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Research Article

(m, n)-Semirings and a Generalized Fault-Tolerance Algebra of Systems

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We propose a new class of mathematical structures called (m, n)-*semirings* (which generalize the usual semirings) and describe their basic properties. We define partial ordering and generalize the concepts of congruence, homomorphism, and so forth, for (m, n)-semirings. Following earlier work by Rao (2008), we consider systems made up of several components whose failures may cause them to fail and represent the set of such systems algebraically as an (m, n)-semiring. Based on the characteristics of these components, we present a formalism to compare the fault-tolerance behavior of two systems using our framework of a partially ordered (m, n)-semiring.

1. Introduction

Fault tolerance is the property of a system to be functional even if some of its components fail. It is a very critical issue in the design of the systems as in Air Traffic Control Systems [1, 2], real-time embedded systems [3], robotics [4, 5], automation systems [6, 7], medical systems [8], mission critical systems [9], and a lot of others. Description of fault-tolerance modeling using algebraic structures is proposed by Beckmann [10] for groups and by Hadjicostis [11] for semigroups and semirings. Semirings are also used in other areas of computer science like cryptography [12], databases [13], graph theory, game theory [14], and so forth. Rao [15] uses the formalism of semirings to analyze the fault tolerance of a system as a function of its composition, with a partial ordering relation between systems used to compare their fault-tolerance behaviors.

The generalization of algebraic structures was in active research for a long time; Timm [16] in 1967 proposed commutative n-groups; later Crombez [17] in 1972 generalized rings and named it as (m, n)-rings. It was further studied by Crombez and Timm [18], Leeson and Butson [19, 20], and by Dudek [21]. Recently the generalization of algebraic structures is studied Davvaz et al. [22, 23].

In this paper, we first define the (m, n)-semiring $(\mathscr{R}, f, g)$ (which is a generalization of the ordinary semiring $(\mathscr{R}, +, \times)$, where $\mathscr{R}$ is a set with binary operations $+$ and $\times$), using f and g which are m-ary and n-ary operations, respectively. We propose identity elements, multiplicatively absorbing elements, idempotents, and homomorphisms for (m, n)-semirings. We also briefly touch on zero-divisor free, zero-sum free, additively cancellative, and multiplicatively cancellative (m, n)-semirings and the congruence relation on (m, n)-semirings. In Section 4, we use the facts that each system consists of components or subsystems and that the fault-tolerance behavior of the system depends on each of the components or subsystems that constitute the system. A system may itself be a module or part of a larger system, so that its fault tolerance affects that of the whole system of which it is a part. We analyze the fault tolerance of a system given its composition, extending earlier work of Rao [15]. Section 2 describes the notations used and the general conventions followed.

Section 3 deals with the definition and properties of (m, n)-semirings. In Section 4, we extend the results of

Rao [15] using a partial ordering on the (m, n)-semiring of systems: the class of systems is algebraically represented by an (m, n)-semiring, and the fault-tolerance behavior of two systems is compared using partially ordered (m, n)-semiring.

2. Preliminaries

The set of integers is denoted by $\mathbb{Z}$, with $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denoting the sets of positive integers and negative integers, respectively, and m and n used are positive integers. Let $\mathscr{R}$ be a set and f a mapping $f : \mathscr{R}^m \to \mathscr{R}$; that is, f is an m-ary operation. Elements of the set $\mathscr{R}$ are denoted by x_i, y_i where $i \in \mathbb{Z}_+$.

Definition 1. A nonempty set $\mathscr{R}$ with an m-ary operation f is called an *m-ary groupoid* and is denoted by $(\mathscr{R}, f)$ (see Dudek [24]).

We use the following general convention.

The sequence $x_i, x_{i+1}, \ldots, x_m$ is denoted by x_i^m where $1 \le i \le m$.

For all $1 \le i \le j \le m$, the following term

$$f\left(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_m\right) \tag{1}$$

is represented as

$$f\left(x_1^i, y_{i+1}^j, z_{j+1}^m\right). \tag{2}$$

In the case when $y_{i+1} = \cdots = y_j = y$, (2) is expressed as

$$f(x_1^i, \overset{(j-i)}{y}, z_{j+1}^m). \tag{3}$$

Definition 2. Let $x_1, x_2, \ldots, x_{2m-1}$ be elements of set $\mathscr{R}$.

(i) Then, the associativity and distributivity laws for the m-ary operation f are defined as follows.

(a) *Associativity*:

$$\begin{aligned} &f\left(x_1^{i-1}, f\left(x_i^{m+i-1}\right), x_{m+i}^{2m-1}\right) \\ &\quad = f\left(x_1^{j-1}, f\left(x_j^{m+j-1}\right), x_{m+j}^{2m-1}\right), \end{aligned} \tag{4}$$

for all $x_1, \ldots, x_{2m-1} \in \mathscr{R}$, for all $1 \le i \le j \le m$ (from Gluskin [25]).

(b) *Commutativity*:

$$\begin{aligned} &f\left(x_1, x_2, \ldots, x_m\right) \\ &\quad = f\left(x_{\eta(1)}, x_{\eta(2)}, \ldots, x_{\eta(m)}\right), \end{aligned} \tag{5}$$

for every permutation η of $\{1, 2, \ldots, m\}$ (from Timm [16]), $\forall x_1, x_2, \ldots, x_m \in \mathscr{R}$.

(ii) An m-ary groupoid $(\mathscr{R}, f)$ is called an *m-ary semigroup* if f is associative (from Dudek [24]); that is, if

$$\begin{aligned} &f\left(x_1^{i-1}, f\left(x_i^{m+i-1}\right), x_{m+i}^{2m-1}\right) \\ &\quad = f\left(x_1^{j-1}, f\left(x_j^{m+j-1}\right), x_{m+j}^{2m-1}\right), \end{aligned} \tag{6}$$

for all $x_1, \ldots, x_{2m-1} \in \mathscr{R}$, where $1 \le i \le j \le m$.

(iii) Let $x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_m$ be elements of set $\mathscr{R}$, and $1 \le i \le n$. The n-ary operation g is *distributive* with respect to the m-ary operation f if

$$\begin{aligned} &g\left(x_1^{i-1}, f\left(a_1^m\right), x_{i+1}^n\right) \\ &\quad = f\left(g\left(x_1^{i-1}, a_1, x_{i+1}^n\right), \ldots,\right. \\ &\qquad \left. g\left(x_1^{i-1}, a_m, x_{i+1}^n\right)\right). \end{aligned} \tag{7}$$

Remark 3. (i) An m-ary semigroup $(\mathscr{R}, f)$ is called a *semiabelian* or $(1, m)$-commutative if

$$f(x, \underbrace{a, \ldots, a}_{m-2}, y) = f(y, \underbrace{a, \ldots, a}_{m-2}, x), \tag{8}$$

for all $x, y, a \in \mathscr{R}$ (from Dudek and Mukhin [26]).

