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Research Article

Two Energy Conserving Numerical Schemes for the Klein-Gordon-Zakharov Equations

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Two new difference schemes are proposed for an initial-boundary-value problem of the Klein-Gordon-Zakharov (KGZ) equations. They have the advantage that there is a discrete energy which is conserved. Their stability and convergence of difference solutions are proved in order $O(h^2 + \tau^2)$ on the basis of the prior estimates. Results of numerical experiments demonstrate the efficiency of the new schemes.

1. Introduction

In this paper, we consider the following initial-boundary-value problem of the KGZ equations (see [1]):

$$U_{tt} - U_{xx} + U + NU + |U|^2 U = 0, \quad -x_L < x < x_R, \quad 0 \le t \le T, \tag{1}$$

$$N_{tt} - N_{xx} = \left(|U|^2\right)_{xx}, \quad -x_L < x < x_R, \quad \le t \le T, \tag{2}$$

$$U|_{t=0} = U_0(x), \qquad U_t|_{t=0} = U_1(x), \qquad N|_{t=0} = N_0(x), \qquad N_t|_{t=0} = N_1(x), \tag{3}$$

$$U|_{x=x_L} = U|_{x=x_R} = 0, \qquad N|_{x=x_L} = N|_{x=x_R} = 0, \tag{4}$$

where a complex unknown function $U(x,t)$ denotes the fast time scale component of electric field raised by electrons and a real unknown function $N(x,t)$ denotes the deviation of ion density from its equilibrium; $U_0(x)$, $U_1(x)$, $N_0(x)$, and $N_1(x)$ are known smooth functions.

The solutions $U(x,t)$ and $N(x,t)$ of the initial-boundary-value problem (1)–(4) formally satisfy the following energy identity:

$$E = \int_{x_L}^{x_R} \left[\left|U_t\right|^2 + \left|U_x\right|^2 + |U|^2 + N|U|^2 + \frac{1}{2}|V|^2 + \frac{1}{2}|N|^2 + \frac{1}{2}|U|^4 \right] = \text{const}, \tag{5}$$

where the potential function V is defined as

$$V = -f_x, f_{xx} = N_t. \tag{6}$$

In [2] Ozawa et al. proved the well-posedness of the equations in three-dimensional space. Adomian discussed the existence of its nonperturbative solutions (see [3]). In [4] Guo and Yuan studied the global smooth solutions for the Cauchy problem of these equations. Furthermore, in [5, 6] the authors proposed three difference schemes for the KGZ equations. It is well known that a conservative scheme performs better than a nonconservative one; for example, Zhang et al. in [7] pointed out that the nonconservative schemes may easily show nonlinear blowup and Li and Vu-Quoc also said, “in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation” (see [8]). Up to now, many conservative finite difference schemes have been studied for the Klein-Gordon equation, Klein-Gordon-Schödinger equations, Sine-Gordon equation, Zakharov equations, and so on (see [9–25]). Numerical results of all the schemes are very good. Therefore, in this paper we will generalize the technique of these methods to propose two new conservative difference schemes which are unconditionally stable and more accurate for the KGZ equations.

The paper is organized as follows. In Section 2, a new difference scheme (i.e., Scheme A) is proposed, and its discrete conservative law is discussed. In Section 3, some prior estimates for difference solutions are made. In Section 4, convergence and stability for the new scheme are proved using discrete energy method. In Section 5, another conservative scheme (i.e., Scheme B) is constructed, and its discrete conservative law is discussed. In Section 6, some prior estimates of Scheme B are obtained by induction, then convergence of the scheme is analyzed. Finally, in Section 7, some numerical results are provided to demonstrate the theoretical results.

2. Finite Difference Scheme and Its Conservative Law

Before we propose the new difference scheme for the KGZ (1)–(4), we give some notations as follows:

$$\begin{gathered} x_j = x_L + jh, \quad 0 \le j \le J = \left[\frac{x_R - x_L}{h}\right], \\ t^n = n\tau, \quad n = 0, 1, 2, \dots, \left[\frac{T}{\tau}\right], \\ \left(w_j^n\right)_x = \frac{w_{j+1}^n - w_j^n}{h}, \qquad \left(w_j^n\right)_{\bar{x}} = \frac{w_j^n - w_{j-1}^n}{h}, \\ \left(w_j^n\right)_t = \frac{w_j^{n+1} - w_j^n}{\tau}, \qquad \left(w_j^n\right)_{\bar{t}} = \frac{w_j^n - w_j^{n-1}}{\tau}, \end{gathered} \tag{7}$$

where h and τ are step size of space and time, respectively.

