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Research Article

Oscillation Criteria of First Order Neutral Delay Differential Equations with Variable Coefficients

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Some new oscillation criteria are given for first order neutral delay differential equations with variable coefficients. Our results generalize and extend some of the well-known results in the literature. Some examples are considered to illustrate the main results.

1. Introduction

In recent years, oscillation of neutral delay differential equations (or NDDEs for short) has received great attention and has been studied extensively. It is a relatively new field with interesting applications from the real world. In fact, NDDEs appear in modeling of the problems as transformation of information, population dynamics, the networks containing lossless transmission lines, and in the theory of automatic control (see, e.g., [1–4] and references cited therein).

Consider the first order NDDE of the form

$$[r(t)(x(t)+p(t)x(t-\tau))]' + q(t)x(t-\sigma) = 0, \quad t \geq t_0, \tag{1}$$

where

$$p \in C[[t_0,\infty),\mathbb{R}], \quad r,q \in C[[t_0,\infty),\mathbb{R}^+], \quad \tau,\sigma \in \mathbb{R}^+. \tag{2}$$

Let $m = \max\{\tau,\sigma\}$. By a solution of (1), we mean a function $x \in C[[t_1-m,\infty),\mathbb{R}]$ for some $t_1 \geq t_0$ such that $x(t)+p(t)x(t-\tau)$ is continuously differentiable, and (1) is satisfied identically for $t_1 \geq t_0$. Such a solution of (1) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative. The NDDE (1) is called oscillatory if all its solutions are oscillatory; otherwise, it is called nonoscillatory.

Recently, some investigations such as [5–7] have appeared which are concerned with the oscillation as well as the nonoscillation behaviour of NDDE (1). In fact, Zahariev and Baĭnov [8] is the first work dealing with oscillation of neutral equations. A systematic development of oscillation theory of NDDEs was initiated by Ladas and Sficas [9]. For the oscillation of (1) when $r(t) = 1$ and $p(t)$ and $q(t)$ are constants, we refer the readers to the articles by Ladas and Schults [10], Sficas and Stavroulakis [11], Grammatikopoulos et al. [12], Zhang [13], and Gopalsamy and Zhang [14]. For the oscillation of (1) when $r(t) = 1$ and $p(t)$ is equal to a constant, we refer the readers to the papers by Grammatikopoulos et al. [15], Zhang [13], Gopalsamy and Zhang [14], and Saker and Elabbasy [16] and the references cited therein. Grammatikopoulos et al. [6], Ladas and Schults [10], Chuanxi and Ladas [17, 18], Kubiaczyk and Saker [19], and Karpuz and Ocalan [20] considered the NDDE (1) when $r(t) = 1$ and established some new oscillation results sorted by the value of function $p(t)$. For further oscillation results on the oscillatory behaviour of solutions of (1), we refer the readers to the monographs by Győri and Ladas [21] and Erbe et al. [22] as well as the papers of Yu et al. [23], Choi and Koo [24], Ocalan [25], and Candan and Dahiya [26].

The purpose of this work is to find some sufficient conditions for the oscillation of all solutions of the first order NDDE (1).

Remark 1. (i) When we write a functional inequality we assume that it holds for all sufficiently large t.

(ii) Without loss of generality, we will deal only with the positive solutions of (1).

In the proof of our main results, we need the following well-known lemmas which can be found in Chuanxi and Ladas [17], Győri and Ladas [21], and Kulenović et al. [27].

Lemma 2. *Assume that ρ is a positive constant. Let $h \in C[[t_0,\infty),\mathbb{R}^+]$, and suppose that*

$$\liminf_{t\to\infty}\int_{t-\rho}^{t} h(s)\,ds > \frac{1}{e}. \tag{3}$$

Then

(i) *the delay differential inequality*

$$x'(t) + h(t)\,x(t-\rho) \le 0, \quad t \ge t_0, \tag{4}$$

has no eventually positive solution;

(ii) *the delay differential inequality*

$$x'(t) + h(t)\,x(t-\rho) \ge 0, \quad t \ge t_0, \tag{5}$$

has no eventually negative solution;

(iii) *the advanced differential inequality*

$$x'(t) - h(t)\,x(t+\rho) \le 0, \quad t \ge t_0, \tag{6}$$

has no eventually negative solution;

(iv) *the advanced differential inequality*

$$x'(t) - h(t)\,x(t+\rho) \ge 0, \quad t \ge t_0, \tag{7}$$

has no eventually positive solution.