(ii) Consider a k-ary group (G, h) in which the k-ary operation h is distributive with respect to itself, that is,

$$\begin{aligned} &h\left(x_1^{i-1}, h\left(a_1^k\right), x_{i+1}^k\right) \\ &\quad = h\left(h\left(x_1^{i-1}, a_1, x_{i+1}^k\right), \ldots, h\left(x_1^{i-1}, a_k, x_{i+1}^k\right)\right), \end{aligned} \tag{9}$$

for all $1 \le i \le k$. These types of groups are called *autodistributive* k-ary groups (see Dudek [27]).

3. (m, n)-Semirings and Their Properties

Definition 4. An (m, n)-semiring is an algebraic structure $(\mathscr{R}, f, g)$ which satisfies the following axioms:

(i) $(\mathscr{R}, f)$ is an m-ary semigroup,

(ii) $(\mathscr{R}, g)$ is an n-ary semigroup,

(iii) the n-ary operation g is *distributive* with respect to the m-ary operation f.

Example 5. Let $\mathscr{B}$ be any Boolean algebra. Then, $(\mathscr{B}, f, g)$ is an (m, n)-semiring where $f(A_1^m) = A_1 \cup A_2 \cup \cdots \cup A_m$ and $g(B_1^n) = B_1 \cap B_2 \cap \cdots \cap B_n$, for all $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n \in \mathscr{B}$.

In general, we have the following.

Theorem 6. *Let $(\mathscr{R}, +, \times)$ be an ordinary semiring. Let f be an m-ary operation and g be an n-ary operation on $\mathscr{R}$ as follows:*

$$\begin{aligned} f\left(x_1^m\right) &= \sum_{i=1}^m x_i, \quad \forall x_1, x_2, \ldots, x_m \in \mathscr{R}, \\ g\left(y_1^n\right) &= \prod_{i=1}^n y_i, \quad \forall y_1, y_2, \ldots, y_n \in \mathscr{R}. \end{aligned} \tag{10}$$

Then, $(\mathscr{R}, f, g)$ is an (m, n)-semiring.

Proof. Omitted as obvious. □

Example 7. The following give us some (m, n)-semirings in different ways indicated by Theorem 6.

(i) Let $(\mathscr{R}, +, \times)$ be an ordinary semiring and $x_1, x_2, \ldots, x_n$ be in $\mathscr{R}$. If we set

$$g(x_1^n) = x_1 \times x_2 \times \cdots \times x_n, \tag{11}$$

we get a $(2, n)$-semiring $(\mathscr{R}, +, g)$.

(ii) In an (m, n)-semiring $(\mathscr{R}, f, g)$, fixing elements a_2^{m-1} and b_2^{n-1}, we obtain two binary operations as follows:

$$\begin{aligned} x \oplus y &= f\left(x, a_2^{m-1}, y\right), \\ x \otimes y &= g\left(x, b_2^{n-1}, y\right). \end{aligned} \tag{12}$$

Obviously, $(\mathscr{R}, \oplus, \otimes)$ is a semiring.

(iii) The set $\mathbb{Z}_-$ of all negative integers is not closed under the binary products; that is, $\mathbb{Z}_-$ does not form a semiring, but it is a $(2, 3)$-semiring.

Definition 8. Let $(\mathscr{R}, f, g)$ be an (m, n)-semiring. Then m-ary semigroup $(\mathscr{R}, f)$ has an *identity element* $\mathbf{0}$ if

$$x = f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{i-1}, x, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-i}), \tag{13}$$

for all $x \in \mathscr{R}$ and $1 \leq i \leq m$. We call $\mathbf{0}$ as an *identity element* of (m, n)-semiring $(\mathscr{R}, f, g)$.

Similarly, n-ary semigroup $(\mathscr{R}, g)$ has an *identity element* $\mathbf{1}$ if

$$y = g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{j-1}, y, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-j}), \tag{14}$$

for all $y \in \mathscr{R}$ and $1 \leq j \leq n$.

We call $\mathbf{1}$ as an *identity element* of (m, n)-semiring $(\mathscr{R}, f, g)$.

We therefore call $\mathbf{0}$ the f-identity, and $\mathbf{1}$ the g-identity.

Remark 9. In an (m, n)-semiring $(\mathscr{R}, f, g)$, placing $\mathbf{0}$ and $\mathbf{1}$, $(m-2)$ and $(n-2)$ times, respectively, we obtain the following binary operations:

$$\begin{gathered} x + y = f(x, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}, y), \\ x \times y = g(x, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, y), \\ \forall x, y \in \mathscr{R}. \end{gathered} \tag{15}$$

Definition 10. Let $(\mathscr{R}, f, g)$ be an (m, n)-semiring with an f-identity element $\mathbf{0}$ and g-identity element $\mathbf{1}$. Then,

(i) $\mathbf{0}$ is said to be *multiplicatively absorbing* if it is absorbing in $(\mathscr{R}, g)$, that is, if

$$g\left(\mathbf{0}, x_1^{n-1}\right) = g\left(x_1^{n-1}, \mathbf{0}\right) = \mathbf{0}, \tag{16}$$

for all $x_1, x_2, \ldots, x_{n-1} \in \mathscr{R}$.

(ii) $(\mathscr{R}, f, g)$ is called *zero-divisor free* if

$$g(x_1, x_2, \ldots, x_n) = \mathbf{0} \tag{17}$$

always implies $x_1 = \mathbf{0}$ or $x_2 = \mathbf{0}$ or $\cdots$ or $x_n = \mathbf{0}$.

Elements $x_1, x_2, \ldots, x_{n-1} \in \mathscr{R}$ are called *left zero-divisors* of (m, n)-semiring $(\mathscr{R}, f, g)$ if there exists $a \neq \mathbf{0}$ and the following holds:

$$g\left(x_1^{n-1}, a\right) = \mathbf{0}. \tag{18}$$

(iii) $(\mathscr{R}, f, g)$ is called *zero-sum free* if

$$f(x_1, x_2, \ldots, x_m) = \mathbf{0} \tag{19}$$

always implies $x_1 = x_2 = \cdots = x_m = \mathbf{0}$.