Also we define the following inner product and norms:

$$\begin{gathered} \left(w^n, u^n\right) = h\sum_{j=0}^{J} w_j^n \bar{u}_j^n, \qquad \left\|w^n\right\|_p^p = h\sum_{j=0}^{J}\left|w_j^n\right|^p, \\ \left\|w^n\right\|_\infty = \sup_{0\le j\le J}\left|w_j^n\right|. \end{gathered} \tag{8}$$

In this paper, C stands for a general positive constant which may take different values on different occasions. For briefness, we omit the subscript 2 of $\| w^n\|_2$.

Lemma 1. *For any two mesh functions $\{w_j\}$ and $\{v_j\}$, $j = 0, 1, \dots, J$, there is the identity*

$$h\sum_{j=1}^{J-1} w_j \left(v_j\right)_{x\bar{x}} = -h\sum_{j=0}^{J-1}\left(w_j\right)_x\left(v_j\right)_x - w_0\left(v_0\right)_x + w_J\left(v_J\right)_{\bar{x}}. \tag{9}$$

It is easy to prove this lemma directly.

Now, we consider the following difference scheme for the KGZ equations (1)–(4).

Scheme A. We consider the following:

$$\begin{aligned} &\left(U_j^n\right)_{t\bar{t}} + \frac{h^2}{12}\left(U_j^n\right)_{x\bar{x}t\bar{t}} - \frac{1}{2}\left(U_j^{n+1} + U_j^{n-1}\right)_{x\bar{x}} \\ &\quad + \frac{1}{2}\left(U_j^{n+1} + U_j^{n-1}\right) + \frac{1}{2}N_j^n\left(U_j^{n+1} + U_j^{n-1}\right) \\ &\quad + \frac{1}{4}\left(\left|U_j^{n+1}\right|^2 + \left|U_j^{n-1}\right|^2\right)\left(U_j^{n+1} + U_j^{n-1}\right) = 0, \end{aligned} \tag{10}$$

$$\left(N_j^n\right)_{t\bar{t}} + \frac{h^2}{12}\left(N_j^n\right)_{x\bar{x}t\bar{t}} - \frac{1}{2}\left(N_j^{n+1} + N_j^{n-1}\right)_{x\bar{x}} = \left(\left|U_j^n\right|^2\right)_{x\bar{x}}, \tag{11}$$

$$\begin{gathered} U_j^0 = U_0\left(x_j\right), \qquad N_j^0 = N_0\left(x_j\right), \\ U_0^n = U_J^n = 0, \qquad N_0^n = N_J^n = 0, \end{gathered} \tag{12}$$

$$U_j^1 - U_j^{-1} = 2\tau U_1\left(x_j\right), \qquad N_j^1 - N_j^{-1} = 2\tau N_1\left(x_j\right). \tag{13}$$

By (10), (12), (11), and (13), we obtain the following:

$$\begin{aligned} &\frac{2}{\tau^2}\left(U_j^1 - U_j^0 - \tau U_1\left(x_j\right)\right) \\ &\quad + \frac{h^2}{6\tau^2}\left(U_j^1 - U_j^0 - \tau U_1\left(x_j\right)\right)_{x\bar{x}} \\ &\quad - \left(U_j^1 - \tau U_1\left(x_j\right)\right)_{x\bar{x}} + \left(U_j^1 - \tau U_1\left(x_j\right)\right) \\ &\quad + N_j^0\left(U_j^1 - \tau U_1\left(x_j\right)\right) \\ &\quad + \frac{1}{2}\left(\left|U_j^1\right|^2 + \left|U_j^1 - 2\tau U_1\left(x_j\right)\right|^2\right) \\ &\quad \times\left(U_j^1 - \tau U_1\left(x_j\right)\right) = 0, \\ &\frac{2}{\tau^2}\left(N_j^1 - N_j^0 - \tau N_1\left(x_j\right)\right) \\ &\quad + \frac{h^2}{6\tau^2}\left(N_j^1 - N_j^0 - \tau N_1\left(x_j\right)\right)_{x\bar{x}} \\ &\quad - \left(N_j^1 - \tau N_1\left(x_j\right)\right)_{x\bar{x}} = \left(\left|U_j^0\right|^2\right)_{x\bar{x}}. \end{aligned} \tag{14}$$

Here, we also define the potential function $\{f_j^n\}$ to be such that

$$\left(f_j^n\right)_{x\bar{x}} = \left(N_j^n\right)_t, \quad j = 1, 2, \dots, J-1, \; f_0^n = f_J^n = 0. \tag{15}$$

