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Research Article

Generalized Hyers-Ulam Stability of a Mixed Type Functional Equation

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We investigate the stability of a functional equation $f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z)=3f(x)+f(-x)+3f(y)+f(-y)+3f(z)+f(-z)$ by applying the direct method in the sense of Hyers and Ulam.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The Ulam's problem for the Cauchy additive functional equation was solved by Hyers under the assumption that G_1 and G_2 are Banach spaces. Indeed, Hyers [2] proved that every solution of the inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ (for all x and y) can be approximated by an additive function. In this case, the Cauchy additive functional equation, $f(x+y)=f(x)+f(y)$, is said to satisfy the Hyers-Ulam stability.

Thereafter, Rassias [3] attempted to weaken the condition for the bound of norm of the Cauchy difference as follows:

$$\|f(x+y)-f(x)-f(y)\| \le \varepsilon\left(\|x\|^p+\|y\|^p\right), \tag{1}$$

and he proved that Hyers' theorem is also true for this case. Indeed, Rassias proved the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability) of the Cauchy additive functional equation between Banach spaces. We here remark that a paper of Aoki [4] was published concerning the generalized Hyers-Ulam stability of the Cauchy functional equation earlier than Rassias' paper.

The stability concept that was introduced by Rassias' theorem provided a large influence to a number of mathematicians to develop the notion of what is known today with the term generalized Hyers-Ulam stability of functional equations. Since then, the stability problems of several functional equations have been extensively investigated by several mathematicians (e.g., see [5–10] and the references therein).

Almost all subsequent proofs in this very active area have used the Hyers' method presented in [2]. Namely, starting from the given mapping f that approximately satisfies a given functional equation, a solution F of the functional equation is explicitly constructed by using the formula:

$$F(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x) \quad \text{or} \quad F(x) := \lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right), \tag{2}$$

which approximates the mapping f. This method of Hyers is called the direct method.

We remark that another method for proving the Hyers-Ulam stability of various functional equations was introduced by Baker [11], which is called the fixed-point method.

This method is very powerful technique of proving the stability of functional equations (see [12, 13]).

Now we consider the following functional equation:

$$\begin{aligned} f(x+y+z) &+ f(x-y+z) \\ &+ f(x+y-z) + f(-x+y+z) \\ &= 3f(x) + f(-x) + 3f(y) \\ &+ f(-y) + 3f(z) + f(-z), \end{aligned} \tag{3}$$

which is called the mixed type functional equation. The mapping $f(x) = ax^2 + bx$ is a solution of this functional equation, where a, b are real constants. Every solution of (3) will be called a quadratic-additive mapping.

In 1998, Jung [14] proved the stability of (3) by decomposing f into the odd and even parts. In his proof, using the direct method, an additive mapping A and a quadratic mapping Q are separately constructed from the odd and even parts of f, and then A and Q are combined to provide a quadratic-additive mapping F which is close to f.

In this paper, we will prove the generalized Hyers-Ulam stability of (3) by making use of the direct method. In particular, we will approximate the given mapping f by a solution F of (3) without decomposing f into its odd and even parts, while in the Jung's paper [14] the mapping f was decomposed into the odd and even parts, and each of them was separately approximated by the corresponding part of a solution F of (3).

2. Main Results

Throughout this paper, let X be a (real or complex) normed space and Y a Banach space. For an arbitrary $p \in \mathbb{R}$, we define $s := \operatorname{sign}\ (2-p)$ and $t := \operatorname{sign}\ (1-p)$.

For a given mapping $f : X \to Y$, we use the following abbreviations:

$$\begin{aligned} f_o(x) &:= \frac{f(x) - f(-x)}{2}, \\ f_e(x) &:= \frac{f(x) + f(-x)}{2}, \\ J_n f(x) &:= \frac{9^{-sn}}{2}\left(f(3^{sn}x) + f(-3^{sn}x)\right) \\ &\quad + \frac{3^{-tn}}{2}\left(f\left(3^{tn}x\right) - f\left(-3^{tn}x\right)\right), \\ Af(x,y) &:= f(x+y) - f(x) - f(y), \\ Qf(x,y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y), \\ Df(x,y,z) &:= f(x+y+z) + f(x-y+z) \\ &\quad + f(x+y-z) + f(-x+y+z) \\ &\quad - 3f(x) - f(-x) - 3f(y) \\ &\quad - f(-y) - 3f(z) - f(-z) \end{aligned} \tag{4}$$

for all $x, y, z \in X$.