(iv) $(\mathscr{R}, f, g)$ is called *additively cancellative* if the m-ary semigroup $(\mathscr{R}, f)$ is cancellative, that is,

$$f\left(x_1^{i-1}, a, x_{i+1}^m\right) = f\left(x_1^{i-1}, b, x_{i+1}^m\right) \quad \Longrightarrow a = b, \tag{20}$$

for all $a, b, x_1, x_2, \ldots, x_m \in \mathscr{R}$ and for all $1 \leq i \leq m$.

(v) $(\mathscr{R}, f, g)$ is called *multiplicatively cancellative* if the n-ary semigroup $(\mathscr{R}, g)$ is cancellative, that is,

$$g\left(x_1^{i-1}, a, x_{i+1}^n\right) = g\left(x_1^{i-1}, b, x_{i+1}^n\right) \quad \Longrightarrow a = b, \tag{21}$$

for all $a, b, x_1, x_2, \ldots, x_n \in \mathscr{R}$ and for all $1 \leq i \leq n$.

Elements $x_1, x_2, \ldots, x_{n-1}$ are called *left cancellable* in an n-ary semigroup $(\mathscr{R}, g)$ if

$$g\left(x_1^{n-1}, a\right) = g\left(x_1^{n-1}, b\right) \quad \Longrightarrow a = b, \tag{22}$$

for all $x_1, x_2, \ldots, x_{n-1}, a, b \in \mathscr{R}$.

$(\mathscr{R}, f, g)$ is called *multiplicatively left cancellative* if elements $x_1, x_2, \ldots, x_{n-1} \in \mathscr{R} \backslash \{\mathbf{0}\}$ are multiplicatively left cancellable in n-ary semigroup $(\mathscr{R}, g)$.

Theorem 11. *Let $(\mathscr{R}, f, g)$ be an (m, n)-semiring with f-identity $\mathbf{0}$.*

(i) If elements $x_1, x_2, \ldots, x_{n-1} \in \mathscr{R}$ are multiplicatively left cancellable, then elements $x_1, x_2, \ldots, x_{n-1}$ are not left divisors.

(ii) If the (m, n)-semiring $(\mathscr{R}, f, g)$ is multiplicatively left cancellative, then it is zero-divisor free.

We have generalized Theorem 11 from Theorem 4.4 of Hebisch and Weinert [28].

We have generalized the definition of idempotents of semirings given by Bourne [29] and Hebisch and Weinert [28]), as follows.

Definition 12. Let $(\mathscr{R}, f, g)$ be an (m, n)-semiring. Then,

(i) it is called *additively idempotent* if $(\mathscr{R}, f)$ is an idempotent m-ary semigroup, that is, if

$$f(\underbrace{x, x, \ldots, x}_{m}) = x, \tag{23}$$

for all $x \in \mathscr{R}$;

(ii) it is called *multiplicatively idempotent* if $(\mathscr{R}, g)$ is an idempotent n-ary semigroup, that is, if

$$g(\underbrace{y, y, \ldots, y}_{n}) = y, \tag{24}$$

for all $y \in \mathscr{R}$, $y \neq \mathbf{0}$.

Theorem 13. *An (m, n)-semiring $(\mathscr{R}, f, g)$ having at least two multiplicatively idempotent elements in the center is not multiplicatively cancellative.*

Proof. Let a and b be two multiplicatively idempotent elements in the center, $a \neq b$. Then,

$$g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, a, b) = g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, b, a), \tag{25}$$

which can be written as follows:

$$g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, g(\overset{(n)}{a}), b) = g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, g(\overset{(n)}{b}), a), \tag{26}$$

which is represented as

$$\begin{aligned} &g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-3}, g(\mathbf{1}, \overset{(n-1)}{a}), a, b) \\ &\quad = g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-3}, g(\mathbf{1}, \overset{(n-1)}{b}), b, a). \end{aligned} \tag{27}$$

If the (m, n)-semiring $(\mathscr{R}, f, g)$ is multiplicatively cancellative, then the following holds true:

$$\begin{aligned} &g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-3}, g(\mathbf{1}, \overset{(n-1)}{a}), \mathbf{1}, \mathbf{1}) \\ &\quad = g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-3}, g(\mathbf{1}, \overset{(n-1)}{b}), \mathbf{1}, \mathbf{1}), \\ &g(\mathbf{1}, \overset{(n-1)}{a}) = g(\mathbf{1}, \overset{(n-1)}{b}), \end{aligned} \tag{28}$$

which implies that $a = b$, which is a contradiction to the assumption that $a \neq b$; therefore, $(\mathscr{R}, f, g)$ is not multiplicatively cancellative. □

We have generalized Exercise 2.7 in Chapter I of Hebisch and Weinert [28] to get the following.

Definition 14. Let $(\mathscr{R}, f, g)$ be an (m, n)-semiring and σ an equivalence relation on $\mathscr{R}$.

(i) Then, σ is called a *congruence relation* or a *congruence* of $(\mathscr{R}, f, g)$, if it satisfies the following properties for all $1 \leq i \leq m$ and $1 \leq j \leq n$:

(a) if $x_i \sigma y_i$ then $f(x_1^m) \sigma f(y_1^m)$,

(b) if $z_j \sigma u_j$ then $g(z_1^n) \sigma g(u_1^n)$,

for all $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m$, $z_1, z_2, \ldots, z_n, u_1, u_2, \ldots, u_n \in \mathscr{R}$.

(ii) Let σ be a congruence on an algebra $\mathscr{R}$. Then, the *quotient* of $\mathscr{R}$ by σ, written as $\mathscr{R}/\sigma$, is the algebra whose universe is $\mathscr{R}/\sigma$ and whose fundamental operation satisfies

$$f^{\mathscr{R}/\sigma}(x_1, x_2, \ldots, x_m) = \frac{f^{\mathscr{R}}(x_1, x_2, \ldots, x_m)}{\sigma}, \tag{29}$$

where $x_1, x_2, \ldots, x_m \in \mathscr{R}$ [30].

Theorem 15. *Let $(\mathscr{R}, f, g)$ be an (m, n)-semiring and the relation σ be a congruence relation on $(\mathscr{R}, f, g)$. Then, the quotient $(\mathscr{R}/\sigma, F, G)$ is an (m, n)-semiring under $F((x_1)/\sigma, \ldots, (x_m)/\sigma) = f(x_1^m)/\sigma$ and $G((y_1)/\sigma, \ldots, (y_n)/\sigma) = g(y_1^n)/\sigma$, for all $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ in $\mathscr{R}$.*

Proof. Omitted as obvious. □

Definition 16. We define homomorphism, isomorphism, and a product of two mappings as follows.

