

## Research Article

# Multivalued Variational Inequalities with $D_J$ -Pseudomonotone Mappings in Reflexive Banach Spaces

A. M. Saddeek<sup>1</sup> and S. A. Ahmed<sup>1,2</sup>

<sup>1</sup> Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt

<sup>2</sup> Department of Mathematics, University College, Umm Al-Qura University, Saudi Arabia

Correspondence should be addressed to A. M. Saddeek; a\_m\_saddeek@yahoo.com

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This paper is concerned with the study of a class of variational inequalities with multivalued  $D_J$ -pseudomonotone mappings in reflexive Banach spaces by using the  $D_J$ -antiresolvent technique. An application to the multivalued nonlinear  $D_J$ -complementarity problem is also presented. The results coincide with the corresponding results announced by many others for the gradient state.

## 1. Introduction and Preliminaries

Variational inequalities give a convenient mathematical framework for discussing a large variety of interesting problems appearing in pure and applied sciences. It is well known that the theory of pseudomonotone mappings plays an important part in the study of the above-mentioned variational inequalities.

In recent years, pseudomonotone theory has become an attractive field for many mathematicians (see [1–8]).

In a very recent paper [9], by using the  $D_J$ -antiresolvent technique (where  $J$  is the duality mapping) devised by the first author, the author introduced a new concept of monotonicity, which is called the  $D_J$ -pseudomonotone type.

In the present paper, the concept of multivalued  $D_J$ -pseudomonotone mappings in reflexive Banach spaces is used to study a wide class of variational inequalities, called the multivalued  $D_J$ -pseudomonotone variational inequalities.

Moreover, the results obtained in this paper can be applied to the multivalued nonlinear  $D_J$ -complementarity problem. This problem contains, in particular, a mathematical model arising in the study of the postcritical equilibrium state of a thin plate resting, without friction on a flat rigid support (see [10–12]). The results coincide with the corresponding results (see [2, 13–15]) in the case of gradient mappings.

Unless otherwise stated,  $V$  stands for a real reflexive Banach space with norm  $\|\cdot\|_V$  and  $V^*$  stands for the uniformly convex dual of  $V$  with the dual norm  $\|\cdot\|_{V^*}$ . The duality pairing between  $V$  and  $V^*$  is denoted by  $\langle \cdot, \cdot \rangle$ . The set of all nonnegative integers is denoted by  $\mathbb{N}$ . The field of real (resp., positive real) numbers is denoted by  $\mathbb{R}$  (resp.,  $\mathbb{R}^+$ ). Notation “ $\rightarrow$ ” stands for strong convergence and “ $\rightharpoonup$ ” for weak convergence.

A mapping  $J : V \rightarrow V^*$  is said to be a duality mapping (see, e.g., [16]) with gauge function  $\Phi$  (i.e.,  $\Phi$  is continuous strictly increasing real-valued function satisfying  $\Phi(0) = 0$  and  $\lim_{t \rightarrow +\infty} \Phi(t) = +\infty$ ) if for every  $u \in V$ ,  $\langle Ju, u \rangle = \|Ju\|_{V^*} \|u\|_V = \Phi(\|u\|_V)(\|u\|_V)$ . If  $V = H$  is a Hilbert space, then  $J \equiv I$ , the identity mapping.

Assume that  $V^*$  has a weakly sequentially continuous duality mapping  $J$  (i.e., if  $\{u_n\}_{n \in \mathbb{N}}$  is a sequence in  $V$  which weakly converges to a point  $u$ , then the sequence  $\{J(u_n)\}_{n \in \mathbb{N}}$  converges to  $J(u)$  (see, e.g., [17])).

Let  $g : V \rightarrow \mathbb{R} \cup \{+\infty\}$  be a function. The domain of  $g$  is  $\text{dom } g = \{u \in V : g(u) < +\infty\}$ . When  $\text{dom } g \neq \emptyset$ ,  $g$  is called proper (see, e.g., [18]). The interior of the domain of  $g$  is denoted by  $\text{int dom } g$ . The function  $g$  is said to be Gâteaux differentiable at  $u \in \text{int dom } g$  (see, e.g., [18]), if

$$g'(u, \eta) = \lim_{t \rightarrow 0} \frac{g(u + t\eta) - g(u)}{t} \quad (1)$$

exists for all  $\eta \in V$ .

