dimensional subspace E_1 of E such that

(i) $\Pi(\bigcup_{\sigma\in\Sigma}\bigcup_{t\geq t_0}\mathscr{U}_\sigma(t,\tau,\theta_{\tau-t}\omega)B(\theta_{\tau-t}\omega))$ is bounded, and

(ii) $\|(I-\Pi)(\bigcup_{\sigma\in\Sigma}\bigcup_{t\geq t_0}\mathscr{U}_\sigma(t,\tau,\theta_{\tau-t}\omega)x)\| \leq \varepsilon$, $\forall x \in B(\theta_{\tau-t}\omega)$,

where $\Pi : E \to E_1$ is a bounded projector.

Therefore, we have the following results.

Theorem 12. *Let Σ be a metric space, and let $T(t)$ be a continuous invariant semigroup $T(t)\Sigma = \Sigma$ on Σ. A family of processes $\{\mathscr{U}(t,\tau,\omega)\}$, $\sigma \in \Sigma$, acting in E is $(E\times\Sigma, E)$-continuous (weakly) and possesses the compact uniformly (with respect to $\sigma \in \Sigma$) random attractor $\mathscr{A}_\Sigma(\omega)$ satisfying*

$$\mathscr{A}_\Sigma(\omega) = \mathscr{W}_{0,\Sigma}(B_0(\omega)) = \mathscr{W}_{\tau,\Sigma}(B_0(\omega)) = \bigcup_{\sigma\in\Sigma}\mathscr{N}_\sigma(0,\omega), \quad \forall \tau \in \mathbb{R}, \tag{19}$$

if it

(i) *has a bounded uniformly (with respect to $\sigma \in \Sigma$) random absorbing set $B_0(\omega)$, and*

(ii) *satisfies uniformly (with respect to $\sigma \in \Sigma$) condition (C).*

Moreover, if E is a uniformly convex Banach space, then the converse is true.

3. Functional Setting

We consider the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where

$$\Omega = \left\{\omega = (\omega_1, \omega_2, \dots, \omega_n)^T \in C(\mathbb{R}, \mathbb{R}^n) : \omega(0) = 0\right\}. \tag{20}$$

$\mathscr{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω and $\mathbb{P}$ the corresponding Wiener measure on $(\Omega, \mathscr{F})$. Then we will identify $W(t)$ with

$$W(t) = (w_1(t), w_2(t), \dots, w_n(t))^T = \omega(t) \quad \text{for } t \in \mathbb{R}. \tag{21}$$

Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}. \tag{22}$$

Then $\{\theta_t\}_{t\in\mathbb{R}}$ is a family of measure preserving transformations on probability space $(\Omega, \mathscr{F}, \mathbb{P})$ in Definition 1.

Next we define a symbol space $\Sigma(\sigma_0)$ for (8). We assume that the function $f(\cdot, t) =: f(t) \in L^2_{\text{loc}}(\mathbb{R}; E)$ is translation bounded. That is, for $f(s) \in L^2_b(\mathbb{R}; E)$, we have

$$\|f\|^2_{L^2_b} = \|f\|^2_{L^2_b(\mathbb{R};E)} = \sup_{t\in\mathbb{R}}\int_t^{t+1}\|f(s)\|^2_E ds < \infty. \tag{23}$$

Definition 13 (cf. [41]). A function $\varphi \in L^2_{\text{loc}}(\mathbb{R}; E)$ is said to be normal if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+\delta}\|\varphi(s)\|^2_E ds \leq \varepsilon. \tag{24}$$

We denote by $L^2_n(\mathbb{R}; E)$ the set of all normal functions in $L^2_{\text{loc}}(\mathbb{R}; E)$. Obviously, $L^2_n(\mathbb{R}; E) \subset L^2_b(\mathbb{R}; E)$, and it is proved in [41] that $L^2_n(\mathbb{R}; E)$ is a closed subset of $L^2_b(\mathbb{R}; E)$. Let a fixed symbol $\sigma_0(s) = f_0(s) = f_0(\cdot, s)$ be normal functions in $L^2_{\text{loc}}(\mathbb{R}; E)$. That is, the family of translation $\{f_0(s+\eta), \eta \in \mathbb{R}\}$ forms a normal function set in $L^2_{\text{loc}}([T_1, T_2]; E)$, where $[T_1, T_2]$ is an arbitrary interval of the time axis $\mathbb{R}$. Therefore,

$$\Sigma(\sigma_0) = \Sigma(f_0) = \{f_0(x, s+\eta) \mid \eta \in \mathbb{R}\}_{L^2_{\text{loc}}(\mathbb{R};E)}. \tag{25}$$

After integrating (8), one can easily see that

$$\frac{d}{dt}\int_{[0,l]^3}\left(\alpha_0^2 u - \alpha_1^2\Delta u\right)dx = \int_{[0,l]^3} f dx + \int_{[0,l]^3} QW_t dx. \tag{26}$$

On the other hand, because of the spatial periodicity of solution, we have $\int_{[0,l]^3}\Delta u dx = 0$. Then we have

$$\frac{d}{dt}\int_{[0,l]^3}\alpha_0^2 u dx = \int_{[0,l]^3} f dx + \int_{[0,l]^3} QW_t dx. \tag{27}$$

That is, the mean of solution is invariant provided that the means of the forcing term and the perturbing term are zero. In this paper, we will consider the forcing term, perturbing term and initial values with spatial means that are zero. That is, we assume $\int_{[0,l]^3} f dx = 0$, $\int_{[0,l]^3} QW_t dx = 0$, and $\int_{[0,l]^3} u_\tau dx = 0$.

Next we introduce some essential functional spaces.

(i) We denote by $\mathscr{V} = \{\varphi : \varphi$ a vector-valued trigonometric polynomial defined on $[0,l]^3$, such that $\nabla\cdot\varphi = 0$ and $\int_{[0,l]^3}\varphi(x)dx = 0\}$, and we let H and V be the closure of $\mathscr{V}$ in $L^2([0,l]^3)^3$ and in $H^1([0,l]^3)^3$, respectively. We can observe that $H^\perp$, the orthogonal complement of H in $L^2([0,l]^3)^3$, is $\{\nabla p : p \in H^1([0,l]^3)\}$ (cf. [38, 41]).

(ii) We denote by $P : L^2([0,l]^3)^3 \to H$ the L^2 orthogonal projection, usually referred to as Helmholtz-Leray projector, and by $A = -P\Delta$ the Stokes operator with domain $D(A) = H^2([0,l]^3)^3 \bigcap V$. Note that, in the case of periodic boundary condition, $A = -\Delta|_{D(A)}$ is a self-adjoint positive operator with compact inverse. Hence, the space H has an orthogonal basis $\{e_j\}_j^\infty$ of eigenfunctions of A, that is, $Ae_j = \lambda_j e_j$, with

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to +\infty, \text{ as } j \to \infty. \tag{28}$$

In fact, these eigenvalues have the form $|k|^2 4\pi/L^2$ with $k \in \mathbb{Z}^3/\{0\}$.

(iii) We denote by $(\cdot,\cdot)$ the $L^2([0,l]^3)^3$ inner product and by $\|\cdot\|_2$ the corresponding $L^2([0,l]^3)^3$ norm. By virtue of Poincaré inequality one can show that there is a constant $c > 0$, such that

$$\begin{gathered} c\|Au\|_2 \le \|u\|_{H^2} \le c^{-1}\|Au\|_2 \\ \text{for every } u \in D(A), \\ c\left\|A^{1/2}u\right\|_2 \le \|u\|_{H^1} \le c^{-1}\left\|A^{1/2}u\right\|_2 \\ \text{for every } u \in V. \end{gathered} \tag{29}$$

Moreover, one can show that $V = D(A^{1/2})$ (cf. [38, 42]). We denote by V' the dual of V. Hereafter c will denote a generic scale invariant positive constant which is independent of the physical parameters in the equation.

(iv) Following the notation for the Navier-Stokes equations we denote yhat $B(u,v) = P[(u\cdot\nabla)v]$, and we set $B(v)u = B(u,v)$ for every u, $v \in V$. That is, for every fixed $v \in V$, $B(v)$ is a linear operator acting on u. Note that

$$\begin{gathered} (B(u,v),w) = -(B(u,w),v) \\ \text{for every } u,v,w \in V. \end{gathered} \tag{30}$$

We also denote that $\widetilde{B}(u,v) = -P(u\times(\nabla\times v))$ for every $u, v \in V$. Using the identity

$$(b\cdot\nabla)a + \sum_{j=1}^{3} a_j\nabla b_j = -b\times(\nabla\times a) + \nabla(a\cdot b), \tag{31}$$

one can easily show that

$$\begin{aligned} \left(\widetilde{B}(u,v),w\right) &= (B(u,v),w) - (B(w,v),u) \\ &= \left(B(v)u - B^*(v)u, w\right) \end{aligned} \tag{32}$$

for every u, v, $w \in V$, where $B^*(v)$ denotes the adjoint operator of the linear operator $B(v)$ defined above. As a result we have

$$\widetilde{B}(u,v) = \left(B(v) - B^*(v)\right)u \quad \text{for every } u, v \in V. \tag{33}$$

The next lemma will present some properties of the bilinear operator $\widetilde{B}$.