(i) A mapping $\varphi : \mathscr{R} \rightarrow \mathscr{S}$ from (m, n)-semiring $(\mathscr{R}, f, g)$ into (m, n)-semiring $(\mathscr{S}, f', g')$ is called a *homomorphism* if

$$\begin{aligned} \varphi(f(x_1^m)) &= f'(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m)), \\ \varphi(g(y_1^n)) &= g'(\varphi(y_1), \varphi(y_2), \ldots, \varphi(y_n)), \end{aligned} \tag{30}$$

for all $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \in \mathscr{R}$.

(ii) The (m, n)-semirings $(\mathscr{R}, f, g)$ and $(\mathscr{S}, f', g')$ are called *isomorphic* if there exists one-to-one homomorphism from $\mathscr{R}$ onto $\mathscr{S}$. One-to-one homomorphism is called *isomorphism*.

(iii) If we apply mapping $\varphi : \mathscr{R} \rightarrow \mathscr{S}$ and then $\psi : \mathscr{S} \rightarrow \mathscr{T}$ on x, we get the mapping $(\psi \circ \varphi)(x)$ which is equal to $\psi(\varphi(x))$, where $x \in \mathscr{R}$. It is called the *product* of ψ and φ [28].

We have generalized Definition 16 from Definition 2 of Allen [31].

We have generalized the following theorem from Theorem 3.3 given by Hebisch and Weinert [28].

Theorem 17. *Let $(\mathscr{R}, f, g)$, $(\mathscr{S}, f', g')$, and $(\mathscr{T}, f'', g'')$ be (m, n)-semirings. Then, if the following mappings $\varphi : (\mathscr{R}, f, g) \rightarrow (\mathscr{S}, f', g')$ and*

$\psi : (\mathscr{S}, f', g') \rightarrow (\mathscr{T}, f'', g'')$ are homomorphisms; then,

$\psi \circ \varphi : (\mathscr{R}, f, g) \rightarrow (\mathscr{T}, f'', g'')$ is also a homomorphism.

Proof. Let $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_n$ be in $\mathscr{R}$. Then

$$\begin{aligned} (\psi \circ \varphi)(f(x_1^m)) &= \psi(\varphi(f(x_1, x_2, \ldots, x_m))) \\ &= \psi(f'(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_m))) \\ &= f''(\psi(\varphi(x_1)), \psi(\varphi(x_2)), \ldots, \psi(\varphi(x_m))) \\ &= f''((\psi \circ \varphi)(x_1), (\psi \circ \varphi)(x_2), \ldots, (\psi \circ \varphi)(x_m)). \end{aligned} \tag{31}$$

In a similar manner, we can deduce that

$$\begin{aligned} &(\psi \circ \varphi)(g(y_1^n)) \\ &\quad = g''((\psi \circ \varphi)(y_1), (\psi \circ \varphi)(y_2), \ldots, (\psi \circ \varphi)(y_n)). \end{aligned} \tag{32}$$

Thus, it is evident that $\psi \circ \varphi$ is a homomorphism from $\mathscr{R} \rightarrow \mathscr{T}$. □

This proof is similar to that of Theorem 6.5 given by Burris and Sankappanavar [30].

Definition 18. Let $(\mathcal{R}, f, g)$ and $(\mathcal{S}, f', g')$ be (m,n)-semirings and $\varphi : \mathcal{R} \to \mathcal{S}$ a homomorphism. Then, the *kernel* of φ, written as $\ker\varphi$, is defined as follows:

$$\ker \varphi = \{(a,b) \in \mathcal{R} \times \mathcal{R} \mid \varphi(a) = \varphi(b)\}. \tag{33}$$

Generalization of Burris and Sankappanavar [30].

Theorem 19. *Let $(\mathcal{R}, f, g)$ and $(\mathcal{S}, f', g')$ be (m,n)-semirings and $\varphi : \mathcal{R} \to \mathcal{S}$ a homomorphism. Then, $\ker\varphi$ is a congruence relation on $\mathcal{R}$, and there exists a unique one-to-one homomorphism ψ from $\mathcal{R}/\ker\varphi$ into $\mathcal{S}$.*

Proof. Omitted as obvious. □

Corollary 20. *Let $(\mathcal{R}, f, g)$ be an (m,n)-semiring and ρ and σ congruence relations on $\mathcal{R}$, with $\rho \subseteq \sigma$. Then, $\sigma/\rho = \{\rho(x), \rho(y) \mid (x,y) \in \sigma\}$ is a congruence relation on $\mathcal{R}/\rho$, and $(\mathcal{R}/\rho)/(\sigma/\rho) \cong \mathcal{R}/\sigma$.*

Lemma 21. *Let $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \in \mathcal{R}$. Then,*

(i) $\underbrace{f(f(\ldots f(f}_{m}(x_1, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}), x_2, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}), \ldots), x_m, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) = f(x_1, x_2, \ldots, x_m)$,

(ii) $\underbrace{g(g(\ldots g(g}_{n}(y_1, \underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-1}), y_2, \underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}), \ldots), y_n, \underbrace{\mathbf{1},\ldots,\mathbf{1}}_{n-2}) = g(y_1, y_2, \ldots, y_n)$.

Proof. (i)

$$\underbrace{f(f(\ldots f(f}_{m}(x_1, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}), x_2, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}), \ldots), x_m, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}). \tag{34}$$

By associativity (Definition 2 (i)), (34) is equal to

$$\begin{aligned}
&\underbrace{f(f(\ldots f(f}_{m}(\mathbf{0}, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}), x_1, x_2, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-3}), \ldots), x_m, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \\
&= \underbrace{f(f(\ldots f(f}_{m-1}(\mathbf{0}, x_1, x_2, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-3}), x_3, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}), \ldots), \\
&\qquad x_m, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \\
&= \underbrace{f(f(\ldots f(f}_{m-1}(\underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m}), x_1, x_2, x_3, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-4}), \ldots), \\
&\qquad x_m, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \\
&= \underbrace{f(f(\ldots f(f}_{m-2}(\mathbf{0}, x_1, x_2, x_3, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-4}), x_4, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}), \ldots), \\
&\qquad x_m, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \\
&\qquad \vdots \\
&= f(f(x_1, x_2, \ldots, x_{m-1}, \mathbf{0}), x_m, \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-2}) \\
&= f(f(x_1, x_2, \ldots, x_{m-1}, x_m), \underbrace{\mathbf{0},\ldots,\mathbf{0}}_{m-1}) \\
&= f(x_1, x_2, \ldots, x_m).
\end{aligned} \tag{35}$$

(ii) Similar to part (i). □

4. Partial Ordering on Fault Tolerance

In this sections we use x_i, y_i, and so forth, where $i \in \mathbb{Z}_+$ to denote individual system components that are assumed to be *atomic* at the level of discussion; that is, they have no components or subsystems of their own. We use *component* to refer to such an atomic part of a system, and *subsystem* to refer to a part of a system that is not necessarily atomic. We assume that components and subsystems are disjoint, in the sense that if they fail, they fail independently and do not affect the functioning of other components.