Let  $g$  be proper, convex, lower semicontinuous, and Gâteaux differentiable at  $u \in \text{int dom } g$ ; then the gradient of  $g$  at  $u$  is the function  $\nabla g(u)$  which is defined by  $\langle \nabla g(u), \eta \rangle = g'(u, \eta)$  for any  $\eta \in V$ . It is known (see, e.g., [19]) that the conjugate  $g^* : V^* \rightarrow \mathbb{R} \cup \{+\infty\}$  is also proper, convex, and lower semicontinuous.

The convex function  $g$  is said to be of Legendre type (see, e.g., [20]) if the following conditions hold:

(L<sub>1</sub>)  $\text{int dom}(g) \neq \emptyset$ ,  $g$  is Gâteaux differentiable on  $\text{int dom}(g)$  and  $\text{dom } \nabla g = \text{int dom } g$ ;

(L<sub>2</sub>)  $\text{int dom}(g^*) \neq \emptyset$ ,  $g^*$  is Gâteaux differentiable on  $\text{int dom}(g^*)$  and  $\text{dom } \nabla g^* = \text{int dom}(g^*)$ .

It is well known (see, e.g., [21]) that if  $g$  is a proper, convex, lower semicontinuous, and Legendre type, then  $\nabla g^* = (\nabla g)^{-1}$  and  $\text{range } \nabla g^* = \text{dom } \nabla g$ .

Throughout this paper, the function  $g : V \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  is proper, convex, and lower semicontinuous which is also Legendre on  $\text{int dom}(g)$ .

The Bregman distance (see, e.g., [22]) is the function  $D_g : V \times \text{int dom}(g) \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$D_g(v, u) = g(v) - g(u) - \langle \nabla g(u), v - u \rangle, \quad (2)$$

with

$$\text{dom } D_g = (\text{dom } g) \times \text{int dom } g. \quad (3)$$

It should be pointed out that if  $V = H$  is a Hilbert space and  $g = (1/2)\|\cdot\|_H^2$ , then  $\nabla g = I$  (the identity mapping) and  $D_J(v, u) = (1/2)\|v - u\|_H^2$ .

For a multivalued mapping  $T : V \rightarrow 2^{V^*}$ , the associated  $D_J$ -antiresolvent (where  $J : V \rightarrow V^*$  is the duality mapping) of  $J - T$  (see [9]) is the mapping  $T^J : V \rightarrow 2^V$ , defined by

$$T^J = \nabla g^*(J - T). \quad (4)$$

Such a mapping is known as (see [23]) a  $D_g$ -antiresolvent mapping of  $T$  when  $J = \nabla g$  (in this case, the mapping  $T^J$  is denoted by  $T^g$ ).

In light of the above-mentioned discussion, we note that if  $J - T = \nabla g$ , then  $T^J$  is the identity mapping  $I$ .

Following [9], the mapping  $J - T : V \rightarrow 2^{V^*}$  is said to be  $D_J$ -pseudomonotone, if for every  $u, \eta \in \text{dom } J \cap \text{dom } T$ ,  $v \in (J - T)(u)$ ,  $\omega \in (J - T)(\eta)$ , and every sequence  $\{u_n\}_{n \in \mathbb{N}} \subset \text{dom } J \cap \text{dom } T$  and  $v_n \in (J - T)(u_n)$  the conditions

$$\begin{aligned} \nabla g^*(v_n) &\rightharpoonup \nabla g^*(v), \\ \limsup_{n \rightarrow \infty} \langle v_n, \nabla g^*(v_n) - \nabla g^*(v) \rangle &\leq 0 \end{aligned} \quad (5)$$

imply that

$$\liminf_{n \rightarrow \infty} \langle v_n, \nabla g^*(v_n) - \nabla g^*(\omega) \rangle \geq \langle \omega, \nabla g^*(v) - \nabla g^*(\omega) \rangle. \quad (6)$$

As remarked in [9], the  $D_J$ -pseudomonotonicity of the mapping  $J - T$  coincides with the pseudomonotonicity (or the

$D_g$ -pseudomonotonicity in the sense of Bregman distance  $D_g$ ) of the mapping  $\nabla g$ , if  $J - T = \nabla g$ .

The multivalued variational inequality defined by the  $D_J$ -mapping (or multivalued  $D_J$ -variational inequality)  $J - T : V \rightarrow 2^{V^*}$  and the set  $K \subset V$  is to find  $u \in K$  such that

$$\begin{aligned} \exists v \in (J - T)(u), \omega \\ \in (J - T)(\eta) : \end{aligned} \quad (7)$$

$$\langle v - f, \nabla g^*(\omega) - \nabla g^*(v) \rangle \geq 0, \quad \forall \eta \in K,$$

where  $f \in 2^{V^*}$ .