Theorem 2. *The difference scheme* (10)–(15) *possesses the following invariant:*

$$E^n = E^{n-1} = \cdots = E^0 = const, \tag{16}$$

where

$$\begin{aligned} E^n &= \|U_t^n\|^2 - \frac{h^2}{12}\|U_{xt}^n\|^2 \\ &\quad + \frac{1}{2}\left(\left\|U_x^{n+1}\right\|^2 + \|U_x^n\|^2\right) + \frac{1}{2}\left(\left\|U^{n+1}\right\|^2 + \|U^n\|^2\right) \\ &\quad + \frac{1}{2}h\sum_{j=1}^{J-1}\left(N_j^n\left|U_j^{n+1}\right|^2 + N_j^{n+1}\left|U_j^n\right|^2\right) \\ &\quad + \frac{1}{4}\left(\left\|U^{n+1}\right\|_4^4 + \|U^n\|_4^4\right) + \frac{1}{2}\|f_x^n\|^2 \\ &\quad - \frac{h^2}{24}\|N_t^n\|^2 + \frac{1}{4}\left(\left\|N^{n+1}\right\|^2 + \|N^n\|^2\right). \end{aligned} \tag{17}$$

Proof. Computing the inner product of (10) with $U^{n+1} - U^{n-1}$ and taking the real part, we have

$$\begin{aligned} &\|U_t^n\|^2 - \left\|U_t^{n-1}\right\|^2 - \frac{h^2}{12}\left(\|U_{xt}^n\|^2 - \left\|U_{xt}^{n-1}\right\|^2\right) \\ &\quad + \frac{1}{2}\left(\left\|U_x^{n+1}\right\|^2 - \left\|U_x^{n-1}\right\|^2\right) \\ &\quad + \frac{1}{2}\left(\left\|U^{n+1}\right\|^2 - \left\|U^{n-1}\right\|^2\right) \\ &\quad + \frac{1}{2}h\sum_{j=1}^{J-1}N_j^n\left(\left|U_j^{n+1}\right|^2 - \left|U_j^{n-1}\right|^2\right) \\ &\quad + \frac{1}{4}\left(\left\|U^{n+1}\right\|_4^4 - \left\|U^{n-1}\right\|_4^4\right) = 0. \end{aligned} \tag{18}$$

Next, computing the inner product of (11) with $(1/2)(f^n + f^{n-1})$ and using (15), we obtain

$$\begin{aligned} &\frac{1}{2}\left(\|f_x^n\|^2 - \left\|f_x^{n-1}\right\|^2\right) - \frac{h^2}{24}\left(\|N_t^n\|^2 - \left\|N_t^{n-1}\right\|^2\right) \\ &\quad + \frac{1}{4}\left(\left\|N^{n+1}\right\|^2 - \left\|N^{n-1}\right\|^2\right) \\ &\quad + \frac{1}{2}h\sum_{j=1}^{J-1}\left|U_j^n\right|^2\left(N_j^{n+1} - N_j^{n-1}\right) = 0. \end{aligned} \tag{19}$$

In the computation of (18) and (19), we have used the boundary conditions and Lemma 1. Then, result (16) follows from (18) and (19). □

3. Some Prior Estimates for Difference Solutions

In this section, we will estimate the difference solutions of Scheme A after introducing two important lemmas proved in [26].

Lemma 3 (discrete Sobolev's inequality). *For any discrete function $u_h = \{u_j \mid j = 0, 1, \ldots, J\}$ on the finite interval $[0, l]$ and for any given $\varepsilon > 0$, there exists a constant C, depending only on ε, such that*

$$\left\|\delta^k u_h\right\|_p \le \varepsilon\left\|\delta^n u_h\right\|_2 + C\left\|u_h\right\|_2, \tag{20}$$

where $2 \le p \le \infty, 0 \le k < n$.

Lemma 4 (Gronwall's inequality). *Suppose that the nonnegative mesh functions $\{w(n), \rho(n), n = 1, 2, \ldots, N, N\tau = T\}$ satisfy the inequality*

$$w(n) \le \rho(n) + \tau\sum_{l=1}^{n}B_l w(l), \tag{21}$$

where B_l $(l = 1, 2, \ldots, N)$ are nonnegative constant. Then, for any $0 \le n \le N$, there is

$$w(n) \le \rho(n)\exp\left(n\tau\sum_{l=1}^{n}B_l\right). \tag{22}$$

Theorem 5. *Assume that $U_0(x) \in H^1, U_1(x) \in L^2, N_0(x) \in H^1$, and $N_1(x) \in L^2$; then the following estimates hold:*

$$\begin{gathered} \|U_t^n\| \le C, \qquad \|U_x^n\| \le C, \qquad \|U^n\| \le C, \\ \|U^n\|_\infty \le C, \qquad \|f_x^n\| \le C, \qquad \|N^n\| \le C, \\ \|U^n\|_4 \le C. \end{gathered} \tag{23}$$