Let $\mathcal{U}$ be a universal set of all systems in the domain of discourse as given by Rao [15], and let f be a mapping $f : \mathcal{U}^m \to \mathcal{U}$, that is, f is an m-ary operation. Likewise, let g be an n-ary operation.

Definition 22. We define f and g operations for systems as follows.

(i) f is an m-ary operation which applies on systems made up of m components or subsystems, where if any one of the components or subsystems fails, then the whole system fails.

If a system made up of m components $x_1, x_2, \ldots, x_m$, then, the system over operation f is represented as $f(x_1, x_2, \ldots, x_m)$ for all $x_1, x_2, \ldots, x_m \in \mathcal{U}$. The system $f(x_1, x_2, \ldots, x_m)$ fails when any of the components $x_1, x_2, \ldots, x_m$ fails.

(ii) g is an n-ary operation which applies on a system consisting of n components or subsystems, which fails if all the components or subsystems fail; otherwise it continues working even if a single component or subsystem is working properly.

Let a system consist of n components $x_1, x_2, \ldots, x_n$, then, the system over operation g is represented as $g(x_1, x_2, \ldots, x_n)$ for all $x_1, x_2, \ldots, x_n \in \mathcal{U}$. The system $g(x_1, x_2, \ldots, x_n)$ fails when all the components $x_1, x_2, \ldots, x_n$ fail.

Consider a partial ordering relation $\preccurlyeq$ on $\mathcal{U}$, such that $(\mathcal{U}, \preccurlyeq)$ is a partially ordered set (poset). This is a *fault-tolerance partial ordering* where $f(x_1^m) \preccurlyeq f(y_1^m)$ means that $f(x_1^m)$ has a lower measure of some fault metric than $f(y_1^m)$ and $f(x_1^m)$ has a better fault tolerance than $f(y_1^m)$, for all $f(x_1^m), f(y_1^m) \in \mathcal{U}$ (see Rao [32] for more details) and $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m$ are disjoint components.

Assume that $\mathbf{0}$ represents the atomic system "which is always up" and $\mathbf{1}$ represents the system "which is always down" (see Rao [32]).

Observation 23. We observe the following for all disjoint components $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$, which are in $\mathcal{U}$.

(i) $g(y_1^{j-1}, \mathbf{0}, y_{j+1}^n) = \mathbf{0}$ for all $1 \le j \le n$.

This is so since $\mathbf{0}$ represents the component or system which never fails, and as per the definition of g, the system as a whole fails if all the components fail, and otherwise it continues working even if a single component is working properly. In a system $g(y_1^{j-1}, \mathbf{0}, y_{j+1}^n)$, even if all other components y_1^{j-1} and y_{j+1}^n fail even then $\mathbf{0}$ is up and the system is always up.

(ii) $f(x_1^{i-1}, \mathbf{1}, x_{i+1}^m) = \mathbf{1}$ for all $1 \le i \le m$.

This is so since $\mathbf{1}$ represents the component or system which is always down, and as per the definition of f if either of the component fails, then the whole system fails. Thus, even though all other components are working properly but due to the component $\mathbf{1}$ the system is always down.

Definition 24. If $(\mathcal{U}, f, g)$ is an (m, n)-semiring and $(\mathcal{U}, \preccurlyeq)$ is a poset, then $(\mathcal{U}, f, g, \preccurlyeq)$ is a *partially ordered* (m, n)*-semiring* if the following conditions are satisfied for all $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n, a, b \in \mathcal{U}$ and $1 \le i \le m$, $1 \le j \le n$.

(i) If $a \preccurlyeq b$, then $f(x_1^{i-1}, a, x_{i+1}^m) \preccurlyeq f(x_1^{i-1}, b, x_{i+1}^m)$.

(ii) If $a \preccurlyeq b$, then $g(y_1^{j-1}, a, y_{j+1}^n) \preccurlyeq g(y_1^{j-1}, b, y_{j+1}^n)$.

Remark 25. As it is assumed that $\mathbf{0}$ is the system which is always up, it is more fault tolerant than any of the other systems or components. Therefore $\mathbf{0} \preccurlyeq a$, for all $a \in \mathcal{U}$. Similarly, $a \preccurlyeq \mathbf{1}$ because $\mathbf{1}$ is the system that always fails, and therefore, it is the least fault tolerant; every other system is more fault tolerant than it.

Observation 26. The following are obtained for all disjoint components r, s, x_i, y_j, a_i, b_j, which are in $\mathcal{U}$, where $1 \le i \le m$, $1 \le j \le n$.

(i) $\mathbf{0} \preccurlyeq f(x_1^{i-1}, r, x_{i+1}^m) \preccurlyeq \mathbf{1}$.

(ii) $\mathbf{0} \preccurlyeq g(y_1^{j-1}, s, y_{j+1}^n) \preccurlyeq \mathbf{1}$.

(iii) $\mathbf{0} \preccurlyeq g(y_1^{j-1}, f(a_1^m), y_{j+1}^n) \preccurlyeq \mathbf{1}$.

(iv) $\mathbf{0} \preccurlyeq f(x_1^{i-1}, g(b_1^n), x_{i+1}^m) \preccurlyeq \mathbf{1}$.

From the above description of $\mathbf{0}$ and $\mathbf{1}$, the observation is quite obvious. Case (i) shows that $\mathbf{0}$ is less faulty than $f(x_1^{i-1}, r, x_{i+1}^m)$, and $f(x_1^{i-1}, r, x_{i+1}^m)$ is less faulty than $\mathbf{1}$. Similarly, case (ii) shows that $\mathbf{0}$ is more fault tolerant than $g(y_1^{j-1}, s, y_{j+1}^n)$ and $g(y_1^{j-1}, s, y_{j+1}^n)$ is more fault tolerant than $\mathbf{1}$. Likewise, case (iii) shows the operation g over y_1^{j-1}, y_{j+1}^n and f of a_1^m to be less faulty than $\mathbf{1}$ and more faulty than $\mathbf{0}$, and a similar interpretation is made for (iv).