The multivalued nonlinear complementarity problem defined by the  $D_J$ -mapping (or multivalued nonlinear  $D_J$ -complementarity problem)  $J - T : V \rightarrow 2^{V^*}$  and the set  $K$  is to find  $u \in K$  such that

$$\langle v - f, \nabla g^*(v) \rangle = 0, \quad (8)$$

$$\langle v - f, \nabla g^*(\omega) - \nabla g^*(v) \rangle \geq 0, \quad \forall \eta \in K,$$

where  $f \in 2^{V^*}$ ,  $v \in (J - T)(u)$ , and  $\omega \in (J - T)(\eta)$ .

The multivalued  $D_J$ -variational inequality and multivalued nonlinear  $D_J$ -complementarity problem are very general in the sense that they include, as special cases, multivalued variational inequality and multivalued nonlinear complementarity problem.

The following definition and results will be used in the sequel.

**Definition 1** (see, e.g., [15, p. 84]). The mapping  $A : V \rightarrow 2^{V^*}$  is continuous on finite dimensional subspaces if for any finite dimensional subspace  $V_0 \subset V$ , the restriction of  $A$  to  $V_0 \cap \text{dom}(A)$  is weakly continuous.

**Corollary 2** (see [24]). Let  $j : V_0 \subset V \rightarrow V$  be the injection mapping. Let  $j^* : V^* \rightarrow V_0^*$  be its dual mapping. Then,  $j^* A j$  :  $\text{dom}(A) \cap jV_0 \rightarrow 2^{V_0^*}$  is continuous.

**Corollary 3** (see [25]). Let  $K$  be a nonempty compact convex set of  $\mathbb{R}^n$  and let  $S : K \rightarrow 2^K$  be continuous. Then  $S$  admits a fixed point.

## 2. Main Results

**Theorem 4.** Let  $K$  be a closed convex set in  $H$  and let  $T : K \rightarrow 2^H$  be a multivalued mapping. Then the following are equivalent:

$$\begin{aligned} (1) \quad \nabla g^*(v') \in \text{Pr}_K(x) \\ = \arg \min \{D_T(\nabla g^*(\omega'), x) \\ = \frac{1}{2} \|\nabla g^*(\omega') - x\|_H^2 : \nabla g^*(\omega') \\ \in K, \omega' \in (I - T)(\eta), \eta \in K\}, \end{aligned} \quad (9)$$

the multivalued projection for  $K$ ;

$$\begin{aligned}
 (2) \quad & \nabla g^*(v') \in K : \langle \nabla g^*(v') - x, \nabla g^*(\omega') \\
 & \quad - \nabla g^*(v') \rangle \geq 0 \\
 & \quad \forall \nabla g^*(\omega') \in K, \\
 & \quad \text{where } \omega' \in (I - T)(\eta), \\
 & \quad v' \in (I - T)(y), \quad y, \eta \in K.
 \end{aligned} \tag{10}$$

*Proof.* Assume that (1) holds. Let  $x \in 2^H$  and  $\nabla g^*(v') \in Pr_K(x) \subset K$ . For every  $\nabla g^*(\omega') \in K$  and  $t \in (0, 1]$ , we have

$$\begin{aligned}
 & D_I(x, \nabla g^*(v')) \\
 & \leq D_I(x, (1 - t)\nabla g^*(v') + t\nabla g^*(\omega')) \\
 & = D_I(x, \nabla g^*(v')) \\
 & \quad - t \langle x - \nabla g^*(v'), \nabla g^*(\omega') - \nabla g^*(v') \rangle \\
 & \quad + t^2 D_I(\nabla g^*(\omega'), \nabla g^*(v')).
 \end{aligned} \tag{11}$$

This implies

$$\begin{aligned}
 & \langle x - \nabla g^*(v'), \nabla g^*(\omega') - \nabla g^*(v') \rangle \\
 & \leq t D_I(\nabla g^*(\omega'), \nabla g^*(v')).
 \end{aligned} \tag{12}$$

Hence,  $t \rightarrow 0^+$  implies (2).