Lemma 3. *Consider the NDDE*

$$\left(x(t) + p(t)\,x(t-\tau)\right)' + q(t)\,x(t-\sigma) = 0, \quad t \ge t_0, \tag{8}$$

where p, q, τ, and σ are as in (2). Assume that

$$\int_{t_0}^{\infty} q(s)\,ds = \infty. \tag{9}$$

Let $x(t)$ be an eventually positive solution of equation and set

$$z(t) = x(t) + p(t)\,x(t-\tau). \tag{10}$$

Then the following statements are true:

(i) *$z(t)$ is an eventually decreasing function;*

(ii) *if $p(t) \le -1$ then $z(t) < 0$;*

(iii) *if $-1 \le p(t) \le 0$ then $z(t) > 0$ and $\lim_{t\to\infty} z(t) = 0$.*

Lemma 4. *Assume that (9) holds and let $x(t)$ be an eventually positive solution of NDDE*

$$\left[\left(x(t) + px(t-\tau)\right)\right]' + q(t)\,x(t-\sigma) = 0, \quad t \ge t_0, \tag{11}$$

where $p \ne 1$, $q \in C[[t_0,\infty),\mathbb{R}^+]$, and $\tau,\sigma \in \mathbb{R}^+$.

Set

$$z(t) = x(t) + px(t-\tau). \tag{12}$$

Then

(a) *$z(t)$ is a decreasing function and either*

$$\lim_{t\to\infty} z(t) = -\infty \tag{13}$$

or

$$\lim_{t\to\infty} z(t) = 0. \tag{14}$$

(b) *The following statements are equivalent:*

(i) *(13) holds;*
(ii) *$p < -1$;*
(iii) *$\lim_{t\to\infty} x(t) = \infty$;*
(iv) *$w(t) > 0$, $w'(t) > 0$.*

(c) *The following statements are equivalent:*

(i) *(14) holds;*
(ii) *$p > -1$;*
(iii) *$\lim_{t\to\infty} x(t) = 0$;*
(iv) *$w(t) > 0$, $w'(t) < 0$.*

2. Main Results

In this section we give some new sufficient conditions for all solutions of NDDE (1) to be oscillatory.

Theorem 5. *Assume that (2) and (9) hold, $p(t) \le -1$, $\tau > \sigma$, and*

$$\liminf_{t\to\infty}\int_{t+\sigma}^{t+\tau}\left[\frac{q(s-\tau)}{-r(s-\sigma)\,p(s-\sigma)}\right]ds > \frac{1}{e}. \tag{15}$$

Then every solution of NDDE (1) is oscillatory.

Proof. Assume, for the sake of a contradiction, that (1) has an eventually positive solution $x(t) > 0$ for all $t \ge t_0 > 0$. Set

$$z(t) = x(t) + p(t)\,x(t-\tau). \tag{16}$$

Then by Lemma 3 we have

$$z(t) < 0. \tag{17}$$

Observe that

$$z(t) > p(t)\, x(t-\tau). \tag{18}$$

From which we find eventually

$$\frac{1}{p(t+\tau-\sigma)} q(t) z(t+\tau-\sigma) < q(t) x(t-\sigma) = -(r(t) z(t))', \tag{19}$$

and hence

$$z'(t) + \frac{r'(t)}{r(t)} z(t) + \frac{q(t)}{r(t)\, p(t+\tau-\sigma)} z(t+\tau-\sigma) < 0. \tag{20}$$

Set

$$z(t) = e^{-\int_{t_0}^{t} (r'(s)/r(s)) ds} y(t). \tag{21}$$

This implies that $y(t) < 0$.

Substituting in (20) yields for all $t \ge t_0$

$$y'(t) + \frac{q(t)}{r(t+\tau-\sigma)\, p(t+\tau-\sigma)} y(t+\tau-\sigma) < 0, \tag{22}$$

or

$$y'(t) - \left[\frac{q(t)}{-r(t+\tau-\sigma)\, p(t+\tau-\sigma)}\right] y(t+(\tau-\sigma)) < 0. \tag{23}$$

In view of (15) and Lemma 2(iii), it is impossible for (23) to have an eventually negative solution. This contradicts the fact that $y(t) < 0$ and the proof is complete. □

Example 6. Consider NDDE

$$\left[\frac{e^{t+1}}{t+1}\left(x(t) - \frac{t+1}{t} x(t-2)\right)\right]' + e^{t+2} x(t-1) = 0, \quad t > 0. \tag{24}$$

Here we have

$$\begin{gathered} p(t) = -\frac{t+1}{t} \le -1, \qquad q(t) = e^{t+2}, \\ r(t) = \frac{e^{t+1}}{t+1}, \qquad \tau = 2, \qquad \sigma = 1. \end{gathered} \tag{25}$$

Then all the hypotheses of Theorem 5 are satisfied where

$$\begin{aligned} &\liminf_{t\to\infty} \int_{t+\sigma}^{t+\tau} \frac{q(s-\tau)}{-r(s-\sigma)\, p(s-\sigma)} ds \\ &= \liminf_{t\to\infty} \int_{t+1}^{t+2} (s-1)\, ds = \liminf_{t\to\infty} \left(t + \frac{9}{2}\right) = \infty > \frac{1}{e}. \end{aligned} \tag{26}$$

Hence every solution of (24) is oscillatory.

Remark 7. Theorem 5 is an extent of [17, Theorem 2], [15, Theorem 7], and [21, Theorem 6.4.3].