Lemma 14 (cf. [19]). *The operator $\widetilde{B}$ can be extended continuously from $V\times V$ with values in V', and it satisfies*

$$\begin{gathered} \left|\left\langle \widetilde{B}(u,v),w\right\rangle_{V'}\right| \le c\|u\|_2^{1/2}\left\|A^{1/2}u\right\|_2^{1/2}\left\|A^{1/2}v\right\|_2\left\|A^{1/2}w\right\|_2, \\ \left|\left\langle \widetilde{B}(u,v),w\right\rangle_{V'}\right| \le c\left\|A^{1/2}u\right\|_2\left\|A^{1/2}v\right\|_2\|w\|_2^{1/2}\left\|A^{1/2}w\right\|_2^{1/2} \end{gathered} \tag{34}$$

for every u, v, $w \in V$. Moreover,

$$\left\langle \widetilde{B}(u,v),w\right\rangle_{V'} = -\left\langle \widetilde{B}(w,v),u\right\rangle_{V'}, \quad \textit{for every } u,v,w \in V \tag{35}$$

and in particular

$$\left\langle \widetilde{B}(u,v),u\right\rangle_{V'} \equiv 0 \quad \textit{for every } u,v \in V. \tag{36}$$

Furthermore, one has

$$\left|\left\langle \widetilde{B}(u,v),w\right\rangle_{D(A)'}\right| \le c\|u\|_2\left\|A^{1/2}v\right\|_2\left\|A^{1/2}w\right\|_2^{1/2}\|Aw\|_2^{1/2} \tag{37}$$

for every $u \in H$, $v \in V$, and $w \in D(A)$ and by symmetry one has

$$\left|\left\langle \widetilde{B}(u,v),w\right\rangle_{D(A)'}\right| \le c\left\|A^{1/2}u\right\|_2^{1/2}\|Au\|_2^{1/2}\left\|A^{1/2}v\right\|_2\|w\|_2 \tag{38}$$

for every $u \in D(A)$, $v \in V$, and $w \in H$. Also

$$\begin{aligned} &\left|\left\langle \widetilde{B}(u,v),w\right\rangle_{D(A)'}\right| \\ &\quad \le c\left(\|u\|_2^{1/2}\left\|A^{1/2}u\right\|_2^{1/2}\|v\|_2\|Aw\|_2\right. \\ &\quad\quad \left. + \|v\|_2\left\|A^{1/2}u\right\|_2\left\|A^{1/2}w\right\|_2^{1/2}\|Aw\|_2^{1/2}\right) \end{aligned} \tag{39}$$

for every $u \in V$, $v \in H$, and $w \in D(A)$. In addition,

$$\begin{aligned} &\left|\left\langle \widetilde{B}(u,v),w\right\rangle_{D(A)'}\right| \\ &\quad \le c\left(\left\|A^{1/2}u\right\|_2^{1/2}\|Au\|_2^{1/2}\|v\|_2\left\|A^{1/2}w\right\|_2\right. \\ &\quad\quad \left. + \|v\|_2\|Au\|_2\|w\|_2^{1/2}\left\|A^{1/2}w\right\|_2^{1/2}\right) \end{aligned} \tag{40}$$

for every $u \in D(A)$, $v \in H$, and $w \in V$.

4. Stochastic Nonautonomous Camassa-Holm Equations

We now apply the result in Section 2 to the stochastic nonautonomous Camassa-Holm equations. To associate a family of processes $\mathscr{U}_\sigma(t,s,\omega)$ with the stochastic equations over $(\Sigma, T(t))$ and $(\Omega,\mathscr{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$, we need to convert the stochastic equations with a random additive term into deterministic equations with a random parameter.

First we define the bounded linear operator Q in (8) as follows:

$$Qx = \sum_{j=1}^{n} x_j h_j, \quad x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n, \tag{41}$$

where $h_j \in D(A)$, $j = 1, 2, \dots, n$.

Given $j = 1, 2, \dots, n$, consider the Ornstein-Uhlenbeck equation:

$$dy_j + \mu y_j dt = dw_j(t). \tag{42}$$

One can easily check that a solution to (42) is given by

$$y_j(t) = y_j\left(\theta_t\omega_j\right) = -\mu\int_{-\infty}^{0} e^{\mu\tau}\left(\theta_t\omega_j\right)(\tau)\,d\tau, \quad t \in \mathbb{R}. \tag{43}$$

It is known that the random variable $|y_j(\omega_j)|$ is tempered and that $y_j(\theta_t\omega_j)$ is $\mathbb{P}$ a.e is continuous. Now we put $z(\theta_t\omega) = (\alpha_0^2 + \alpha_1^2 A)^{-1}\sum_{j=1}^{m} h_j y_j(\theta_t\omega_j)$. By (42) we have

$$d\left(\alpha_0^2 z(\theta_t\omega) + \alpha_1^2 Az(\theta_t\omega)\right) + \mu\left(\alpha_0^2 z(\theta_t\omega)\,dt + \alpha_1^2 Az(\theta_t\omega)\right)dt = \sum_{j=1}^{n} h_j dw_j(t). \tag{44}$$

Employing Cauchy-Schwarz's inequality, we get

$$\|z(\theta_t\omega)\|_2^2 + \left\|A^{1/2}z(\theta_t\omega)\right\|_2^2 + \|Az(\theta_t\omega)\|_2^2 \le c\sum_{j=1}^{n}\left|y_j(\theta_t\omega_j)\right|^2, \tag{45}$$

To show that (8) corresponds to a process $\{\mathcal{U}_\sigma(t,\tau,\omega)\}$, we let $v = u - z(\theta_t\omega)$, where u is a solution of (8). Then for $v(t,\omega)$, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\alpha_0^2 v + \alpha_1^2 Av\right)\\&\quad + \nu A\left(\alpha_0^2 v + \alpha_1^2 Av + \alpha_0^2 z(\theta_t\omega) + \alpha_1^2 Az(\theta_t\omega)\right)\\&\quad - (v + z(\theta_t\omega))\\&\quad \times\left(\nabla\times\left(\alpha_0^2 v + \alpha_1^2 Av + \alpha_0^2 z(\theta_t\omega) + \alpha_1^2 Az(\theta_t\omega)\right)\right)\\&\quad + \frac{1}{\rho_0}\nabla p\\&= f(x,t) + \mu\alpha_0^2 z(\theta_t\omega) + \mu\alpha_1^2 Az(\theta_t\omega)\end{aligned} \tag{46}$$

defined in the periodic box $[0,l]^3$, satisfying

$$\begin{aligned}&\nabla\cdot v = 0,\\&v(x,\tau,\omega) = v_\tau(x,\omega) = u_\tau(x) - z(\theta_\tau\omega) \in H.\end{aligned} \tag{47}$$

We apply P to (46) and use the notation in Section 3 to obtain the equivalent system of equations

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\alpha_0^2 v + \alpha_1^2 Av\right)\\&\quad + \nu A\left(\alpha_0^2 v + \alpha_1^2 Av + \alpha_0^2 z(\theta_t\omega) + \alpha_1^2 Az(\theta_t\omega)\right)\\&\quad + \widetilde{B}\left(v + z(\theta_t\omega), \alpha_0^2 v + \alpha_1^2 Av\right.\\&\qquad\left. + \alpha_0^2 z(\theta_t\omega) + \alpha_1^2 Az(\theta_t\omega)\right)\\&= Pf(x,t) + \mu\alpha_0^2 z(\theta_t\omega) + \mu\alpha_1^2 Az(\theta_t\omega)\end{aligned} \tag{48}$$

satisfying the initial condition

$$v(x,\tau,\omega) = v_\tau(x,\omega). \tag{49}$$

By a Galerkin method as in [19], it can be proved that if $f(x,t) \in L^2_{\text{loc}}((0,T);H)$ and $v_\tau \in V$, for $\mathbb{P}$ a.e. $\omega \in \Omega$, (48) has a unique solution satisfying for any $T > \tau$

$$\begin{aligned}&v(t,\omega,v_\tau) \in C([\tau,T);V)\bigcap L^2([\tau,T);D(A)),\\&\frac{dv}{dt} \in L^2([\tau,T);H)\end{aligned} \tag{50}$$

and such that, for almost all $t \in [\tau,T)$ and for any $w \in D(A)$,

$$\begin{aligned}&\left\langle\frac{d}{dt}\left(\alpha_0^2 v + \alpha_1^2 Av\right), w\right\rangle_{D(A)'}\\&\quad + \nu\left\langle A\left(\alpha_0^2 v + \alpha_1^2 Av\right.\right.\\&\qquad\left.\left. + \alpha_0^2 z(\theta_t\omega) + \alpha_1^2 Az(\theta_t\omega)\right), w\right\rangle_{D(A)'}\\&\quad + \left\langle\widetilde{B}\left(v + z(\theta_t\omega), \alpha_0^2 v + \alpha_1^2 Av\right.\right.\\&\qquad\left.\left. + \alpha_0^2 z(\theta_t\omega) + \alpha_1^2 Az(\theta_t\omega)\right), w\right\rangle_{D(A)'}\\&= \left(Pf(x,t) + \mu\alpha_0^2 z(\theta_t\omega) + \mu\alpha_1^2 Az(\theta_t\omega), w\right).\end{aligned} \tag{51}$$

Now for any $f(x,t) \in \Sigma(f_0)$, (48) with f instead of f_0 possesses a corresponding process $\{\mathcal{U}_f(t,\tau,\omega)\}$ acting on V. It is analogous to the proof in [27] to prove that, for $\mathbb{P}$ a.e $\omega \in \Omega$, the family $\{\mathcal{U}_f(t,\tau,\omega) \mid f \in \Sigma(f_0)\}$ of processes is $(V\times\Sigma(f_0);V)$-continuous. Let

$$\begin{aligned}\mathcal{N}_f(\omega) = \Big\{&v_f(x,t,\omega)\ \text{for}\ t\in\mathbb{R} \mid v_f(x,t,\omega)\\&\text{is solution of (48) satisfying}\\&\left\|v_f(x,t,\omega)\right\|_H \le M_f(\omega)\ \forall t\in\mathbb{R}\Big\}\end{aligned} \tag{52}$$

be the so-called kernel of the process $\{\mathcal{U}_f(t,\tau,\omega)\}$.