On the other hand, assume that (2) holds. For every  $\nabla g^*(\omega') \in K$ , we have

$$\begin{aligned}
 & D_I(\nabla g^*(\omega'), x) \\
 & = D_I(\nabla g^*(\omega'), \nabla g^*(v')) \\
 & \quad + \langle \nabla g^*(\omega') - x, \nabla g^*(\omega') - \nabla g^*(v') \rangle \\
 & \quad + D_I(\nabla g^*(v'), x) \geq D_I(\nabla g^*(v'), x).
 \end{aligned} \tag{13}$$

This implies (1). □

*Remark 5.* In the particular situation when  $I - T = \nabla g$  Theorem 4 coincides (in gradient setting) with Theorem 2.3 in [15] and also with Proposition 2.1 (1) and (2) in [2].

**Theorem 6.** *In addition to conditions on  $V, V^*, g^*$ , and  $J$ , one assumes that  $V$  is separable,  $K \subset V$  is nonempty closed and convex,  $T : K \rightarrow 2^{V^*}, \nabla g^*$  are weakly continuous mappings, and either  $T$  or  $\nabla g^*$  is continuous. Moreover, assume that the mapping  $J - T : K \rightarrow 2^{V^*}$  is a bounded  $D_J$ -pseudomonotone mapping and that, for each  $\eta \in K$ , there exist  $\eta_0 \in K, \omega_0 \in (J - T)(\eta_0), \omega \in (J - T)(\eta)$ , and  $r > 0$  such that*

$$\begin{aligned}
 & \langle \omega, \nabla g^*(\omega) - \nabla g^*(\omega_0) \rangle > \langle f, \nabla g^*(\omega) - \nabla g^*(\omega_0) \rangle, \\
 & f \in 2^{V^*}, \quad \|\nabla g^*(\omega)\|_V \geq r.
 \end{aligned} \tag{14}$$

Then there exists a solution to the multivalued  $D_J$ -variational inequality (7).

*Proof.* Suppose that  $\{x_1, x_2, x_3, \dots\}$  is an infinite dense set in  $K$  and  $V_m, m \in \mathbb{N}$ , is the linear span of  $\{x_1, x_2, x_3, \dots, x_m\}$ .

Let  $K_m = \text{con}\{x_1, x_2, x_3, \dots, x_m\} = \{\sum_{i=1}^m \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1\}$ . Let  $j_m : V_m \rightarrow V$  be the injection mapping and let  $j_m^* : V^* \rightarrow V_m^*$  be its restriction dual. Observe that  $\cup K_m$  is dense in  $K$ , for  $m \in \mathbb{N}$ .

Now, fix an integer  $m \geq 1$  and consider the finite dimensional problem.

Find  $u_m \in K_m$  such that for each  $\eta \in K_m$ , there exist  $s_m \in j_m^*(J - T)j_m(u_m), v_m \in (J - T)(u_m)$ , and  $\omega \in (J - T)(\eta)$  such that

$$\langle s_m - j_m^* f, \nabla g^*(\omega) - \nabla g^*(v_m) \rangle \geq 0. \tag{15}$$

The equivalent form of problem (15) is to find  $u_m \in K_m$  such that for each  $\eta \in K_m$ , there exist  $s_m \in j_m^*(J - T)j_m(u_m), v_m \in (J - T)(u_m)$ , and  $\omega \in (J - T)(\eta)$  such that

$$\begin{aligned}
 & \langle \nabla g^*(v_m), \nabla g^*(\omega) - \nabla g^*(v_m) \rangle \\
 & \geq \langle \nabla g^*(v_m) + j_m^* f - s_m, \nabla g^*(\omega) - \nabla g^*(v_m) \rangle.
 \end{aligned} \tag{16}$$

Using the identification of  $V_m$  with  $\mathbb{R}^m$  and  $V_m^*$  and Theorem 4 (with  $\mathbb{R}^m = H$  and  $J = I$ ), we see that (16) is equivalent to  $\nabla g^*(v_m) \in Pr_{K_m}(\nabla g^*(v_m) + j_m^* f - s_m)$ .

Let  $\bar{B}_r(0)$  be any closed ball containing  $K_m$ . It is well known (see, e.g., [26, p. 54, 224]) that  $\bar{B}_r(0)$  is compact and convex in  $K_m$ ; thus it is weakly closed.