Lemma 27. *If $\preccurlyeq$ is a fault-tolerance partial order and $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m$, $z_1, z_2, \ldots, z_n$, $u_1, u_2, \ldots, u_n$ are disjoint components, which are in $\mathcal{U}$, where $m, n \in \mathbb{Z}_+$, then for all $1 \le i \le m$ and $1 \le j \le n$ the following holds true:*

(i) *if $x_i \preccurlyeq y_i$, then $f(x_1^m) \preccurlyeq f(y_1^m)$,*

(ii) *if $z_j \preccurlyeq u_j$, then $g(z_1^n) \preccurlyeq g(u_1^n)$.*

Proof. (i) Since $x_i \preccurlyeq y_i$ for all $1 \le i \le m$, we have

$$x_1 \preccurlyeq y_1, \tag{36}$$

which is represented as follows:

$$f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, x_1) \preccurlyeq f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, y_1), \tag{37}$$

$$f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, x_2) \preccurlyeq f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, y_2). \tag{38}$$

By f operation on both sides of (37) with y_2, we get

$$\begin{aligned} & f(f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, x_1), y_2, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \\ & \quad \preccurlyeq f(f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, y_1), y_2, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}). \end{aligned} \tag{39}$$

By f operation on both sides of (38) with x_1

$$\begin{aligned} & f(f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, x_2), x_1, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \\ & \quad \preccurlyeq f(f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, y_2), x_1, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}). \end{aligned} \tag{40}$$

From (39) and (40), we get

$$\begin{aligned} & f(f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, x_1), y_2, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \\ & \quad \preccurlyeq f(f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, y_1), y_2, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}). \end{aligned} \tag{41}$$

Similarly, we find for m terms

$$\begin{aligned} & \underbrace{f(\ldots(f(f}_{m}(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, x_1), x_2, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}), \ldots), x_m, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \\ & \quad \preccurlyeq \underbrace{f(\ldots(f(f}_{m}(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-1}, y_1), y_2, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}), \ldots), y_m, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}). \end{aligned} \tag{42}$$

From Lemma 21, (42) may be represented as

$$f(x_1, x_2, \ldots, x_m) \preccurlyeq f(y_1, y_2, \ldots, y_m) \tag{43}$$

so

$$f(x_1^m) \preccurlyeq f(y_1^m). \tag{44}$$

(ii) Since $z_j \preccurlyeq y_j$, for all $1 \le j \le n$

$$\begin{aligned} & g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-1}, z_1) \preccurlyeq g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-1}, u_1) \\ & g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-1}, z_2) \preccurlyeq g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-1}, u_2). \end{aligned} \tag{45}$$

After following similar steps as seen in part (i), we use the g operation for n terms,

$$\underbrace{g(\dots(g(g}_{n}(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-1},z_1),z_2,\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-2}),\dots),z_n,\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-2}) \preccurlyeq \underbrace{g(\dots(g(g}_{n}(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-1},u_1),u_2,\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-2}),\dots),u_n,\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-2}), \quad (46)$$

which is represented as

$$g(z_1,z_2,\dots,z_n) \preccurlyeq g(u_1,u_2,\dots,u_n), \quad (47)$$

and so

$$g(z_1^n) \preccurlyeq g(u_1^n). \quad (48)$$

□

Theorem 28. *If $\preccurlyeq$ is a fault-tolerance partial order and given disjoint components a_i, c_j, b_i, d_j in $\mathcal{U}$, where $1 \le i \le m$, $1 \le j \le n$ and $1 \le k \le m$, the following obtain.*

(i) *If $a_i \preccurlyeq b_i$, then*

$$g(y_1^{j-1}, f(a_1^m), y_{j+1}^n) \preccurlyeq g(y_1^{j-1}, f(b_1^m), y_{j+1}^n) \quad \forall y_1,y_2,\dots,y_n \in \mathcal{U}. \quad (49)$$

(ii) *If $c_j \preccurlyeq d_j$, then*

$$f(x_1^{k-1}, g(c_1^n), x_{k+1}^m) \preccurlyeq f(x_1^{k-1}, g(d_1^n), x_{k+1}^m) \quad \forall x_1,x_2,\dots,x_m \in \mathcal{U}. \quad (50)$$

Proof. (i) Since $a_i \preccurlyeq b_i$, for all $1 \le i \le m$.

Therefore, from Lemma 27 (i)

$$f(a_1^m) \preccurlyeq f(b_1^m), \quad \forall a_1,a_2,\dots,a_m,b_1,b_2,\dots,b_m \in \mathcal{U}. \quad (51)$$

From Definition 24 of a partially ordered (m,n)-semiring, we deduce that

$$g(y_1^{j-1}, f(a_1^m), y_{j+1}^n) \preccurlyeq g(y_1^{j-1}, f(b_1^m), y_{j+1}^n), \quad (52)$$

for all $1 \le j \le n$.

(ii) Since $c_j \preccurlyeq d_j$, for all $1 \le j \le n$, from Lemma 27 (ii), we find that

$$g(c_1^n) \preccurlyeq g(d_1^n), \quad \forall c_1,c_2,\dots,c_n,d_1,d_2,\dots,d_n \in \mathcal{U}. \quad (53)$$

From Definition 24 of a partially ordered (m,n)-semiring, we deduce that

$$f(x_1^{k-1}, g(c_1^n), x_{k+1}^m) \preccurlyeq f(x_1^{k-1}, g(d_1^n), x_{k+1}^m), \quad (54)$$

for all $1 \le k \le m$. □

Lemma 29. *If $\preccurlyeq$ is a fault-tolerance partial order and x_i, y_j are disjoint components which are in $\mathcal{U}$, where $1 \le i \le m$ and $1 \le j \le n$, one gets the following:*

(i) $x_i \preccurlyeq f(x_1,x_2,\dots,x_m)$,

(ii) $g(y_1,y_2,\dots,y_n) \preccurlyeq y_j$.

Proof. (i) As

$$\mathbf{0} \preccurlyeq x_1, \quad (55)$$

by f operation on both sides of (55) with x_i, we get

$$f(\mathbf{0},x_i,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}) \preccurlyeq f(x_1,x_i,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}). \quad (56)$$

Therefore,

$$x_i \preccurlyeq f(x_1,x_i,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}). \quad (57)$$

Similarly, we obtain

$$x_i \preccurlyeq f(x_1,x_i,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}) \preccurlyeq \cdots \preccurlyeq f(x_1,x_2,x_i,\dots,x_{m-1},\mathbf{0}) \preccurlyeq f(x_1,x_2,\dots,x_m). \quad (58)$$

Hence,

$$x_i \preccurlyeq f(x_1,x_2,\dots,x_m), \quad (59)$$

for all $1 \le i \le m$.