As we stated in the previous section, f is called a quadratic-additive mapping provided that f satisfies the functional equation $Df(x, y, z) = 0$ for all $x, y, z \in X$.

Proposition 1. *A mapping $f : X \to Y$ is a solution of (3) if and only if f_e is a quadratic mapping and f_o is an additive mapping.*

Proof. Assume that $f : X \to Y$ is a solution of (3). Then we have

$$\begin{aligned} Qf_e(x,y,z) &= \frac{Df_e(x,y,0)}{2} = 0, \\ Af_o(x,y,z) &= \frac{Df_o(x,y,0)}{2} = 0 \end{aligned} \tag{5}$$

for all $x, y, z \in X$, that is, f_e is a quadratic mapping and f_o is an additive mapping.

Conversely, assume that f_e is a quadratic mapping and f_o is an additive mapping. Then we get

$$\begin{aligned} Df(x,y,z) &= Df_e(x,y,z) + Df_o(x,y,z) \\ &= Qf_e(x+y,z) + Qf_e(x-y,z) \\ &\quad + 2Qf_e(x,y) + Af_o(x+y,z) \\ &\quad + Af_o(x+y,-z) + Af_o(x-y,z) \\ &\quad + Af_o(-x+y,z) + 2Af_o(x,y) \\ &= 0 \end{aligned} \tag{6}$$

for all $x, y, z \in X$; that is, f is a solution of (3). □

We first prove the following lemma.

Lemma 2. *If a mapping $f : X \to Y$ satisfies $Df(x, y, z) = 0$ for all $x, y, z \in X \setminus \{0\}$ and $f(0) = 0$, then f is a quadratic-additive mapping.*

Proof. Using the hypothesis, we have

$$\begin{aligned} &f(2x) - 3f(x) - f(-x) \\ &\quad = \frac{11}{112}\left(Df(4x,3x,x) - Df(4x,2x,2x)\right. \\ &\qquad - Df(2x,2x,2x) + 2Df(2x,x,x) \\ &\qquad \left. + 3Df(x,x,x) + Df(-x,-x,-x)\right) \\ &\quad - \frac{3}{112}\left(Df(-4x,-3x,-x) - Df(-4x,-2x,-2x)\right. \\ &\qquad - Df(-2x,-2x,-2x) + 2Df(-2x,-x,-x) \\ &\qquad \left. + 3Df(-x,-x,-x) + Df(x,x,x)\right) \\ &\quad = 0 \end{aligned} \tag{7}$$

for all $x \in X \setminus \{0\}$. Furthermore, by the last equality, we get

$$\begin{aligned} Df(x,y,0) &= Df\left(x,\frac{y}{2},\frac{y}{2}\right) + Df\left(y,\frac{x}{2},\frac{x}{2}\right) \\ &\quad - 2f(x) + 6f\left(\frac{x}{2}\right) + 2f\left(-\frac{x}{2}\right) \\ &\quad - 2f(y) + 6f\left(\frac{y}{2}\right) + 2f\left(-\frac{y}{2}\right) \\ &= 0 \end{aligned} \tag{8}$$

for all $x, y \in X\setminus\{0\}$. Since $Df(x,y,z)$ is invariant with respect to the permutation of (x,y,z), it holds that $Df(0,y,z) = Df(x,0,z) = 0$ for all $x,y,z \in X \setminus \{0\}$. It is also easy to show that $Df(x,0,0) = 0$, $Df(0,y,0) = 0$, $Df(0,0,z) = 0$, and $Df(0,0,0) = 0$ for all $x,y,z \in X \setminus \{0\}$ as we desired. □

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation (3) by making use of the direct method.

Theorem 3. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|Df(x,y,z)\| \le \|x\|^p + \|y\|^p + \|z\|^p \tag{9}$$

for all $x,y,z \in X \setminus \{0\}$ with a real constant $p \notin \{1,2\}$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \dfrac{3\|x\|^p}{3^p - 9} & (\mathit{for}\ p > 2), \\ \left(\dfrac{3}{9-3^p} + \dfrac{3}{3^p - 3}\right)\|x\|^p & (\mathit{for}\ 1 < p < 2), \\ \dfrac{3\|x\|^p}{3 - 3^p} & (\mathit{for}\ 0 \le p < 1) \end{cases} \tag{10}$$

for all $x \in X \setminus \{0\}$. Moreover, if $p < 0$, then f itself is a quadratic-additive mapping.