5. Uniformly Random Attractor for Stochastic Nonautonomous Camassa-Holm Equation

In [19], the authors have shown that the semigroup corresponding to the autonomous system possesses a global attractor. In [21–24], the authors have proved that the deterministic version of nonautonomous system has a uniform attractor. The main objective of this section is to obtain the existence of uniformly (with respect to $f \in \Sigma(f_0)$) random attractor for the stochastic nonautonomous Camassa-Holm equations in V and $D(A)$.

Lemma 15. *Let $\{B(\omega)\}_{\omega\in\Omega}$ be tempered and $v_\tau(\omega) \in B(\omega)$. Then the process $\{\mathcal{U}_f(t,\tau,\omega)\}$ corresponding to* (48) *possesses a uniformly (with respect to $f \in \Sigma(f_0)$) random absorbing set $\mathcal{K}_0(\omega)$ in V.*

Proof. Letting $w = v$ in (51), we have

$$\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)+\nu\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right)$$
$$=-\nu\left(\sum_{j=1}^{n}Ah_jy_j\left(\theta_t\omega_j\right),v\right)$$
$$-\left(\widetilde{B}\left(v+z\left(\theta_t\omega\right),\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right),v\right)$$
$$+\left(Pf\left(x,t\right),v\right)+\left(\sum_{j=1}^{n}\mu h_jy_j\left(\theta_t\omega_j\right),v\right). \tag{53}$$

Now we estimate the second term of (53) on the right-hand side. Applying Lemma 14 we have

$$\left|\left(\widetilde{B}\left(v+z\left(\theta_t\omega\right),\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right),v\right)\right|$$
$$=\left|\left(\widetilde{B}\left(z\left(\theta_t\omega\right),\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right),v\right)\right|$$
$$\leq c\left\|A^{1/2}z(\theta_t\omega)\right\|_2^{1/2}\left\|Az(\theta_t\omega)\right\|_2^{1/2}$$
$$\times\left\|\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right\|_2\left\|A^{1/2}v\right\|_2$$
$$+c\left\|Az\left(\theta_t\omega\right)\right\|_2\left\|\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right\|_2$$
$$\times\|v\|_2^{1/2}\left\|A^{1/2}v\right\|_2^{1/2}$$
$$\leq c\left\|A^{1/2}z(\theta_t\omega)\right\|_2\|v\|_2^2+c\left\|Az(\theta_t\omega)\right\|_2\left\|A^{1/2}v\right\|_2^2$$
$$+\frac{\nu\alpha_1^2}{4}\|Av\|_2^2$$
$$+c\left\|A^{1/2}z(\theta_t\omega)\right\|_2\left\|Az(\theta_t\omega)\right\|_2\left\|A^{1/2}v\right\|_2^2$$
$$+c\left\|A^{1/2}z(\theta_t\omega)\right\|_2\left\|Az(\theta_t\omega)\right\|_2$$
$$+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2\left\|A^{1/2}v\right\|_2^2$$
$$+c\left\|Az\left(\theta_t\omega\right)\right\|_2\|v\|_2^2+\frac{\nu\alpha_1^2}{4}\|Av\|_2^2$$
$$+c\left\|Az(\theta_t\omega)\right\|_2^2\|v\|_2^2+c\left\|Az(\theta_t\omega)\right\|_2^2\left\|A^{1/2}v\right\|_2^2$$
$$+c\left\|Az(\theta_t\omega)\right\|_2^2+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2\left\|A^{1/2}v\right\|_2^2$$
$$+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2$$
$$\leq\frac{\nu\alpha_1^2}{2}\|Av\|_2^2+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)$$
$$+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)$$
$$+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2. \tag{54}$$

By Poincaré's inequality, we have

$$\nu\left|\left(\sum_{j=1}^{n}Ah_jy_j\left(\theta_t\omega_j\right),v\right)\right|\leq\frac{\nu\alpha_0^2}{6}\left\|A^{1/2}v\right\|_2^2+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2,$$
$$\left|\left(Pf\left(x,t\right),v\right)\right|\leq\|f\|_{V'}\|v\|_2\leq\frac{\nu\alpha_0^2}{6}\left\|A^{1/2}v\right\|_2^2+c\|f\|_{V'}^2,$$
$$\left|\left(\sum_{j=1}^{n}\mu h_jy_j\left(\theta_t\omega_j\right),v\right)\right|\leq\frac{\nu\alpha_0^2}{6}\left\|A^{1/2}v\right\|_2^2+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2. \tag{55}$$

Associating with the above inequalities and employing Poincaré's inequality, we have

$$\frac{d}{dt}\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)$$
$$+\nu\lambda_1\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)$$
$$\leq c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)$$
$$+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)$$
$$+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2+c\|f\|_{V'}^2. \tag{56}$$

Applying Gronwall's lemma, we have

$$\alpha_0^2\|v\left(t,\omega\right)\|_2^2+\alpha_1^2\left\|A^{1/2}v\left(t,\omega\right)\right\|_2^2$$
$$\leq e^{J_1(\tau,t,\omega)}\left(\alpha_0^2\left\|v_\tau\left(\omega\right)\right\|_2^2+\alpha_1^2\left\|A^{1/2}v_\tau\left(\omega\right)\right\|_2^2\right)$$
$$+\int_\tau^t\left(c\sum_{j=1}^{n}\left|y_j\left(\theta_s\omega_j\right)\right|^2+c\left\|f\left(s\right)\right\|_{V'}^2\right)e^{J_1(s,t,\omega)}ds, \tag{57}$$

where

$$J_1(\tau,t,\omega) = -\nu\lambda_1 t + \nu\lambda_1\tau + c\sum_{j=1}^{n}\int_\tau^t \left|y_j\left(\theta_s\omega_j\right)\right| ds + c\sum_{j=1}^{n}\int_\tau^t \left|y_j\left(\theta_s\omega_j\right)\right|^2 ds. \quad (58)$$

Replacing ω by $\theta_{-t}\omega$ in (57) and (58), we have

$$\begin{aligned}&\alpha_0^2\|v(t,\theta_{-t}\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(t,\theta_{-t}\omega)\right\|_2^2 \\ &\le e^{J_1(\tau-t,0,\omega)}\left(\alpha_0^2\|v_\tau(\theta_{-t}\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v_\tau(\theta_{-t}\omega)\right\|_2^2\right) \\ &\quad + \int_{\tau-t}^0\left(c\sum_{j=1}^{n}\left|y_j\left(\theta_s\omega_j\right)\right|^2 + c\|f(s+t)\|_{V'}^2\right)e^{J_1(s,0,\omega)}ds.\end{aligned} \quad (59)$$

Note that $\{|y_j(\theta_t\omega_j)|\}_{j=1}^n$ are stationary and ergodic (cf. [24]), then it follows from ergodic theorem that

$$\lim_{t\to\infty}\frac{1}{t}\int_{-t}^0\left|y_j\left(\theta_\tau\omega_j\right)\right|d\tau = E\left(\left|y_j\left(\omega_j\right)\right|\right). \quad (60)$$

On the other hand, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_{-t}^0\left|y_j\left(\theta_\tau\omega_j\right)\right|d\tau &= E\left(\left|y_j\left(\omega_j\right)\right|\right) = \frac{c}{\sqrt{\mu}}, \\ \lim_{t\to\infty}\frac{1}{t}\int_{-t}^0\left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau &= E\left(\left|y_j\left(\omega_j\right)\right|^2\right) = \frac{c}{\mu}.\end{aligned} \quad (61)$$

Employing (61), we have

$$\begin{aligned}\lim_{s\to-\infty}\frac{J_1(s,0,\omega)}{s} &= \lim_{s\to-\infty}\left(\nu\lambda_1 + c\sum_{j=1}^{m}\frac{1}{s}\int_s^0\left|y_j\left(\theta_\tau\omega_j\right)\right|d\tau + c\sum_{j=1}^{m}\frac{1}{s}\int_s^0\left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau\right) \\ &= \left(\nu\lambda_1 - \frac{c}{\sqrt{\mu}} - \frac{c}{\mu}\right).\end{aligned} \quad (62)$$

So for $\mathbb{P}$ a.e. $\omega\in\Omega$, there are $T_0(\omega) > 0$ and $\mu_0 > 0$ such that, for $s \ge T_1(\omega)$ and $\mu \ge \mu_0$,

$$\begin{aligned}\frac{J_1(s,0,\omega)}{s} &= \nu\lambda_1 + c\sum_{j=1}^{m}\frac{1}{s}\int_s^0\left|y_j\left(\theta_\tau\omega_j\right)\right|d\tau + c\sum_{j=1}^{m}\frac{1}{s}\int_s^0\left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau < 0, \\ J_1(s,0,\omega) &\longrightarrow -\infty, \quad \text{as } s\longrightarrow-\infty.\end{aligned} \quad (63)$$

Since $\{|y_j(\omega_j)|\}_{j=1}^n$ are tempered, the integral

$$r_1(\omega) = \int_{-\infty}^0\sum_{j=1}^{n}\left|y_j\left(\theta_s\omega_j\right)\right|^2 e^{J_1(s,0,\omega)}ds \quad (64)$$

is convergent. For any $t > \tau$, there exists $K \in \mathbb{N}$ such that $-K < \tau - t$, and we have

$$\begin{aligned}\int_{\tau-t}^0\|f(s+t)\|_{V'}^2 e^{J_1(s,0,\omega)}ds &= \int_{\tau-t}^0\|f_0(s+t+\eta)\|_{V'}^2 e^{J_1(s,0,\omega)}ds \\ &\le \int_{-K}^0\|f_0(s+t+\eta)\|_{V'}^2 e^{J_1(s,0,\omega)}ds.\end{aligned} \quad (65)$$