(ii) As

$$y_1 \preccurlyeq \mathbf{1}, \quad (60)$$

by g operation on both sides of (60) with y_j, we get

$$g(y_1,y_j,\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-2}) \preccurlyeq y_j. \quad (61)$$

Similarly, we obtain

$$g(y_1,y_2,\dots,y_n) \preccurlyeq g(y_1,y_2,y_j,\dots,y_{n-1},\mathbf{1}) \preccurlyeq \cdots \preccurlyeq g(y_1,y_j,\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-2}) \preccurlyeq y_j. \quad (62)$$

Hence,

$$g(y_1,y_2,\dots,y_n) \preccurlyeq y_j, \quad (63)$$

for all $1 \le j \le n$. □

Corollary 30. *If $\preccurlyeq$ is a fault-tolerance partial order, then the following hold for all disjoint components x_i, y_j which are elements of $\mathcal{U}$, where $1 \le i \le m$, $1 \le j \le n$ and $k, t \in \mathbb{Z}_+$:*

(i) $f(x_1,x_2,\dots,x_k,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-k}) \preccurlyeq f(x_1^m)$, *where $k < m$,*

(ii) $g(y_1^n) \preccurlyeq g(y_1,y_2,\dots,y_t,\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-t})$, *where $t < n$.*

Proof. (i) From (58), we deduce that

$$f(x_1,\dots,x_k,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-k}) \preccurlyeq f(x_1,\dots,x_{k+1},\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-k-1}) \preccurlyeq \cdots \preccurlyeq f(x_1,x_2,\dots,x_m). \tag{64}$$

Therefore,

$$f(x_1,\dots,x_k,\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-k}) \preccurlyeq f(x_1^m). \tag{65}$$

(ii) As in part (i), we deduce from (62) that

$$g(y_1^n) \preccurlyeq g(y_1,y_2,\dots,y_t,\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-t}). \tag{66}$$

□

$f(\overset{(m)}{f}(a_1^m))$ represents the system which is obtained after applying the f operation on m repeated $f(a_1^m)$ systems or subsystems. Similarly, $g(\overset{(n)}{g}(b_1^n))$ represents the system which is obtained after applying the g operation on n repeated $g(b_1^n)$ systems or subsystems.

Theorem 31. *If $\preccurlyeq$ is a fault-tolerance partial order, and components $x_1,x_2,\dots,x_m$, $y_1,y_2,\dots,y_n$ are disjoint components and are in $\mathcal{U}$, then*

(i) $f(x_1^m) \preccurlyeq f(\overset{(m)}{f}(x_1^m))$,

(ii) $g(\overset{(n)}{g}(y_1^n)) \preccurlyeq g(y_1^n)$.

Corollary 32. *The following hold for all disjoint components $x_1,\dots,x_m$, $z_1,\dots,z_n$, $y_1,\dots,y_m$, $u_1,\dots,u_n$, which are elements of $\mathcal{U}$, where $m,n\in\mathbb{Z}_+$.*

(i) *If $f(\overset{(m)}{f}(x_1^m)) \preccurlyeq f(y_1^m)$, then*

$$f(x_1^m) \preccurlyeq f(y_1^m). \tag{67}$$

(ii) *If $g(z_1^n) \preccurlyeq g(\overset{(n)}{g}(u_1^n))$, then*

$$g(z_1^n) \preccurlyeq g(u_1^n). \tag{68}$$

Proof. (i) $f(\overset{(m)}{f}(x_1^m)) \preccurlyeq f(y_1^m)$ and from Theorem 31, $f(x_1^m) \preccurlyeq f(\overset{(m)}{f}(x_1^m))$. Therefore, $f(x_1^m) \preccurlyeq f(y_1^m)$.

(ii) The proof is very similar to that of part (i). □

Corollary 33. *Let k and t be positive integers and $k < m$, $t < n$. Given disjoint components $x_1,\dots,x_m$, $y_1,y_2,\dots,y_m$, $z_1,z_2,\dots,z_n$, $u_1,u_2,\dots,u_n$ that are in $\mathcal{U}$, the following hold:*

(i) *If $f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-k},\overset{(k)}{f}(x_1^m)) \preccurlyeq f(y_1^m)$, then $f(x_1^m) \preccurlyeq f(y_1^m)$.*

(ii) *If $g(z_1^n) \preccurlyeq g(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-t},\overset{(t)}{g}(u_1^n))$, then $g(z_1^n) \preccurlyeq g(u_1^n)$.*

Proof. Similar to Corollary 32. □

Theorem 34. *Let $\preccurlyeq$ be a fault-tolerance partial order and $x_i \preccurlyeq y_i$ and $z_j \preccurlyeq u_j$ for all $x_i,y_i,z_j,u_j\in\mathcal{U}$, where $1\le i\le m$ and $1\le j\le n$. Then, the following obtain:*

(i) $f(\overset{(m)}{f}(x_1^m)) \preccurlyeq f(\overset{(m)}{f}(y_1^m))$,

(ii) $g(\overset{(n)}{g}(z_1^n)) \preccurlyeq g(\overset{(n)}{g}(u_1^n))$,

(iii) $f(\overset{(m)}{g}(z_1^n)) \preccurlyeq f(\overset{(m)}{g}(u_1^n))$,

(iv) $g(\overset{(n)}{f}(x_1^m)) \preccurlyeq g(\overset{(n)}{f}(y_1^m))$.