Proof. Let us define $\tau_{s,n} := s(n+1/2) - 1/2$, where $s \in \{-1,1\}$. From the definitions of $J_n f(x)$ and $Df(x,y,z)$, we have

$$\begin{aligned} J_n f(x) - J_{n+1} f(x) &= -\frac{1}{2}\left(9^{\tau_{-s,n}}\left(Df\left(3^{\tau_{s,n}}x, 3^{\tau_{s,n}}x, 3^{\tau_{s,n}}x\right)\right.\right. \\ &\qquad \left. + Df\left(-3^{\tau_{s,n}}x, -3^{\tau_{s,n}}x, -3^{\tau_{s,n}}x\right)\right) s \\ &\quad + 3^{\tau_{-t,n}}\left(Df\left(3^{\tau_{t,n}}x, 3^{\tau_{t,n}}x, 3^{\tau_{t,n}}x\right)\right. \\ &\qquad \left.\left. - Df\left(-3^{\tau_{t,n}}x, -3^{\tau_{t,n}}x, -3^{\tau_{t,n}}x\right)\right) t\right) \end{aligned} \tag{11}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}_0$. It follows from (9) and (11) that

$$\begin{aligned} &\|J_n f(x) - J_{n+m} f(x)\| \\ &= \sum_{j=n}^{n+m-1} \|J_j f(x) - J_{j+1} f(x)\| \\ &\le \frac{1}{2}\sum_{j=n}^{n+m-1}\left(\left\|9^{\tau_{-s,j}} Df\left(3^{\tau_{s,j}}x, 3^{\tau_{s,j}}x, 3^{\tau_{s,j}}x\right)s\right.\right. \\ &\qquad \left. + 3^{\tau_{-t,j}} Df\left(3^{\tau_{t,j}}x, 3^{\tau_{t,j}}x, 3^{\tau_{t,j}}x\right)t\right\| \\ &\qquad + \left\|9^{\tau_{-s,j}} Df\left(-3^{\tau_{s,j}}x, -3^{\tau_{s,j}}x, -3^{\tau_{s,j}}x\right)s\right. \\ &\qquad \left.\left. - 3^{\tau_{-t,j}} Df\left(-3^{\tau_{t,j}}x, -3^{\tau_{t,j}}x, -3^{\tau_{t,j}}x\right)t\right\|\right) \\ &\le \begin{cases} \displaystyle\sum_{j=n}^{n+m-1} 3^{-j}\left\|3^j x\right\|^p & (\text{for } p < 1), \\ \displaystyle\sum_{j=n}^{n+m-1} \left(3^{-2j-1}\left\|3^j x\right\|^p + 3^{j+1}\left\|3^{-j-1}x\right\|^p\right) & (\text{for } 1 < p < 2), \\ \displaystyle\sum_{j=n}^{n+m-1} 3^{2j+1}\left\|3^{-j-1}x\right\|^p & (\text{for } p > 2) \end{cases} \\ &\le \begin{cases} \dfrac{3^{np}\|x\|^p}{3^{n-1}(3-3^p)} & (\text{for } p < 1), \\ \dfrac{3^{np}\|x\|^p}{3^{2n-1}(9-3^p)} + \dfrac{3^{n+1}\|x\|^p}{3^{np}(3^p-3)} & (\text{for } 1 < p < 2), \\ \dfrac{3^{2n+1}\|x\|^p}{3^{np}(3^p-9)} & (\text{for } p > 2) \end{cases} \end{aligned} \tag{12}$$

for all $x \in X \setminus \{0\}$. So, it is easy to show that the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in X \setminus \{0\}$.

Since Y is complete and $f(0) = 0$, the sequence $\{J_n f(x)\}$ converges for all $x \in X$. Hence, we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_n f(x) \tag{13}$$

for all $x \in X$. Moreover, if we put $n = 0$ and let $m \to \infty$ in (12), we obtain the inequality (10).

From the definition of F, we get

$$\begin{aligned} &DF(x,y,z) \\ &= \lim_{n\to\infty} \frac{9^{-sn}}{2}\left(Df\left(3^{sn}x, 3^{sn}y, 3^{sn}z\right)\right. \\ &\qquad \left. + Df\left(-3^{sn}x, -3^{sn}y, -3^{sn}z\right)\right) \\ &\quad + \lim_{n\to\infty} \frac{3^{-tn}}{2}\left(Df\left(3^{tn}x, 3^{tn}y, 3^{tn}z\right)\right. \\ &\qquad \left. - Df\left(-3^{tn}x, -3^{tn}y, -3^{tn}z\right)\right) \\ &\le \lim_{n\to\infty}\left(3^{-s(2-p)n} + 3^{-tn(1-p)}\right)\left(\|x\|^p + \|y\|^p + \|z\|^p\right) \\ &= 0 \end{aligned} \tag{14}$$

for all $x, y, z \in X \setminus \{0\}$. By Lemma 2, F is a quadratic-additive mapping.