Then by piecewise integration, we have

$$\begin{aligned}\int_{-K}^0\|f_0(s+t+\eta)\|_{V'}^2 e^{J_1(s,0,\omega)}ds = \sum_{j=1}^{K}\int_{-n+j-1}^{-n+j}\|f_0(s+t+\eta)\|_{V'}^2 e^{J_1(s,0,\omega)}ds.\end{aligned} \quad (66)$$

By differential mean value theorem, for each $1 \le j \le K$, there exists $s_j \in [-n+j-1, -n+j]$, such that

$$\begin{aligned}\sum_{j=1}^{K}\int_{-n+j-1}^{-n+j}\|f_0(s+t+\eta)\|_{V'}^2 e^{J_1(s,0,\omega)}ds = \sum_{j=1}^{K}e^{J_1(s_j,0,\omega)}\int_{-n+j-1}^{-n+j}\|f_0(s+t+\eta)\|_{V'}^2 ds.\end{aligned} \quad (67)$$

By the property of normal function, there exists M_{f_0}, which just depends on f_0, such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}\|f_0(s)\|_{V'}^2 ds < M_{f_0}. \quad (68)$$

It is easy to check that the series $\sum_{j=1}^{\infty}e^{J_1(s_j,0,\omega)}$ is convergent. Associating with (65)–(68), for any $t > \tau$, we have

$$\int_{\tau-t}^0\|f(s+t)\|_{V'}^2 e^{J_1(s,0,\omega)}ds \le r_2(\omega), \quad (69)$$

where

$$r_2(\omega) = M_{f_0}\sum_{j=1}^{\infty}e^{J_1(s_j,0,\omega)}. \quad (70)$$

Therefore (64) and (69) imply that, for $t > \tau$, the second term on the right-hand side of (59) can be bounded by

$$\int_{\tau-t}^0\left(c\sum_{j=1}^{n}\left|y_j\left(\theta_s\omega_j\right)\right|^2 + c\|f(s+t)\|_{V'}^2\right)e^{J_1(s,0,\omega)}ds \le r_3(\omega), \quad (71)$$

where $r_3(\omega) = cr_1(\omega) + cr_2(\omega)$. For $v_\tau(\omega) \in B(\omega)$ being tempered, there exists $T_1(\omega) > 0$ such that, when $t \ge T_2(\omega)$,

$$e^{J_1(\tau-t,0,\omega)}\left(\alpha_0^2\|v_\tau(\theta_{-t}\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v_\tau(\theta_{-t}\omega)\right\|_2^2\right) < r_3(\omega). \quad (72)$$

Letting $T_3(\omega) = \max\{T_1(\omega), T_2(\omega)\}$ and $r_4(\omega) = 2r_3(\omega)$, when $t \geq T_3(\omega)$, we have

$$\alpha_0^2\|v(t,\theta_{-t}\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(t,\theta_{-t}\omega)\right\|_2^2 \leq r_4(\omega), \quad \forall t \geq T_3(\omega), f \in \Sigma(f_0). \tag{73}$$

We define

$$\mathcal{K}_1(\omega) = \left\{v(\omega) \in V \mid \alpha_0^2\|v(\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(\omega)\right\|_2^2 \leq r_4(\omega)\right\}. \tag{74}$$

In conclusion, $\mathcal{K}_1(\omega)$ is a uniformly random absorbing set for $\{\mathcal{U}(t,\tau,\omega)\}$ in V, which complete the proof. □

Lemma 16. *Let $\{B(\omega)\}_{\omega\in\Omega}$ be tempered and $v_\tau(\omega) \in B(\omega)$. Then the process $\{\mathcal{U}_f(t,\tau,\omega)\}$ corresponding to (48) possesses a uniformly (with respect to $f \in \Sigma(f_0)$) random absorbing set $\mathcal{K}_2(\omega)$ in $D(A)$.*

Proof. Integrating (56) into $[t, t+1]$ where $t \geq T_3(\omega)$, we have

$$\begin{aligned}
&\alpha_0^2\|v(t+1,\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(t+1,\omega)\right\|_2^2 \\
&\quad + \nu\int_t^{t+1}\left(\alpha_0^2\left\|A^{1/2}v(s,\omega)\right\|_2^2 + \alpha_1^2\|Av(s,\omega)\|_2^2\right)ds \\
&\leq \left(\alpha_0^2\|v(t,\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(t,\omega)\right\|_2^2\right) \\
&\quad + c\sum_{j=1}^n\int_t^{t+1}\left|y_j(\theta_s\omega_j)\right| \\
&\qquad \times\left(\alpha_0^2\|v(s,\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(s,\omega)\right\|_2^2\right)ds \\
&\quad + c\sum_{j=1}^n\int_t^{t+1}\left|y_j(\theta_s\omega_j)\right|^2 \\
&\qquad \times\left(\alpha_0^2\|v(s,\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(s,\omega)\right\|_2^2\right)ds \\
&\quad + c\sum_{j=1}^n\int_t^{t+1}\left|y_j(\theta_s\omega_j)\right|^2ds.
\end{aligned} \tag{75}$$

Replacing ω by $\theta_{-t-1}\omega$ in (75), we have

$$\begin{aligned}
&\nu\int_t^{t+1}\left(\alpha_0^2\left\|A^{1/2}v(s,\theta_{-t-1}\omega)\right\|_2^2 + \alpha_1^2\|Av(s,\theta_{-t-1}\omega)\|_2^2\right)ds \\
&\leq c\sum_{j=1}^n\int_t^{t+1}\left|y_j(\theta_{s-t-1}\omega_j)\right|\left(\alpha_0^2\|v(s,\theta_{-t-1}\omega)\|_2^2\right. \\
&\qquad\left. + \alpha_1^2\left\|A^{1/2}v(s,\theta_{-t-1}\omega)\right\|_2^2\right)ds \\
&\quad + c\sum_{j=1}^n\int_t^{t+1}\left|y_j(\theta_{s-t-1}\omega_j)\right|^2\left(\alpha_0^2\|v(s,\theta_{-t-1}\omega)\|_2^2\right. \\
&\qquad\left. + \alpha_1^2\left\|A^{1/2}v(s,\theta_{-t-1}\omega)\right\|_2^2\right)ds \\
&\quad + c\sum_{j=1}^n\int_t^{t+1}\left|y_j(\theta_{s-t-1}\omega_j)\right|^2ds \\
&\quad + \left(\alpha_0^2\|v(t,\theta_{-t-1}\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(t,\theta_{-t-1}\omega)\right\|_2^2\right).
\end{aligned} \tag{76}$$

As the previous consideration in Lemma 15, for $s \in [t, t+1)$ where $t \geq T_2(\omega)$, we have

$$\alpha_0^2\|v(s,\theta_{-t-1}\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(s,\theta_{-t-1}\omega)\right\|_2^2 \leq r_5(\omega), \tag{77}$$

where $r_5(\omega) = e^{\nu\lambda_1}r_4(\omega)$.

Associating (76) with (77), we have

$$\int_t^{t+1}\left(\alpha_0^2\left\|A^{1/2}v(s,\theta_{-t-1}\omega)\right\|_2^2 + \alpha_1^2\|Av(s,\theta_{-t-1}\omega)\|_2^2\right)ds \leq r_6(\omega), \tag{78}$$

where

$$\begin{aligned}
r_6(\omega) &= cr_5(\omega)\sum_{j=1}^n\int_{-1}^0\left|y_j(\theta_s\omega_j)\right|ds \\
&\quad + cr_5(\omega)\sum_{j=1}^n\int_{-1}^0\left|y_j(\theta_s\omega_j)\right|^2ds \\
&\quad + c\sum_{j=1}^n\int_{-1}^0\left|y_j(\theta_s\omega_j)\right|^2ds + r_5(\omega).
\end{aligned} \tag{79}$$

Now we let $w = Av$ in (51), and we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2 + \alpha_1^2\|Av\|_2^2\right) \\
&\quad + \nu\left(\alpha_0^2\|Av\|_2^2 + \alpha_1^2\left\|A^{3/2}v\right\|_2^2\right) \\
&= -\nu\left(\sum_{j=1}^n Ah_jy_j(\theta_t\omega_j), Av\right) \\
&\quad - \left(\widetilde{B}\left(v + z(\theta_t\omega), \alpha_0^2v + \alpha_1^2Av + \sum_{j=1}^n h_jy_j(\theta_t\omega_j)\right), Av\right) \\
&\quad + (Pf(x,t), Av) + \left(\sum_{j=1}^n \mu h_jy_j(\theta_t\omega_j), Av\right).
\end{aligned} \tag{80}$$

Employing Lemma 14, the bilinear term in (80) can be bounded by

$$\begin{aligned}
&\left(\widetilde{B}\left(v+z(\theta_t\omega),\alpha_0^2 v+\alpha_1^2 Av+\sum_{j=1}^n h_j y_j(\theta_t\omega_j)\right),Av\right)\\
&\le c\left(\left\|A^{1/2}v+A^{1/2}z(\theta_t\omega)\right\|_2^{1/2}\|Av+Az(\theta_t\omega)\|_2^{1/2}\right.\\
&\qquad\left.\times\left\|\alpha_0^2 v+\alpha_1^2 Av+\sum_{j=1}^n h_j y_j(\theta_t\omega_j)\right\|_2\left\|A^{3/2}v\right\|_2\right)\\
&\quad+c\left(\|Av+Az(\theta_t\omega)\|_2\left\|\alpha_0^2 v+\alpha_1^2 Av+\sum_{j=1}^n h_j y_j(\theta_t\omega_j)\right\|_2\right.\\
&\qquad\left.\times\|Av\|_2^{1/2}\left\|A^{3/2}v\right\|_2^{1/2}\right)\\
&\le \nu\alpha_1^2\left\|A^{3/2}v\right\|_2^2+\frac{\nu\alpha_0^2}{4}\|Av\|_2^2\\
&\quad+c\left(\alpha_0^2\|v\|_2^2+\alpha_0^2\left\|A^{1/2}v\right\|_2^2\right)^2\\
&\quad+c\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_0^2\|Av\|_2^2\right)^2\\
&\quad+c\left(\alpha_0^2\|v\|_2^2+\alpha_0^2\left\|A^{1/2}v\right\|_2^2\right)\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_0^2\|Av\|_2^2\right)\\
&\quad+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2\left(\alpha_0^2\|v\|_2^2+\alpha_0^2\left\|A^{1/2}v\right\|_2^2\right)\\
&\quad+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_0^2\|Av\|_2^2\right)\\
&\quad+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^4.
\end{aligned}\tag{81}$$