Proof. (i) As

$$x_i \preccurlyeq y_i, \quad 1\le i\le m, \tag{69}$$

from Lemma 27 (i), we get

$$f(x_1^m) \preccurlyeq f(y_1^m). \tag{70}$$

This is written as

$$f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-1},f(x_1^m)) \preccurlyeq f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-1},f(y_1^m)). \tag{71}$$

So by f operation on both sides of (71) with $f(x_1^m)$, we get

$$f(f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-1},f(x_1^m)),f(x_1^m),\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}) \preccurlyeq f(f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-1},f(x_1^m)),f(y_1^m),\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}). \tag{72}$$

So by f operation on both sides of (71) with $f(y_1^m)$, we get

$$f(f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-1},f(x_1^m)),f(y_1^m),\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2}) \preccurlyeq f(f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-1},f(y_1^m)),\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2},f(y_1^m)). \tag{73}$$

From (72) and (73), we get

$$f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2},\overset{(2)}{f}(x_1^m)) \preccurlyeq f(\underbrace{\mathbf{0},\dots,\mathbf{0}}_{m-2},\overset{(2)}{f}(y_1^m)). \tag{74}$$

Similarly, we get for m terms

$$f(\overset{(m)}{f}(x_1^m)) \preccurlyeq f(\overset{(m)}{f}(y_1^m)). \tag{75}$$

(ii) We know that

$$z_j \preccurlyeq u_j, \quad 1\le j\le n. \tag{76}$$

From Lemma 27 (ii), we get

$$g(z_1^n) \preccurlyeq g(u_1^n). \tag{77}$$

Which is represented as follows

$$g(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-1},g(z_1^n)) \preccurlyeq g(\underbrace{\mathbf{1},\dots,\mathbf{1}}_{n-1},g(u_1^n)). \tag{78}$$

Now by g operation on both sides of (78) with $g(z_1^n)$, we get

$$g(\overset{(2)}{g}(z_1^n), \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}) \preccurlyeq g(g(z_1^n), g(u_1^n), \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}). \quad (79)$$

So by g operation on both sides of (78) with $g(u_1^n)$, we get

$$g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, g(z_1^n), g(u_1^n)) \preccurlyeq g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, \overset{(2)}{g}(u_1^n)). \quad (80)$$

So now from (79) and (80), we get

$$g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, \overset{(2)}{g}(z_1^n)) \preccurlyeq g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-2}, \overset{(2)}{f}(u_1^n)). \quad (81)$$

Similarly, we find for n terms

$$g\,(\overset{(n)}{g}(z_1^n)) \preccurlyeq g\,(\overset{(n)}{g}(u_1^n)). \quad (82)$$

(iii) From Lemma 27 (ii)

$$g(z_1^n) \preccurlyeq g(u_1^n). \quad (83)$$

Similar to part (i), we find f operation of m terms and get

$$f(\underbrace{g(z_1^n), g(z_1^n), \ldots, g(z_1^n)}_{m}) \preccurlyeq f(\underbrace{g(u_1^n), g(u_1^n), \ldots, g(u_1^n)}_{m}), \quad (84)$$

$$f\,(\overset{(m)}{g}(z_1^n)) \preccurlyeq f\,(\overset{(m)}{g}(u_1^n)).$$

(iv) We know that

$$x_i \preccurlyeq y_i, \quad 1 \le i \le m, \quad (85)$$

so from Lemma 27 (i), we get

$$f(x_1^m) \preccurlyeq f(y_1^m). \quad (86)$$

As proved in part (ii), we find g operations of n terms and get

$$g(\underbrace{f(x_1^m), f(x_1^m), \ldots, f(x_1^m)}_{n}) \preccurlyeq g(\underbrace{f(y_1^m), f(y_1^m), \ldots, f(y_1^m)}_{n}). \quad (87)$$

Thus, we get

$$g\,(\overset{(n)}{f}(x_1^m)) \preccurlyeq g\,(\overset{(n)}{f}(y_1^m)). \quad (88)$$

□

Corollary 35. *If $\preccurlyeq$ is a fault-tolerance partial order and $k < m$, $t < n$ where $k, t \in \mathbb{Z}_+$, if $x_i \preccurlyeq y_i$, $z_j \preccurlyeq u_j$ for all disjoint components x_i, z_j, y_i, u_j, which are in $\mathcal{U}$, where $1 \le i \le m$ and $1 \le j \le n$, then*

(i) $f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-k}, \overset{(k)}{f}(x_1^m)) \preccurlyeq f(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-k}, \overset{(k)}{f}(y_1^m))$,

(ii) $g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-t}\ \overset{(t)}{g}(z_1^n)) \preccurlyeq g(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-t}, \overset{(t)}{g}(u_1^n))$.

Proof. (i) Proof is similar to that of Theorem 34 (i). We find the f operation of k terms where $\forall k \in \mathbb{Z}_+$, and $k < m$.

(ii) Proof is similar to that of Theorem 34 (ii). We find the g operation of t terms where $\forall t \in \mathbb{Z}_+$, and $t < n$. □

We propose the following theorem for very complex systems.

Theorem 36. *If $\preccurlyeq$ is a fault-tolerance partial order, disjoint components a_i, b_i, c_j, d_j, x_k, y_k, z_t, u_t are in $\mathcal{U}$ and $a_i \preccurlyeq b_i$, $c_j \preccurlyeq d_j$, $x_k \preccurlyeq y_k$ and $z_t \preccurlyeq u_t$, where $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le m$ and $1 \le t \le n$, then*

(i) $f(x_1^{k-1}, f(a_1^m), x_{k+1}^m) \preccurlyeq f(y_1^{k-1}, f(b_1^m), y_{k+1}^m)$, *for all* $1 \le k \le m$;

(ii) $f(x_1^{k-1}, g(c_1^n), x_{k+1}^m) \preccurlyeq f(y_1^{k-1}, g(d_1^n), y_{k+1}^m)$, *for all* $1 \le k \le m$;

(iii) $g(z_1^{t-1}, f(a_1^m), z_{t+1}^n) \preccurlyeq g(u_1^{t-1}, f(b_1^m), u_{t+1}^n)$, *for all* $1 \le t \le n$; *and*

(iv) $g(z_1^{t-1}, g(c_1^n), z_{t+1}^n) \preccurlyeq g(u_1^{t-1}, g(d_1^n), u_{t+1}^n)$, *for all* $1 \le t \le n$.

Proof. (i) From Lemma 27 (i), if $a_i \preccurlyeq b_i$, then $f(a_1^m) \preccurlyeq f(b_1^m)$ for all $1 \le i \le m$.

We prove in a similar manner as Lemma 27 (i) that

$$f(f(a_1^m), x_1, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}) \preccurlyeq f(f(b_1^m), y_1, \underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{m-2}). \quad (89)$$

Similarly, we get

$$f\left(f\left(a_1^m\right), x_1^{k-1}, x_{k+1}^m\right) \preccurlyeq f\left(f\left(b_1^m\right), y_1^{k-1}, y_{k+1}^m\right). \quad (90)$$

Thus,

$$f\left(x_1^{k-1}, f\left(a_1^m\right), x_{k+1}^m\right) \preccurlyeq f\left(y_1^{k-1}, f\left(b_1^m\right), y_{k+1}^m\right). \quad (91)$$

Similar to the above, we can prove (ii), (iii), and (iv). □

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