Now, we will show that F is uniquely determined. Let $F' : X \to Y$ be another quadratic-additive mapping satisfying (10). It is easy to show that $F'(0) = 0$ for all quadratic-additive mapping F'. It follows from (11) that

$$\begin{aligned} F'(x) - J_n F'(x) &= \sum_{j=0}^{n-1} \left(J_j F'(x) - J_{j+1} F'(x)\right) \\ &= -\frac{1}{2} \sum_{j=0}^{n-1} \Big(9^{\tau_{-s,j}} \left(DF'\left(3^{\tau_{s,j}} x, 3^{\tau_{s,j}} x, 3^{\tau_{s,j}} x\right) + DF'\left(-3^{\tau_{s,j}} x, -3^{\tau_{s,j}} x, -3^{\tau_{s,j}} x\right)\right) s \\ &\quad + 3^{\tau_{-t,j}} \left(DF'\left(3^{\tau_{t,j}} x, 3^{\tau_{t,j}} x, 3^{\tau_{t,j}} x\right) - DF'\left(-3^{\tau_{t,j}} x, -3^{\tau_{t,j}} x, -3^{\tau_{t,j}} x\right)\right) t\Big) \\ &= 0 \end{aligned} \tag{15}$$

for all $n \in \mathbb{N}$ and $x \in X$. Since F and F' are quadratic-additive, if we replace x with $3^n x$ in (10), then we have

$$\begin{aligned} &\left\|F(x) - F'(x)\right\| \\ &= \left\|J_n F(x) - J_n F'(x)\right\| \\ &\le \frac{9^{-sn}}{2} \Big(\left\|(F - f)(3^{sn} x)\right\| + \left\|(f - F')(3^{sn} x)\right\| + \left\|(F - f)(-3^{sn} x)\right\| + \left\|(f - F')(-3^{sn} x)\right\|\Big) \\ &\quad + \frac{2^{-tn}}{2} \Big(\left\|(F - f)(3^{tn} x)\right\| + \left\|(F' - f)(3^{tn} x)\right\| + \left\|(F - f)(-3^{tn} x)\right\| + \left\|(F' - f)(-3^{tn} x)\right\|\Big) \\ &\le \left(\frac{6}{|9 - 3^p|} + \frac{6}{|3^p - 3|}\right) \left(3^{-sn(2-p)} + 3^{-tn(1-p)}\right) \|x\|^p \end{aligned} \tag{16}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F(x) = F'(x)$ for all $x \in X$, which proves the uniqueness of F.

Since

$$\begin{aligned} &\|f(x) - F(x)\| \\ &\le \|Df((2k-1)x, kx, kx) - DF((2k-1)x, kx, kx)\| \\ &\quad + \|F((4k-1)x) - f((4k-1)x)\| \\ &\quad + \|f((2k-1)x) - F((2k-1)x)\| \\ &\quad + \|f((1-2k)x) - F((1-2k)x)\| \\ &\quad + 2\|f(-kx) - F(-kx)\| \\ &\quad + 6\|f(kx) - F(kx)\| \\ &\le \left((2k-1)^p + 2k^p + \frac{3\left((4k-1)^p + 2(2k-1)^p + 8k^p\right)}{3 - 3^p}\right) \\ &\quad \times \|x\|^p \end{aligned} \tag{17}$$

for all $x \in X \setminus \{0\}$ and $k \in \mathbb{N}$, if $p < 0$, then we conclude that $f(x) = F(x)$ for all $x \in X \setminus \{0\}$ by letting $k \to \infty$ in the previous inequality. From the fact that $f(0) = 0$, f is a quadratic-additive mapping. □

Theorem 4. *If a mapping $f : X \to Y$ satisfies*

$$\|Df(x, y, z)\| \le \|x\|^p + \|y\|^p + \|z\|^p \tag{18}$$

for all $x, y, z \in X$ and for a nonnegative real constant $p \notin \{1, 2\}$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \dfrac{\|x\|^p}{2^p - 4} & (\mathit{for}\ p > 2), \\ \left(\dfrac{1}{4 - 2^p} + \dfrac{1}{2^p - 2}\right)\|x\|^p & (\mathit{for}\ 1 < p < 2), \\ \dfrac{\|x\|^p}{2 - 2^p} & (\mathit{for}\ 0 < p < 1), \\ \dfrac{25}{16} & (\mathit{for}\ p = 0) \end{cases} \tag{19}$$

for all $x \in X$.