The other terms are bounded by

$$\begin{aligned}
&\nu\left|\left(\sum_{j=1}^n Ah_j y_j(\theta_t\omega_j),Av\right)\right|\le\frac{\nu\alpha_0^2}{4}\|Av\|_2^2+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2.\\
&\left|(Pf(x,t),Av)\right|\le\|f\|_{V'}\|Av\|_2\le\frac{\nu\alpha_0^2}{4}\|Av\|_2^2+c\|f\|_{V'}^2.\\
&\mu\left|\left(\sum_{j=1}^n h_j y_j(\theta_t\omega_j),Av\right)\right|\le\frac{\nu\alpha_0^2}{4}\|Av\|_2^2+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2.
\end{aligned}\tag{82}$$

Associating with all the above inequalities, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right)\\
&\le c\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)^2\\
&\quad+c\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right)^2\\
&\quad+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2\\
&\quad+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^4\\
&\quad+c\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)\\
&\quad\times\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right)\\
&\quad+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)\\
&\quad+c\|f\|_{V'}^2\\
&\quad+c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right).
\end{aligned}\tag{83}$$

Applying Gronwall's lemma on $[s,t+1]$ where $s\in(t,t+1)$ and $t\ge T_3(\omega)$, we have

$$\begin{aligned}
&\alpha_0^2\left\|A^{1/2}v(t+1,\omega)\right\|_2^2+\alpha_1^2\|Av(t+1,\omega)\|_2^2\\
&\le\left(\alpha_0^2\left\|A^{1/2}v(s,\omega)\right\|_2^2+\alpha_1^2\|Av(s,\omega)\|_2^2\right)e^{J_2(s,t+1,\omega)}\\
&\quad+c\int_s^{t+1}\sum_{j=1}^n\left|y_j(\theta_\tau\omega_j)\right|^2e^{J_2(\tau,t+1,\omega)}d\tau\\
&\quad+c\int_s^{t+1}\sum_{j=1}^n\left|y_j(\theta_\tau\omega_j)\right|^4e^{J_2(\tau,t+1,\omega)}d\tau\\
&\quad+c\int_s^{t+1}\|f(\tau)\|_{V'}^2e^{J_2(\tau,t+1,\omega)}d\tau\\
&\quad+c\int_s^{t+1}\sum_{j=1}^n\left|y_j(\theta_\tau\omega_j)\right|^2\\
&\qquad\times\left(\alpha_0^2\|v(\tau,\omega)\|_2^2+\alpha_1^2\left\|A^{1/2}v(\tau,\omega)\right\|_2^2\right)e^{J_2(\tau,t+1,\omega)}d\tau\\
&\quad+c\int_s^{t+1}\left(\alpha_0^2\|v(\tau,\omega)\|_2^2+\alpha_1^2\left\|A^{1/2}v(\tau,\omega)\right\|_2^2\right)^2e^{J_2(\tau,t+1,\omega)}d\tau,
\end{aligned}\tag{84}$$

where

$$J_2(s,t,\omega) = c\int_s^t \left(\alpha_0^2\|v(\tau,\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(\tau,\omega)\right\|_2^2\right)d\tau + c\int_s^t \left(\alpha_0^2\left\|A^{1/2}v(\tau,\omega)\right\|_2^2 + \alpha_1^2\|Av(\tau,\omega)\|_2^2\right)d\tau + c\int_s^t \sum_{j=1}^n \left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau. \quad (85)$$

Replacing ω by $\theta_{-t-1}\omega$ in (84) and (85), we have

$$\begin{aligned} J_2(\tau,t+1,\theta_{-t-1}\omega) &\le c\int_t^{t+1} \left(\alpha_0^2\|v(\tau,\theta_{-t-1}\omega)\|_2^2 + \alpha_1^2\left\|A^{1/2}v(\tau,\theta_{-t-1}\omega)\right\|_2^2\right)d\tau \\ &+ c\int_t^{t+1}\left(\alpha_0^2\left\|A^{1/2}v(\tau,\theta_{-t-1}\omega)\right\|_2^2 + \alpha_1^2\|Av(\tau,\theta_{-t-1}\omega)\|_2^2\right)d\tau \\ &+ c\sum_{j=1}^n\int_t^{t+1}\left|y_j\left(\theta_{\tau-t-1}\omega_j\right)\right|^2 d\tau \\ &\le cr_5(\omega) + cr_6(\omega) + c\sum_{j=1}^n\int_{-1}^0\left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau \\ &\triangleq r_7(\omega). \end{aligned} \quad (86)$$

And then

$$\begin{aligned} &\alpha_0^2\left\|A^{1/2}v(t+1,\theta_{-t-1}\omega)\right\|_2^2 + \alpha_1^2\|Av(t+1,\theta_{-t-1}\omega)\|_2^2 \\ &\le \left(\alpha_0^2\left\|A^{1/2}v(s,\theta_{-t-1}\omega)\right\|_2^2 + \alpha_1^2\|Av(s,\theta_{-t-1}\omega)\|_2^2\right)e^{r_7(\omega)} \\ &+ ce^{r_7(\omega)}r_5(\omega)^2 + ce^{r_7(\omega)}\int_{-1}^0\sum_{j=1}^n\left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau \\ &+ ce^{r_7(\omega)}\int_{-1}^0\sum_{j=1}^n\left|y_j\left(\theta_\tau\omega_j\right)\right|^4 d\tau \\ &+ ce^{r_7(\omega)}\int_t^{t+1}\|f(\tau)\|_{V'}^2 d\tau \\ &+ ce^{r_7(\omega)}r_5(\omega)\int_{-1}^0\sum_{j=1}^n\left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau. \end{aligned} \quad (87)$$

Integrating (87) with respect to s over $(t,t+1)$ where $t \ge T_3(\omega)$ and employing the property of normal function in (68), we have

$$\alpha_0^2\left\|A^{1/2}v(t+1,\theta_{-t-1}\omega)\right\|_2^2 + \alpha_1^2\|Av(t+1,\theta_{-t-1}\omega)\|_2^2 \le r_8(\omega), \quad \forall t \ge T_3(\omega),\ f \in \Sigma(f_0), \quad (88)$$

where

$$\begin{aligned} r_8(\omega) &= r_6(\omega)e^{r_7(\omega)} + ce^{r_7(\omega)}r_5(\omega)^2 \\ &+ ce^{r_7(\omega)}\int_{-1}^0\sum_{j=1}^n\left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau \\ &+ ce^{r_7(\omega)}\int_{-1}^0\sum_{j=1}^n\left|y_j\left(\theta_\tau\omega_j\right)\right|^4 d\tau \\ &+ ce^{r_7(\omega)}M_{f_0} \\ &+ ce^{r_7(\omega)}r_5(\omega)\int_{-1}^0\sum_{j=1}^n\left|y_j\left(\theta_\tau\omega_j\right)\right|^2 d\tau. \end{aligned} \quad (89)$$

We define

$$\mathscr{K}_2(\omega) = \left\{v(\omega) \in D(A) \mid \alpha_0^2\left\|A^{1/2}v(\omega)\right\|_2^2 + \alpha_1^2\|Av(\omega)\|_2^2 \le r_8(\omega)\right\}. \quad (90)$$

In conclusion, $\mathscr{K}_2(\omega)$ is a uniformly random absorbing set for $\{\mathscr{U}(t,\tau,\omega)\}$ in $D(A)$, which complete the proof. □

So much for that we have proved the existence of bounded uniformly (with respect to $f \in \Sigma(f_0)$) random absorbing sets in V and $D(A)$. Next we derive the existence of uniformly (with respect to $f \in \Sigma(f_0)$) random attractors in V and $D(A)$.

Theorem 17. *If $f_0(x,s)$ is a normal function in $L^2_{\mathrm{loc}}(\mathbb{R},V')$, then the process $\{\mathscr{U}_{f_0}(t,\tau,\omega)\}$ corresponding to (48) possess a compact uniformly (with respect to $\tau \in \mathbb{R}$) random attractor $\mathscr{A}_1(\omega)$ in V which coincides with the uniformly (with respect to $f \in \Sigma(f_0)$) random attractor $\mathscr{A}_{\Sigma(f_0)}$ of the family of processes $\{\mathscr{U}_f(t,\tau,\omega)\}$, $f \in \Sigma(f_0)$:*

$$\mathscr{A}_1(\omega) = \mathscr{A}_{\Sigma(f_0)}(\omega) = \mathscr{W}_{0,\Sigma(f_0)}(\mathscr{K}_1) = \bigcup_{f\in\Sigma(f_0)}\mathscr{N}_f(0,\omega), \quad (91)$$

where $\mathscr{K}_1(\omega)$ is the uniformly (with respect to $f \in \Sigma(f_0)$) random absorbing set in V and $\mathscr{N}_f(\omega)$ is the kernel of the process $\{\mathscr{U}_f(t,\tau,\omega)\}$. Furthermore, the kernel $\mathscr{N}_f(\omega)$ is nonempty for all $f \in \Sigma(f_0)$.