From Corollary 2,  $j^*(J - T)j : \bar{B}_r(0) \rightarrow 2^{\bar{B}_r(0)}$  is continuous; hence the function  $\nabla g^*(v) \mapsto Pr_{\bar{B}_r(0)}(\nabla g^*(v) + j_m^* f - s_m)$  is continuous from  $\bar{B}_r(0)$  into  $2^{\bar{B}_r(0)}$ .

Hence, by Corollary 3, this equation admits a solution. If the closed convex set  $K$  is assumed to be bounded, then by the reflexivity of  $V$  it is weakly compact (by employing the Banach-Alaoglu theorem (see, e.g., [16, p. 3])).

Then we have a subsequence denoted by  $\{u_m\}_{m \in \mathbb{N}}$  such that  $u_m \rightharpoonup u \in K$ . Since  $J - T$  is bounded, we have  $\|v_m\|_{V^*} \leq M$  for all  $m \in \mathbb{N}$ . Since  $J - T$  is weakly continuous and since either  $J - T$  or  $\nabla g^*$  is continuous by hypothesis, it follows that  $T^J$  is weakly continuous by [27, Lemma 1]. So, we have  $\nabla g^*(v_m) \rightharpoonup \nabla g^*(v)$ .

Now, we prove that

$$\limsup_{m \rightarrow \infty} \langle v_m, \nabla g^*(v_m) - \nabla g^*(v) \rangle \leq 0. \tag{17}$$

For any  $\epsilon > 0$ , choose  $N$  so large and  $\tilde{u} \in K_N$  such that

$$\|\nabla g^*(v) - \nabla g^*(\tilde{v})\|_V < \epsilon \quad \text{for } \tilde{v} \in (J - T)(\tilde{u}). \tag{18}$$

Therefore, we have

$$\langle v_m - f, \nabla g^*(v_m) - \nabla g^*(\tilde{v}) \rangle \leq 0 \quad \text{for } m \geq N. \tag{19}$$

Since  $K_N \subset K_m$ , we have

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \langle v_m, \nabla g^*(v_m) - \nabla g^*(v) \rangle \\ &= \limsup_{m \rightarrow \infty} [\langle v_m, \nabla g^*(v_m) - \nabla g^*(\tilde{v}) \rangle \\ & \quad + \langle v_m, \nabla g^*(\tilde{v}) - \nabla g^*(v) \rangle] \\ & \leq (\|f\|_{V^*} + M) \epsilon. \end{aligned} \quad (20)$$

Since  $\epsilon$  is arbitrary, this shows the desired inequality.

By the  $D_J$ -pseudomonotonicity of  $J - T$ , it follows that

$$\liminf_{m \rightarrow \infty} \langle v_m, \nabla g^*(v_m) - \nabla g^*(\omega) \rangle \geq \langle v, \nabla g^*(v) - \nabla g^*(\omega) \rangle \quad (21)$$

for all  $\eta \in \text{dom } J \cap \text{dom } T$  and  $\omega \in (J - T)(\eta)$ .

If  $\eta \in K_n$ ,  $m \geq n$ , we have

$$\langle v_m, \nabla g^*(v_m) - \nabla g^*(\omega) \rangle \leq \langle f, \nabla g^*(v_m) - \nabla g^*(\omega) \rangle. \quad (22)$$

Hence

$$\langle v, \nabla g^*(v) - \nabla g^*(\omega) \rangle \leq \langle f, \nabla g^*(v) - \nabla g^*(\omega) \rangle \quad (23)$$

for every  $\eta$  in  $K_n$ ,  $n \in \mathbb{N}$ ,  $\omega \in (J - T)(\eta)$ .

Since  $\cup_n K_n$  is dense in  $K_n$ , so we have that  $u$  is a solution to (7).

Now, to complete the proof, we consider the case when  $K$  is unbounded.

In this case we consider the set  $K_\rho = \{\eta \in K : \|\nabla g^*(\omega)\|_V \leq \rho\}$ , where  $\rho = \max\{\|\nabla g^*(\omega_0)\|_V, r\}$ .

Since  $K_\rho$  is bounded, there exists at least one  $u_\rho \in K_\rho$ :

$$\langle v_\rho - f, \nabla g^*(\omega) - \nabla g^*(v_\rho) \rangle \geq 0 \quad (24)$$

for  $v_\rho \in (J - T)(u_\rho)$  and  $\eta \in K_\rho$ .

Since  $\eta_0 \in K_\rho$ , we have

$$\langle v_\rho - f, \nabla g^*(v_\rho) - \nabla g^*(\omega_0) \rangle \leq 0 \quad \text{for } \omega_0 \in (J - T)(\eta_0). \quad (25)$$

This, together with (14), implies that  $\|\nabla g^*(v_\rho)\|_V < \rho$ .