Proof. Since

$$\|f(0)\| = \left\|\frac{1}{8} Df(0,0,0)\right\| \le \frac{3\|0\|^p}{8}, \tag{20}$$

we get $f(0) = 0$ for $p \notin \{0, 1, 2\}$ and $\|f(0)\| \le 3/8$ for $p = 0$. From the definitions of $J_n f(x)$ and $Df(x, y, z)$, we have

$$\begin{aligned} J_n f(x) - J_{n+1} f(x) &= -\frac{1}{4} \Big(4^{\tau_{-s,n}} \left(Df\left(2^{\tau_{s,n}} x, 2^{\tau_{s,n}} x, 0\right) + Df\left(-2^{\tau_{s,n}} x, -2^{\tau_{s,n}} x, 0\right)\right) s \\ &\quad + 2^{\tau_{-t,n}} \left(Df\left(2^{\tau_{t,n}} x, 2^{\tau_{t,n}} x, 0\right) - Df\left(-2^{\tau_{t,n}} x, -2^{\tau_{t,n}} x, 0\right)\right) t\Big) \\ &\quad + \frac{1}{2} 4^{\tau_{-s,n}} f(0) \end{aligned} \tag{21}$$

for all $x \in X$ and $n \in \mathbb{N}_0$, where $\tau_{s,n}$ is defined by $\tau_{s,n} = s(n + 1/2) - 1/2$ and $s \in \{-1, 1\}$.

It follows from (18) and (21) that

$$\begin{aligned}
&\left\|J_n f(x) - J_{n+m} f(x)\right\| \\
&= \sum_{j=n}^{n+m-1} \left\|J_j f(x) - J_{j+1} f(x)\right\| \\
&\leq \frac{1}{4} \sum_{j=n}^{n+m-1} \Big(\left\|4^{\tau_{-s,j}} Df\left(2^{\tau_{s,j}} x, 2^{\tau_{s,j}} x, 0\right) s \right. \\
&\qquad\qquad \left. + 2^{\tau_{-t,j}} Df\left(2^{\tau_{t,j}} x, 2^{\tau_{t,j}} x, 0\right) t \right\| \\
&\qquad + \left\|4^{\tau_{-s,j}} Df\left(-2^{\tau_{s,j}} x, -2^{\tau_{s,j}} x, 0\right) s \right. \\
&\qquad\qquad \left. -2^{\tau_{-t,j}} Df\left(-2^{\tau_{t,j}} x, -2^{\tau_{t,j}} x, 0\right) t \right\| \\
&\qquad + 2 \cdot 4^{\tau_{-s,j}} \left\|f(0)\right\| \Big) \\
&\leq \begin{cases}
\displaystyle\sum_{j=n}^{n+m-1} \frac{3}{2^{j+2}} + \frac{3 \cdot 2^{-2j-3}}{8} & (\text{for } p = 0), \\
\displaystyle\sum_{j=n}^{n+m-1} 2^{-j-1} \left\|2^j x\right\|^p & (\text{for } 0 < p < 1), \\
\displaystyle\sum_{j=n}^{n+m-1} 2^{-2j-2} \left\|2^j x\right\|^p + 2^j \left\|2^{-j-1} x\right\|^p & (\text{for } 1 < p < 2), \\
\displaystyle\sum_{j=n}^{n+m-1} 2^{2j} \left\|2^{-j-1} x\right\|^p & (\text{for } p > 2)
\end{cases} \\
&\leq \begin{cases}
\dfrac{3}{2^{n+1}} + \dfrac{1}{2^{2n+4}} & (\text{for } p = 0), \\
\dfrac{2^{np} \|x\|^p}{2^n (2 - 2^p)} & (\text{for } 0 < p < 1), \\
\dfrac{2^{np} \|x\|^p}{4^n (4 - 2^p)} + \dfrac{2^n \|x\|^p}{2^{np} (2^p - 2)} & (\text{for } 1 < p < 2), \\
\dfrac{4^n \|x\|^p}{2^{np} (2^p - 4)} & (\text{for } p > 2)
\end{cases}
\end{aligned} \tag{22}$$

for all $x \in X$. So, it is easy to show that the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in X$.

Since Y is complete, the sequence $\{J_n f(x)\}$ converges for all $x \in X$. Hence, we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n \to \infty} J_n f(x) \tag{23}$$

for all $x \in X$. Moreover, putting $n = 0$ and letting $m \to \infty$ in (22), we get the inequality (19). □

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