Proof. We only have to verify condition(C). As the previous section, for fixed m, let H_1 be the subspace spanned by $\{e_j\}_{j=1}^m$ and H_2 the orthogonal complement of H_1 in H. We write

$$v = v_1 + v_2, \quad v_1 \in H_1,\ v_2 \in H_2 \text{ for any } v \in H. \quad (92)$$

The proof of boundary of v_1 is similar to the proof in Lemma 15. We need to estimate v_2, where $v = v_1 + v_2$ is a solution of (48). Letting $w = v_2$ in (51), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\|v_2\|_2^2+\alpha_1^2\left\|A^{1/2}v_2\right\|_2^2\right)+\nu\left(\alpha_0^2\left\|A^{1/2}v_2\right\|_2^2+\alpha_1^2\|Av_2\|_2^2\right)\\
&\quad=-\nu\left(\sum_{j=1}^{n}Ah_jy_j\left(\theta_t\omega_j\right),v_2\right)\\
&\qquad-\left(\widetilde{B}\left(v+z\left(\theta_t\omega\right),\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right),v_2\right)\\
&\qquad+\left(Pf\left(x,t\right),v_2\right)+\mu\left(\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right),v_2\right).
\end{aligned}\tag{93}$$

Employing Lemma 14, the second term on the right-hand side of (93) can be bounded by

$$\begin{aligned}
&\left|\left(\widetilde{B}\left(v+z\left(\theta_t\omega\right),\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right),v_2\right)\right|\\
&\quad\le c\|v+z\left(\theta_t\omega\right)\|_2^{1/2}\left\|A^{1/2}v+A^{1/2}z\left(\theta_t\omega\right)\right\|_2^{1/2}\\
&\qquad\times\left\|\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right\|_2\|Av_2\|_2\\
&\qquad+c\left\|A^{1/2}v+A^{1/2}z\left(\theta_t\omega\right)\right\|_2\\
&\qquad\times\left\|\alpha_0^2v+\alpha_1^2Av+\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right)\right\|_2\\
&\qquad\times\left\|A^{1/2}v_2\right\|_2^{1/2}\|Av_2\|_2^{1/2}\\
&\quad\le\frac{\nu\alpha_0^2}{8}\left\|A^{1/2}v_2\right\|_2^2+\frac{\nu\alpha_1^2}{2}\|Av_2\|_2^2\\
&\qquad+c\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)\\
&\qquad\times\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right)\\
&\qquad+c\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)^2\\
&\qquad+c\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right)^2\\
&\qquad+c\sum_{j=1}^{n}\left|y_j\left(\theta_j\omega_j\right)\right|^2+c\sum_{j=1}^{n}\left|y_j\left(\theta_j\omega_j\right)\right|^4\\
&\qquad+c\sum_{j=1}^{n}\left|y_j\left(\theta_j\omega_j\right)\right|^2\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)\\
&\qquad+c\sum_{j=1}^{n}\left|y_j\left(\theta_j\omega_j\right)\right|^2\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right).
\end{aligned}\tag{94}$$

By Poincaré's inequality, note that

$$\begin{aligned}
&\nu\left|\left(\sum_{j=1}^{n}Ah_jy_j\left(\theta_j\omega_j\right),v_2\right)\right|\\
&\quad\le\frac{\nu\alpha_0^2}{8}\left\|A^{1/2}v_2\right\|_2^2+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2,\\
&\left|\left(Pf,v_2\right)\right|\le\|f\|_{V'}\|v_2\|\le\frac{\nu\alpha_0^2}{8}\left\|A^{1/2}v_2\right\|_2^2+c\|f\|_{V'}^2,\\
&\mu\left|\left(\sum_{j=1}^{n}h_jy_j\left(\theta_t\omega_j\right),v_2\right)\right|\\
&\quad\le\frac{\nu\alpha_0^2}{8}\left\|A^{1/2}v_2\right\|_2^2+c\sum_{j=1}^{n}\left|y_j\left(\theta_j\omega_j\right)\right|^2.
\end{aligned}\tag{95}$$

Associating with (93)–(95) and applying Poincaré inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\alpha_0^2\|v_2\|_2^2+\alpha_1^2\left\|A^{1/2}v_2\right\|_2^2\right)\\
&\quad+\nu\lambda_{m+1}\left(\alpha_0^2\|v_2\|_2^2+\alpha_1^2\left\|A^{1/2}v_2\right\|_2^2\right)\\
&\quad\le c\|f\|_{V'}^2+G\left(t,\omega\right),
\end{aligned}\tag{96}$$

where

$$\begin{aligned}
G\left(t,\omega\right)&=c\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)^2\\
&\quad+c\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right)^2\\
&\quad+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2\\
&\quad+c\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)\\
&\quad\times\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right)\\
&\quad+c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^4
\end{aligned}$$

$$+ c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2\left(\alpha_0^2\|v\|_2^2+\alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)$$
$$+ c\sum_{j=1}^{n}\left|y_j\left(\theta_t\omega_j\right)\right|^2\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2+\alpha_1^2\|Av\|_2^2\right). \quad (97)$$

Applying Gronwall Lemma on $[t, t+1]$ where $t \geq T_3(\omega)$, we have

$$\begin{aligned}&\alpha_0^2\|v_2(t+1,\omega)\|_2^2+\alpha_1^2\left\|A^{1/2}v_2(t+1,\omega)\right\|_2^2\\&\quad\leq e^{-\nu\lambda_{m+1}}\left(\alpha_0^2\|v_2(t,\omega)\|_2^2+\alpha_1^2\left\|A^{1/2}v_2(t,\omega)\right\|_2^2\right)\\&\qquad+\int_t^{t+1}G(s,\omega)e^{-\nu\lambda_{m+1}}ds\\&\qquad+c\int_t^{t+1}\|f(s)\|_{V'}^2e^{-\nu\lambda_{m+1}}ds.\end{aligned} \quad (98)$$

Replacing ω by $\theta_{-t-1}\omega$ in (98), we have

$$\begin{aligned}&\alpha_0^2\|v_2(t+1,\theta_{-t-1}\omega)\|_2^2\\&\qquad+\alpha_1^2\left\|A^{1/2}v_2(t+1,\theta_{-t-1}\omega)\right\|_2^2\\&\quad\leq e^{-\nu\lambda_{m+1}}\left(\alpha_0^2\|v_2(t,\theta_{-t-1}\omega)\|_2^2\right.\\&\qquad\left.+\alpha_1^2\left\|A^{1/2}v_2(t,\theta_{-t-1}\omega)\right\|_2^2\right)\\&\qquad+\int_t^{t+1}G(s,\theta_{-t-1}\omega)e^{-\nu\lambda_{m+1}}ds\\&\qquad+c\int_t^{t+1}\|f(s)\|_{V'}^2e^{-\nu\lambda_{m+1}}ds.\end{aligned} \quad (99)$$

Now we need to estimate the second term on the right-hand side of (99). As the previous consideration in Lemma 15, for $s \in [t, t+1)$ where $t \geq T_3(\omega)$, we have

$$\alpha_0^2\|v(s,\theta_{-t-2})\|_2^2+\alpha_1^2\left\|A^{1/2}v(s,\theta_{-t-2})\right\|_2^2\leq r_9(\omega), \quad (100)$$

where $r_9(\omega) = e^{2\nu\lambda_1}r_4(\omega)$. As the previous consideration in Lemma 16, we have

$$\begin{aligned}&\int_t^{t+1}\left(\alpha_0^2\left\|A^{1/2}v(s,\theta_{-t-2}\omega)\right\|_2^2\right.\\&\qquad\left.+\alpha_1^2\|Av(s,\theta_{-t-2}\omega)\|_2^2\right)ds\leq r_{10}(\omega),\end{aligned} \quad (101)$$

where

$$\begin{aligned}r_{10}(\omega)&=cr_9(\omega)\sum_{j=1}^{n}\int_{-2}^{0}\left|y_j\left(\theta_s\omega_j\right)\right|ds\\&\quad+cr_9(\omega)\sum_{j=1}^{n}\int_{-2}^{0}\left|y_j\left(\theta_s\omega_j\right)\right|^2ds\\&\quad+c\sum_{j=1}^{n}\int_{-2}^{0}\left|y_j\left(\theta_s\omega_j\right)\right|^2ds+r_9(\omega).\end{aligned} \quad (102)$$

Replacing ω by $\theta_{-t-2}\omega$ in (84), we have

$$\begin{aligned}&\alpha_0^2\left\|A^{1/2}v(t+1,\theta_{-t-2}\omega)\right\|_2^2+\alpha_1^2\|Av(t+1,\theta_{-t-2}\omega)\|_2^2\\&\leq\left(\alpha_0^2\left\|A^{1/2}v(s,\theta_{-t-2}\omega)\right\|_2^2\right.\\&\qquad\left.+\alpha_1^2\|Av(s,\theta_{-t-2}\omega)\|_2^2\right)e^{J_2(s,t+1,\theta_{-t-2}\omega)}\\&\quad+c\int_t^{t+1}\sum_{j=1}^{n}\left|y_j\left(\theta_{\tau-t-2}\omega_j\right)\right|^2e^{J_2(\tau,t+1,\theta_{-t-2}\omega)}d\tau\\&\quad+c\int_t^{t+1}\sum_{j=1}^{n}\left|y_j\left(\theta_{\tau-t-2}\omega_j\right)\right|^4e^{J_2(\tau,t+1,\theta_{-t-2}\omega)}d\tau\\&\quad+c\int_t^{t+1}\sum_{j=1}^{n}\left|y_j\left(\theta_{\tau-t-2}\omega_j\right)\right|^2\\&\qquad\times\left(\alpha_0^2\|v(\tau,\theta_{-t-2}\omega)\|_2^2\right.\\&\qquad\left.+\alpha_1^2\left\|A^{1/2}v(\tau,\theta_{-t-2}\omega)\right\|_2^2\right)e^{J_2(\tau,t+1,\theta_{-t-2}\omega)}d\tau\\&\quad+c\int_t^{t+1}\left(\alpha_0^2\|v(\tau,\theta_{-t-2}\omega)\|_2^2\right.\\&\qquad\left.+\alpha_1^2\left\|A^{1/2}v(\tau,\theta_{-t-2}\omega)\right\|_2^2\right)^2e^{J_2(\tau,t+1,\theta_{-t-2}\omega)}d\tau\\&\quad+c\int_t^{t+1}\|f(\tau)\|_{V'}^2e^{J_2(\tau,t+1,\theta_{-t-2}\omega)}d\tau.\end{aligned} \quad (103)$$