To clarify that  $u_\rho$  is also a solution to original problem on  $K$ , for any  $\eta \in K$ , set  $\nabla g^*(\omega_t) = (1 - t)\nabla g^*(v_\rho) + t\nabla g^*(\omega)$  for  $t > 0$  is sufficiently small, where  $\omega_t \in (J - T)(\eta_t)$  and  $\eta_t \in K_\rho$ . Consequently

$$\begin{aligned} u_\rho \in K_\rho \subset K : 0 & \leq \langle v_\rho - f, \nabla g^*(\omega_t) - \nabla g^*(v_\rho) \rangle \\ & = t \langle v_\rho - f, \nabla g^*(\omega) - \nabla g^*(v_\rho) \rangle \end{aligned} \quad (26)$$

for  $\eta \in K$ .

This completes the proof.  $\square$

*Remark 7.* In the particular situation when  $J - T = \nabla g$ , Theorem 6 coincides with the Brezis Theorem (see, e.g., [13, 14]) for the case of gradient mapping.

We are now in a position to state and prove the following theorem.

**Theorem 8.** Let all assumptions of Theorem 6 hold, except for condition (14) let it be replaced by the  $D_J$ -coercive condition: for  $\omega \in (J - T)(\eta)$ ,

$$\lim_{\|\nabla g^*(\omega)\|_V \rightarrow \infty} \left[ \frac{\langle \omega, \nabla g^*(\omega) - \nabla g^*(\omega_0) \rangle}{\|\nabla g^*(\omega)\|_V} \right] = +\infty, \quad (27)$$

$$\omega_0 \in (J - T)(\eta_0), \quad \eta, \eta_0 \in K.$$

Suppose further that  $K$  has the following property (W):  $\alpha \nabla g^*(\omega) \in K$  for all  $\nabla g^*(\omega) \in K$  and  $\alpha \geq 0$ .

Then for every  $f \in 2^{V^*}$  there exist  $u \in K$ ,  $v \in (J - T)(u)$  such that

$$\langle v - f, \nabla g^*(v) \rangle = 0, \quad \langle v - f, \nabla g^*(\omega) - \nabla g^*(v) \rangle \geq 0 \quad (28)$$

for all  $\eta \in K$ ,  $\omega \in (J - T)(\eta)$ .

*Proof.* Let  $f \in 2^{V^*}$  satisfy  $\|f\|_{V^*} < M$  and

$$\|\nabla g^*(\omega_0)\|_V < \frac{2M - \|f\|_{V^*}}{\|f\|_{V^*}} \rho. \quad (29)$$

The  $D_J$ -coercivity of  $J - T$  implies that there exists  $\rho > 0$  such that

$$\langle \omega, \nabla g^*(\omega) - \nabla g^*(\omega_0) \rangle \geq 2M \|\nabla g^*(\omega)\|_V \quad (30)$$

for  $\omega \in (J - T)(\eta)$ ,  $\eta \in K$  with  $\|\nabla g^*(\omega)\|_V \geq \rho$ .

So we conclude

$$\begin{aligned} & \langle \omega - f, \nabla g^*(\omega) - \nabla g^*(\omega_0) \rangle \\ & \geq 2M \|\nabla g^*(\omega)\|_V - \|f\|_{V^*} \|\nabla g^*(\omega)\|_V - \|f\|_{V^*} \|\nabla g^*(\omega_0)\|_V \\ & \geq (2M - \|f\|_{V^*}) \rho - \|f\|_{V^*} \|\nabla g^*(\omega_0)\|_V \\ & > 0 \quad \text{for } \eta \in K, \omega \in (J - T)(\eta) \text{ with } \|\nabla g^*(\omega)\|_V \geq \rho. \end{aligned} \quad (31)$$

The second part of (28) thus follows from Theorem 6.

To prove the first part of (28), observe that we can choose a point  $\eta$  in  $K$  and  $\omega \in (J - T)(\eta)$  and assume that  $\nabla g^*(\omega) = 0$ .

Therefore, from Theorem 6, we have

$$\langle v - f, \nabla g^*(v) \rangle \leq 0 \quad (32)$$

for all  $v \in (J - T)(u)$ ,  $u \in K$ .