Theorem 8. *Assume that* (2) *and* (9) *hold,* $-1 \le p(t) \le 0$, *and*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \frac{q(s)}{r(s-\sigma)} ds > \frac{1}{e}. \tag{27}$$

Then every solution of NDDE (1) *oscillates.*

Proof. Assume, for the sake of contradiction, that (1) has an eventually positive solution $x(t) > 0$ for all $t \ge t_0 > 0$. Set

$$z(t) = x(t) + p(t)\, x(t-\tau). \tag{28}$$

Then by Lemma 3, it follows that

$$z(t) > 0. \tag{29}$$

As $x(t) > z(t)$, it follows from (1) that

$$(r(t) z(t))' + q(t) z(t-\sigma) \le 0. \tag{30}$$

Dividing the last inequality by $r(t) > 0$, we obtain

$$z'(t) + \frac{r'(t)}{r(t)} z(t) + \frac{q(t)}{r(t)} z(t-\sigma) \le 0. \tag{31}$$

Let

$$z(t) = e^{-\int_{t_0}^{t} (r'(s)/r(s)) ds} y(t). \tag{32}$$

This implies that $y(t) > 0$.

Substituting in (31) yields for all $t \ge t_0$

$$y'(t) + \frac{q(t)}{r(t-\sigma)} y(t-\sigma) \le 0, \quad t \ge t_0. \tag{33}$$

In view of Lemma 2(i) and (27), it is impossible for (33) to have an eventually positive solution. This contradicts the fact that $y(t) > 0$ and the proof is complete. □

Example 9. Consider the NDDE

$$\left[\frac{1}{t}\left(x(t) - \frac{t}{t+1} x(t-\tau)\right)\right]' + \frac{1}{t-(5\pi/2)} x\left(t - \frac{5\pi}{2}\right) = 0, \quad t > \frac{5\pi}{2}. \tag{34}$$

Note that all the hypotheses of Theorem 8 are satisfied:

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \frac{q(s)}{r(s-\sigma)} ds = \liminf_{t\to\infty} \int_{t-(5\pi/2)}^{t} ds = \frac{5\pi}{2} > \frac{1}{e}. \tag{35}$$

Therefore every solution of (34) is oscillatory.

Remark 10. Theorem 8 is an extent of [17, Theorem 3] and [21, Theorem 6.4.2].

Theorem 11. *Assume that* (2) *holds with* $p(t) \equiv p \ne \pm 1$, $r(t) \equiv r > 0$, $q(t)$ *being* τ *periodic, and*

$$\frac{1}{r(1+p)} \liminf_{t\to\infty} \int_{t-\sigma}^{t-\tau} q(s)\, ds > \frac{1}{e}. \tag{36}$$

Then every solution of NDDE

$$[r(x(t)+px(t-\tau))]'+q(t)x(t-\sigma)=0, \quad t\geq t_0, \tag{37}$$

is oscillatory.

Proof. Assume, for the sake of contradiction, that (37) has an eventually positive solution $x(t)>0$ for all $t\geq t_0>0$. Set

$$\begin{aligned} z(t) &= x(t)+px(t-\tau), \\ w(t) &= z(t)+pz(t-\tau). \end{aligned} \tag{38}$$

It is easily seen, by direct substituting, that $z(t)$ and $w(t)$ are also solutions of (37). That is,

$$rz'(t)+prz'(t-\tau)+q(t)z(t-\sigma)=0, \tag{39}$$

$$rw'(t)+prw'(t-\tau)+q(t)w(t-\sigma)=0. \tag{40}$$

By Lemma 4, $z(t)$ is decreasing and either (13) or (14) holds. In either case we claim that

$$w'(t-\tau)\geq w'(t). \tag{41}$$

Indeed,

$$\begin{aligned} w'(t) &= -\frac{1}{r}q(t)z(t-\sigma)\leq -\frac{1}{r}q(t)z(t-\sigma-\tau) \\ &= -\frac{1}{r}q(t-\tau)z(t-\sigma-\tau)=w'(t-\tau). \end{aligned} \tag{42}$$

Furthermore, we have by Lemma 4 that as long as $p\neq \pm 1$,

$$w(t)>0. \tag{43}$$

Using (41) in (40) implies

$$r(1+p)w'(t-\tau)+q(t)w(t-\sigma)\leq 0 \tag{44}$$

or

$$w'(t-\tau)+\frac{1}{r(1+p)}q(t)w(t-\sigma)\leq 0. \tag{45}$$

Since $q(t)$ is periodic of period τ, we find

$$w'(t)+\frac{1}{r(1+p)}q(t)w(t-(\sigma-\tau))\leq 0, \quad \text{if } 1+p>0, \tag{46}$$

or

$$w'(t)-\left[\frac{1}{-r(1+p)}\right]q(t)w(t+(\tau-\sigma))\geq 0, \quad \text{if } 1+p<0. \tag{47}$$

In view of Lemma 2((i) and (iv)) and (36), it is impossible for (46) and (47) to have eventually positive solutions. This contradicts the fact that $w(t)>0$ and the proof is complete. □

Remark 12. Theorem 11 extends [15, Theorems 8 and 10]. See also [21, Theorem 6.4.4].

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