Analogously to the previous estimates, we have

$$\begin{aligned}&J_2(\tau,t+1,\theta_{-t-2}\omega)\\&\leq c\int_t^{t+1}\left(\alpha_0^2\|v(\tau,\theta_{-t-2}\omega)\|_2^2\right.\\&\qquad\left.+\alpha_1^2\left\|A^{1/2}v(\tau,\theta_{-t-2}\omega)\right\|_2^2\right)d\tau\\&\quad+c\int_t^{t+1}\left(\alpha_0^2\left\|A^{1/2}v(\tau,\theta_{-t-2}\omega)\right\|_2^2\right.\\&\qquad\left.+\alpha_1^2\|Av(\tau,\theta_{-t-2}\omega)\|_2^2\right)d\tau\\&\quad+c\sum_{j=1}^{n}\int_t^{t+1}\left|y_j\left(\theta_{\tau-t-2}\omega_j\right)\right|^2d\tau\\&\leq r_{11}(\omega),\end{aligned} \quad (104)$$

where

$$r_{11}(\omega)=cr_9(\omega)+cr_{10}(\omega)+c\sum_{j=1}^{n}\int_{-2}^{0}\left|y_j\left(\theta_\tau\omega_j\right)\right|^2d\tau. \quad (105)$$

Integrating (103) with respect to $s \in (t, t+1)$ associating (100), (101), and (104) with (68), we have

$$\alpha_0^2 \left\| A^{1/2} v(t+1, \theta_{-t-2}\omega) \right\|_2^2 + \alpha_1^2 \left\| A v(t+1, \theta_{-t-2}\omega) \right\|_2^2 \leq r_{12}(\omega), \tag{106}$$

where

$$\begin{aligned} r_{12}(\omega) = {} & c r_{10}(\omega) e^{r_{11}(\omega)} \\ & + c e^{r_{11}(\omega)} \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_\tau \omega_j) \right|^2 d\tau \\ & + c e^{r_{11}(\omega)} \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_\tau \omega_j) \right|^4 d\tau \\ & + c r_9(\omega) e^{r_{11}(\omega)} \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_\tau \omega_j) \right|^2 d\tau \\ & + c r_9(\omega)^2 e^{r_{11}(\omega)} + c M_{f_0} e^{r_{11}(\omega)}. \end{aligned} \tag{107}$$

Replacing s, $t+1$, and ω by t, s, and $\theta_{-t-1}\omega$, respectively, in (84) where $s \in (t, t+1)$ and $t \geq T_3(\omega)$, we have

$$\begin{aligned} & \alpha_0^2 \left\| A^{1/2} v(s, \theta_{-t-1}\omega) \right\|_2^2 + \alpha_1^2 \left\| A v(s, \theta_{-t-1}\omega) \right\|_2^2 \\ & \leq \Big(\alpha_0^2 \left\| A^{1/2} v(t, \theta_{-t-1}\omega) \right\|_2^2 \\ & \qquad + \alpha_1^2 \left\| A v(t, \theta_{-t-1}\omega) \right\|_2^2 \Big) e^{J_2(t, t+1, \theta_{-t-1}\omega)} \\ & \quad + c \int_t^{t+1} \sum_{j=1}^{n} \left| y_j(\theta_{\tau-t-1}\omega_j) \right|^2 e^{J_2(\tau, t+1, \theta_{-t-1}\omega)} d\tau \\ & \quad + c \int_t^{t+1} \sum_{j=1}^{n} \left| y_j(\theta_{\tau-t-1}\omega_j) \right|^4 e^{J_2(\tau, t+1, \theta_{-t-1}\omega)} d\tau \\ & \quad + c \int_t^{t+1} \sum_{j=1}^{n} \left| y_j(\theta_{\tau-t-1}\omega_j) \right|^2 \\ & \qquad \times \Big(\alpha_0^2 \left\| v(\tau, \theta_{-t-1}\omega) \right\|_2^2 \\ & \qquad\quad + \alpha_1^2 \left\| A^{1/2} v(\tau, \theta_{-t-1}\omega) \right\|_2^2 \Big) e^{J_2(\tau, t+1, \theta_{-t-1}\omega)} d\tau \\ & \quad + c \int_t^{t+1} \Big(\alpha_0^2 \left\| v(\tau, \theta_{-t-1}\omega) \right\|_2^2 \\ & \qquad + \alpha_1^2 \left\| A^{1/2} v(\tau, \theta_{-t-1}\omega) \right\|_2^2 \Big)^2 e^{J_2(\tau, t+1, \theta_{-t-1}\omega)} d\tau \\ & \quad + c \int_t^{t+1} \left\| f(\tau) \right\|_{V'}^2 e^{J_2(\tau, t+1, \theta_{-t-1}\omega)} d\tau \\ & \leq r_{12}(\omega) e^{r_7(\omega)} \\ & \quad + c e^{r_7(\omega)} \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_\tau \omega_j) \right|^2 d\tau \\ & \quad + c e^{r_7(\omega)} \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_\tau \omega_j) \right|^4 d\tau + c e^{r_7(\omega)} M_{f_0} \\ & \quad + c r_5(\omega)^2 e^{r_7(\omega)} \\ & \quad + c r_5(\omega) e^{r_7(\omega)} \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_\tau \omega_j) \right|^2 d\tau \\ & \triangleq r_{13}(\omega). \end{aligned} \tag{108}$$

Associating with the above inequalities, we have

$$\int_t^{t+1} G(s, \theta_{-t-1}\omega) \, ds \leq r_{14}(\omega), \tag{109}$$

where

$$\begin{aligned} r_{14}(\omega) = {} & c r_5(\omega)^2 + c r_{13}(\omega)^2 \\ & + c \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_s \omega_j) \right|^2 ds \\ & + c r_5(\omega) r_{13}(\omega) \\ & + c \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_s \omega_j) \right|^4 ds \\ & + c r_5(\omega) \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_s \omega_j) \right|^2 ds \\ & + c r_{13}(\omega) \int_{-1}^{0} \sum_{j=1}^{n} \left| y_j(\theta_s \omega_j) \right|^2 ds. \end{aligned} \tag{110}$$

According to the property of $\{\lambda_j\}_{j=1}^{\infty}$ mentioned in Section 3, for $\mathbb{P}$ a.e $\omega \in \Omega$ and $\forall \varepsilon > 0$, there exists M such that for $m \geq M$

$$\begin{gathered} e^{-\nu\lambda_{m+1}} \left(\alpha_0^2 \left\| v_2(t, \theta_{-t-1}\omega) \right\|_2^2 + \alpha_1^2 \left\| A^{1/2} v_2(t, \theta_{-t-1}\omega) \right\|_2^2 \right) \\ \leq \frac{r_5(\omega)}{e^{\nu\lambda_{m+1}}} \leq \varepsilon r_5(\omega), \\ \int_t^{t+1} G(s, \theta_{-t-1}\omega) e^{-\nu\lambda_{m+1}} ds \leq \frac{r_{14}(\omega)}{e^{\nu\lambda_{m+1}}} \leq \varepsilon r_{14}(\omega), \\ c \int_t^{t+1} \left\| f(s) \right\|_{V'}^2 e^{-\nu\lambda_{m+1}} ds \leq \frac{c M_{f_0}}{e^{\nu\lambda_{m+1}}} \leq \varepsilon. \end{gathered} \tag{111}$$

Therefore, we deduce from (99) that, for $\mathbb{P}$ a.e. $\omega \in \Omega$,

$$\alpha_0^2 \|v_2(t,\theta_{-t}\omega)\|_2^2 + \alpha_1^2 \left\|A^{1/2} v_2(t,\theta_{-t}\omega)\right\|_2^2 \leq \varepsilon + \varepsilon r_5(\omega) + \varepsilon r_{14}(\omega), \quad \forall t \geq T_3(\omega), \ f \in \Sigma(f_0), \tag{112}$$

which indicates $\{\mathscr{U}_f(t,\tau,\omega)\}$, $f \in \Sigma(f_0)$ satisfying uniform (with respect to $f \in \Sigma(f_0)$) condition(C) in V. According to Theorem 12, the proof is completed. □

In the following we prove the existence of uniformly random attractor for the families of processes $\{\mathscr{U}_f(t,\tau,\omega)\}$, $f \in \Sigma(f_0)$ corresponding to (48) in $D(A)$.