On the other hand, setting  $\nabla g^*(\omega) = \alpha \nabla g^*(v)$ , where  $\alpha > 1$ , we get

$$0 \leq \langle v - f, \nabla g^*(\omega) - \nabla g^*(v) \rangle = (\alpha - 1) \langle v - f, \nabla g^*(v) \rangle. \quad (33)$$

This implies

$$\langle v - f, \nabla g^*(v) \rangle \geq 0. \quad (34)$$

So,

$$\langle v - f, \nabla g^*(v) \rangle = 0. \quad (35)$$

This completes the proof of (28).  $\square$

The following proposition gives a characterization of the sum of two  $D_J$ -Pseudomonotone mappings.

**Proposition 9.** *Let  $V, V^*$ , and  $J$  be as above and let  $T_i : V \rightarrow 2^{V^*}, i = 1, 2$ , and  $\nabla g^*$  be weakly continuous mappings. If  $J - T_i : V \rightarrow 2^{V^*}, i = 1, 2$ , are  $D_J$ -pseudomonotone mappings such that  $\text{dom } J \cap \text{dom } T_i \neq \emptyset, i = 1, 2$ , then  $\sum_{i=1}^2 (J - T_i)$  is  $D_J$ -pseudomonotone.*

*Proof.* Let  $\tilde{y}_n \in \sum_{i=1}^2 (J - T_i)(\eta_n), \tilde{y} \in \sum_{i=1}^2 (J - T_i)(\eta), \eta_n, \eta \in \text{dom } J \cap \text{dom } T_i, i = 1, 2$ , with  $\nabla g^*(\tilde{y}_n) \rightarrow \nabla g^*(\tilde{y})$  and

$$\limsup_{n \rightarrow \infty} \langle \tilde{y}_n, \nabla g^*(\tilde{y}_n) - \nabla g^*(\tilde{y}) \rangle \leq 0. \quad (36)$$

Now, we prove for  $y_n^{(i)} \in (J - T_i)(\eta_n), y^{(i)} \in (J - T_i)(\eta), i = 1, 2$ , that

$$\limsup_{n \rightarrow \infty} \langle y_n^{(1)}, \nabla g^*(y_n^{(1)}) - \nabla g^*(y^{(1)}) \rangle \leq 0, \quad (37)$$

$$\limsup_{n \rightarrow \infty} \langle y_n^{(2)}, \nabla g^*(y_n^{(2)}) - \nabla g^*(y^{(2)}) \rangle \leq 0.$$

If

$$\limsup_{n \rightarrow \infty} \langle y_n^{(2)}, \nabla g^*(y_n^{(2)}) - \nabla g^*(y^{(2)}) \rangle = \epsilon > 0, \quad (38)$$

(note that otherwise, by symmetry), then there exists a subsequence  $\{y_{n_k}^{(2)}\}_{k \in \mathbb{N}} \subset \{y_n^{(2)}\}_{n \in \mathbb{N}}$  such that

$$\limsup_{k \rightarrow \infty} \langle y_{n_k}^{(2)}, \nabla g^*(y_{n_k}^{(2)}) - \nabla g^*(y^{(2)}) \rangle = \epsilon. \quad (39)$$

This implies that

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \langle y_{n_k}^{(1)}, \nabla g^*(y_{n_k}^{(1)}) - \nabla g^*(y^{(1)}) \rangle \\ &= \limsup_{k \rightarrow \infty} [ \langle \tilde{y}_{n_k}, \nabla g^*(\tilde{y}_{n_k}) - \nabla g^*(\tilde{y}) \rangle \\ & \quad - \langle y_{n_k}^{(2)}, \nabla g^*(y_{n_k}^{(2)}) - \nabla g^*(y^{(2)}) \rangle ] \\ &\leq 0 - \epsilon. \end{aligned} \quad (40)$$

From the  $D_J$ -pseudomonotonicity of  $J - T_1$ , we get for all  $y' \in (J - T_1)(\eta'), \eta' \in \text{dom } J \cap \text{dom } T_1$

$$\begin{aligned} & \langle y^{(1)}, \nabla g^*(y^{(1)}) - \nabla g^*(y') \rangle \\ &\leq \liminf_{k \rightarrow \infty} \langle y_{n_k}^{(1)}, \nabla g^*(y_{n_k}^{(1)}) - \nabla g^*(y') \rangle. \end{aligned} \quad (41)$$

Letting  $y' = y^{(1)}$ , we obtain

$$0 \leq \liminf_{k \rightarrow \infty} \langle y_{n_k}^{(1)}, \nabla g^*(y_{n_k}^{(1)}) - \nabla g^*(y^{(1)}) \rangle \leq 0 - \epsilon, \quad (42)$$

a contradiction.

Hence,

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \langle y_{n_k}^{(1)}, \nabla g^*(y_{n_k}^{(1)}) - \nabla g^*(y^{(1)}) \rangle \leq 0, \\ & \limsup_{k \rightarrow \infty} \langle y_{n_k}^{(2)}, \nabla g^*(y_{n_k}^{(2)}) - \nabla g^*(y^{(2)}) \rangle \leq 0. \end{aligned} \quad (43)$$

This holds for any subsequence, so (37) holds and the proof follows immediately by the superadditivity of the  $\lim \inf$ .  $\square$

### 3. Application to Multivalued Nonlinear $D_J$ -Complementarity Problem

As applications of Theorem 8 we consider the multivalued nonlinear  $D_J$ -complementarity problem (8) with  $T = \lambda T_1 + T_2 - \lambda J$ , where  $T_i, i = 1, 2$  are two nonlinear multivalued mappings from  $K$  to  $2^{V^*}$ , and  $\lambda \in (0, \infty)$ .

**Theorem 10.** *Let  $V, V^*, g^*, J$ , and  $K$  be the same as in Theorem 6, and suppose that  $K$  has the property (W). Let the mappings  $\nabla g^*$  and  $T_i, i = 1, 2$  be weakly continuous and let either  $\nabla g^*$  or both  $T_i, i = 1, 2$  be continuous. Let  $J - T_i : K \rightarrow 2^{V^*}, i = 1, 2$  be two bounded  $D_J$ -pseudomonotone mappings.*

Let

$$\begin{aligned} \frac{1}{\rho_1} &= \inf_{\substack{\eta \in K \\ \|\nabla g^*(\lambda w_1 + w_2)\|_V \neq 0}} \left( \langle w_1, \nabla g^*(\lambda w_1 + w_2) \right. \\ & \quad \left. - \nabla g^*(\lambda w_0^{(1)} + w_0^{(2)}) \right) \\ & \quad \times \left( \|\nabla g^*(\lambda w_1 + w_2)\|_V^2 \right)^{-1}, \\ a &= \liminf_{\substack{\eta \in K \\ \|\nabla g^*(\lambda w_1 + w_2)\|_V \rightarrow \infty}} \left( \langle w_2, \nabla g^*(\lambda w_1 + w_2) \right. \\ & \quad \left. - \nabla g^*(\lambda w_0^{(1)} + w_0^{(2)}) \right) \\ & \quad \times \left( \|\nabla g^*(\lambda w_1 + w_2)\|_V^2 \right)^{-1}, \\ b &= \liminf_{t \rightarrow \infty} \frac{\Phi(t)}{t} \end{aligned} \quad (44)$$

Be such that  $a < b$ , where  $w_i \in (J - T_i)(\eta), w_0^{(i)} \in (J - T_i)(\eta_0), i = 1, 2, \eta, \eta_0 \in K, \rho_1 > 0$  and  $\Phi$  is the gauge function. Then for every  $\lambda > \rho_1(b - a)$  problem (8) with  $T = \lambda T_1 + T_2 - \lambda J$  has a solution in  $K$ .

*Proof.* By Proposition 9,  $J - T = \lambda(J - T_1) + (J - T_2)$  is  $D_J$ -pseudomonotone for every  $\lambda \geq 0$ . Set  $\lambda > \rho(b - a)$ . Then

$$\begin{aligned} & \liminf_{\substack{\eta \in K \\ \|\nabla g^*(\lambda w_1 + w_2)\|_V \rightarrow \infty}} \left( \langle \lambda w_1 + w_2, \nabla g^*(\lambda w_1 + w_2) \right. \\ & \quad \left. - \nabla g^*(\lambda w_0^{(1)} + w_0^{(2)}) \right) \\ & \quad \times \left( \|\nabla g^*(\lambda w_1 + w_2)\|_V^2 \right)^{-1} \end{aligned} \quad (45)$$

$$\geq \frac{\lambda}{\rho_1} + a > b > 0.$$

This implies that the mapping  $J - T = \lambda(J - T_1) + (J - T_2)$  is  $D_J$ -coercive.  $\square$

The conclusion follows from Theorem 8.

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