Theorem 18. *If $f_0(x,s)$ is a normal function in $L^2_{\mathrm{loc}}(\mathbb{R}, V')$, then the process $\{\mathscr{U}_{f_0}(t,\tau,\omega)\}$ corresponding to (48) possess a compact uniformly (with respect to $\tau \in \mathbb{R}$) random attractor $\mathscr{A}_2(\omega)$ in $D(A)$ which coincides with the uniformly (with respect to $f \in \Sigma(f_0)$) random attractor $\mathscr{A}_{\Sigma(f_0)}$ of the family of processes $\{\mathscr{U}_f(t,\tau,\omega)\}$, $f \in \Sigma(f_0)$:*

$$\mathscr{A}_2(\omega) = \mathscr{A}_{\Sigma(f_0)}(\omega) = \mathscr{W}_{0,\Sigma(f_0)}(\mathscr{K}_2) = \bigcup_{f\in\Sigma(f_0)} \mathscr{N}_f(0,\omega), \tag{113}$$

where $\mathscr{K}_2(\omega)$ is the uniformly (with respect to $f \in \Sigma(f_0)$) random absorbing set in $D(A)$ and $\mathscr{N}_f(\omega)$ is the kernel of the process $\{\mathscr{U}_f(t,\tau,\omega)\}$. Furthermore, the kernel $\mathscr{N}_f(\omega)$ is nonempty for all $f \in \Sigma(f_0)$.

Proof. In Lemma 16, we have proved that the semigroup of processes $\{\mathscr{U}_f(t,\tau,\omega)\}$, $f \in \Sigma(f_0)$, has a uniformly random absorbing set in $D(A)$. Now we testify that the semigroup of processes corresponding to (48) satisfies uniform (with respect to $f \in \Sigma(f_0)$) condition (C). Analogously to the proof in Lemma 16, we easily check that v_1 is bounded in $D(A)$. Letting $w = Av_2$ in (51), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\alpha_0^2\left\|A^{1/2}v_2\right\|_2^2 + \alpha_1^2\|Av_2\|_2^2\right) + \nu\left(\alpha_0^2\|Av_2\|_2^2 + \alpha_1^2\left\|A^{3/2}v_2\right\|_2^2\right) \\
&= -\nu\left(\sum_{j=1}^n Ah_j y_j(\theta_t\omega_j), Av_2\right) - \left(\widetilde{B}\left(v + z(\theta_t\omega), \alpha_0^2 v + \alpha_1^2 Av + \sum_{j=1}^n h_j y_j(\theta_t\omega_j)\right), Av_2\right) \\
&\quad + (Pf(x,t), Av_2) + \mu\left(\sum_{j=1}^n h_j y_j(\theta_t\omega_j), Av_2\right).
\end{aligned} \tag{114}$$

Applying Lemma 14, the second term on the right-hand side of (114) can be bounded by

$$\begin{aligned}
&\left|\left(\widetilde{B}\left(v + z(\theta_t\omega), \alpha_0^2 v + \alpha_1^2 Av + \sum_{j=1}^n h_j y_j(\theta_t\omega_j)\right), Av_2\right)\right| \\
&\leq \frac{\nu\alpha_0^2}{8}\|Av_2\|_2^2 + \frac{\nu\alpha_1^2}{2}\left\|A^{3/2}v_2\right\|_2^2 + c\left(\alpha_0^2\|v\|_2^2 + \alpha_1^2\left\|A^{1/2}v\right\|_2^2\right) \times \left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2 + \alpha_1^2\|Av\|_2^2\right) \\
&\quad + c\left(\alpha_0^2\|v\|_2^2 + \alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)^2 + c\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2 + \alpha_1^2\|Av\|_2^2\right)^2 \\
&\quad + c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2 + c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^4 \\
&\quad + c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2\left(\alpha_0^2\|v\|_2^2 + \alpha_1^2\left\|A^{1/2}v\right\|_2^2\right)^2 \\
&\quad + c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2\left(\alpha_0^2\left\|A^{1/2}v\right\|_2^2 + \alpha_1^2\|Av\|_2^2\right)^2.
\end{aligned} \tag{115}$$

Note that

$$\begin{aligned}
\nu\left|\left(\sum_{j=1}^n h_j y_j(\theta_t\omega_j), Av_2\right)\right| &\leq \frac{\nu\alpha_0^2}{8}\|Av_2\|_2^2 + c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2, \\
|(Pf(x,t), Av_2)| \leq \|f\|_{V'}\|Av_2\|_2 &\leq \frac{\nu\alpha_0^2}{8}\|Av_2\|_2^2 + c\|f\|_{V'}^2, \\
\mu\left|\left(\sum_{j=1}^n h_j y_j(\theta_t\omega_j), Av_2\right)\right| &\leq \frac{\nu\alpha_0^2}{8}\|Av_2\|_2^2 + c\sum_{j=1}^n\left|y_j(\theta_t\omega_j)\right|^2.
\end{aligned} \tag{116}$$

Associating with (114)–(116) and applying Poincaré's inequality, we have

$$\frac{d}{dt}\left(\alpha_0^2\left\|A^{1/2}v_2\right\|_2^2+\alpha_1^2\left\|Av_2\right\|_2^2\right)+\nu\lambda_{m+1}\left(\alpha_0^2\left\|A^{1/2}v_2\right\|_2^2+\alpha_1^2\left\|Av_2\right\|_2^2\right)\leq G(t,\omega)+c\|f\|_{V'}^2. \tag{117}$$

Applying Gronwall's lemma over $s\in[t,t+1]$ where $t\geq T_2(\omega)$, we have

$$\alpha_0^2\left\|A^{1/2}v_2(t+1,\omega)\right\|_2^2+\alpha_1^2\left\|Av_2(t+1,\omega)\right\|_2^2\leq\left(\alpha_0^2\left\|A^{1/2}v_2(t,\omega)\right\|_2^2+\alpha_1^2\left\|Av_2(t,\omega)\right\|_2^2\right)e^{-\nu\lambda_{m+1}}+\int_t^{t+1}G(s,\omega)e^{-\nu\lambda_{m+1}}ds+c\int_t^{t+1}\|f\|_{V'}^2e^{-\nu\lambda_{m+1}}ds. \tag{118}$$

Replacing ω by $\theta_{-t-1}\omega$ in (118), we have

$$\begin{aligned}&\alpha_0^2\left\|A^{1/2}v_2(t+1,\theta_{-t-1}\omega)\right\|_2^2+\alpha_1^2\left\|Av_2(t+1,\theta_{-t-1}\omega)\right\|_2^2\\&\leq\left(\alpha_0^2\left\|A^{1/2}v_2(t,\theta_{-t-1}\omega)\right\|_2^2+\alpha_1^2\left\|Av_2(t,\theta_{-t-1}\omega)\right\|_2^2\right)e^{-\nu\lambda_{m+1}}\\&\quad+\int_t^{t+1}G(s,\theta_{-t-1}\omega)e^{-\nu\lambda_{m+1}}ds+c\int_t^{t+1}\|f\|_{V'}^2e^{-\nu\lambda_{m+1}}ds.\end{aligned} \tag{119}$$

Analogously to the consideration in Theorem 17, for $\mathbb{P}$ a.e. $\omega\in\Omega$ and $\forall\varepsilon>0$, there exists M such that, for $m>M$,

$$\begin{aligned}&e^{-\nu\lambda_{m+1}}\left(\alpha_0^2\left\|A^{1/2}v_2(t,\theta_{-t-1}\omega)\right\|_2^2+\alpha_1^2\left\|Av_2(t,\theta_{-t-1}\omega)\right\|_2^2\right)\leq\frac{r_{12}(\omega)}{e^{\nu\lambda_{m+1}}}\leq\varepsilon r_{12}(\omega),\\&\int_t^{t+1}G(s,\theta_{-t-1}\omega)e^{-\nu\lambda_{m+1}}ds\leq\frac{r_{14}(\omega)}{e^{\nu\lambda_{m+1}}}\leq\varepsilon r_{14}(\omega),\\&c\int_t^{t+1}\|f(s)\|_{V'}^2e^{-\nu\lambda_{m+1}}ds\leq\frac{cM_{f_0}}{e^{\nu\lambda_{m+1}}}\leq\varepsilon.\end{aligned} \tag{120}$$

Therefore, we deduce from (119) that, for $\mathbb{P}$ a.e $\omega\in\Omega$,

$$\begin{aligned}&\alpha_0^2\left\|A^{1/2}v_2(t+1,\theta_{-t-1}\omega)\right\|_2^2+\alpha_1^2\left\|Av_2(t+1,\theta_{-t-1}\omega)\right\|_2^2\\&\leq\varepsilon+\varepsilon r_{12}(\omega)+\varepsilon r_{14}(\omega),\quad\forall t\geq T_3(\omega),\ f\in\Sigma(f_0).\end{aligned} \tag{121}$$

which indicates $\{\mathscr{U}_f(t,\tau,\omega)\}$, $f\in\Sigma(f_0)$ satisfying uniform (with respect to $f\in\Sigma(f_0)$) condition (C) in $D(A)$. According to Theorem 12, the proof is completed. □

Now we introduce a homeomorphism $\Phi(\theta_t\omega)u=u+z(\theta_t\omega), u\in E$, whose inverse homeomorphism $\Phi^{-1}(\theta_t\omega)u=u-z(\theta_t\omega)$. Then the transformation

$$\mathscr{V}_\sigma(t,\tau,\omega)=\Phi(\theta_t\omega)\circ\mathscr{U}_\sigma(t,\tau,\omega)\circ\Phi^{-1}(\theta_\tau\omega) \tag{122}$$

generates a process corresponding to (8). Note that the two processes are equivalent by (122). It is easy to check that $\mathscr{V}_\sigma(t,\tau,\omega)$ has a uniformly (with respect to $\sigma\in\Sigma$) random attractor provided $\mathscr{U}_\sigma(t,\tau,\omega)$ possesses a uniformly (with respect to $\sigma\in\Sigma$) random attractor. As a result Theorem 17 and Theorem 18 imply that (8) has a uniformly (with respect to $f\in\Sigma(f_0)$) random attractor in V and $D(A)$.

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