Proof. Applying Young's inequality, it is easy to see that

$$\begin{aligned} -\frac{h^2}{12}\|U_{xt}^n\|^2 &= -\frac{h^3}{12}\sum_{j=1}^{J-1}\left|\left(U_j^n\right)_{xt}\right|^2 = -\frac{h^3}{12}\sum_{j=1}^{J-1}\left(U_j^n\right)_{xt}\left(\overline{U}_j^n\right)_{xt} \\ &= -\frac{h}{12}\sum_{j=1}^{J-1}\left(U_{j+1}^n - U_j^n\right)_t\left(\overline{U}_{j+1}^n - \overline{U}_j^n\right)_t \\ &\ge -\frac{h}{6}\sum_{j=1}^{J-1}\left(\left|\left(U_{j+1}^n\right)_t\right|^2 + \left|\left(U_j^n\right)_t\right|^2\right) = -\frac{1}{3}\|U_t^n\|^2, \end{aligned} \tag{24}$$

and by (15), we have

$$\begin{aligned} -\frac{h^2}{24}\|N_t^n\|^2 &= -\frac{h^3}{24}\sum_{j=1}^{J-1}\left[\left(N_j^n\right)_t\right]^2 = -\frac{h^3}{24}\sum_{j=1}^{J-1}\left[\left(f_j^n\right)_{x\bar{x}}\right]^2 \\ &= -\frac{h}{24}\sum_{j=1}^{J-1}\left[\left(f_j^n\right)_x - \left(f_{j-1}^n\right)_x\right]^2 \\ &\ge -\frac{h}{12}\sum_{j=1}^{J-1}\left\{\left[\left(f_j^n\right)_x\right]^2 + \left[\left(f_{j-1}^n\right)_x\right]^2\right\} \\ &= -\frac{1}{6}\|f_x^n\|^2; \end{aligned} \tag{25}$$

then from (16) we get

$$\frac{2}{3}\|U_t^n\|^2+\frac{1}{2}\left(\|U_x^{n+1}\|^2+\|U_x^n\|^2\right)+\frac{1}{2}\left(\|U^{n+1}\|^2+\|U^n\|^2\right)+\frac{1}{2}h\sum_{j=1}^{J-1}\left(N_j^n|U_j^{n+1}|^2+N_j^{n+1}|U_j^n|^2\right)+\frac{1}{4}\left(\|U^{n+1}\|_4^4+\|U^n\|_4^4\right)+\frac{1}{3}\|f_x^n\|^2+\frac{1}{4}\left(\|N^{n+1}\|^2+\|N^n\|^2\right)\le E^n=C. \tag{26}$$

Since

$$\left|\frac{1}{2}h\sum_{j=1}^{J-1}N_j^n|U_j^{n+1}|^2\right|\le\frac{1}{4}h\sum_{j=1}^{J-1}\left((N_j^n)^2+|U_j^{n+1}|^4\right)\le\frac{1}{4}\left(\|N^n\|^2+\|U^{n+1}\|_4^4\right), \tag{27}$$

$$\left|\frac{1}{2}h\sum_{j=1}^{J-1}N_j^{n+1}|U_j^n|^2\right|\le\frac{1}{4}\left(\|N^{n+1}\|^2+\|U^n\|_4^4\right),$$

it follows from (26) that

$$\frac{2}{3}\|U_t^n\|^2+\frac{1}{2}\left(\|U_x^{n+1}\|^2+\|U_x^n\|^2\right)+\frac{1}{2}\left(\|U^{n+1}\|^2+\|U^n\|^2\right)+\frac{1}{3}\|f_x^n\|^2\le C. \tag{28}$$

Therefore

$$\|U_t^n\|\le C,\qquad \|U_x^n\|\le C,\qquad \|U^n\|\le C,\qquad \|f_x^n\|\le C. \tag{29}$$

Besides, we can obtain the following estimates by Lemma 3:

$$\|U^n\|_\infty\le C,\qquad \|U^n\|_4\le C. \tag{30}$$

On the other hand, by inequality $ab\le(1/4)\ a^2+b^2$, we have

$$\left|h\sum_{j=1}^{J-1}N_j^n|U_j^{n+1}|^2\right|\le h\sum_{j=1}^{J-1}\left[\frac{1}{4}(N_j^n)^2+|U_j^{n+1}|^4\right]=\frac{1}{4}\|N^n\|^2+\|U^{n+1}\|_4^4, \tag{31}$$

$$\left|h\sum_{j=1}^{J-1}N_j^{n+1}|U_j^n|^2\right|\le h\sum_{j=1}^{J-1}\left[\frac{1}{4}(N_j^{n+1})^2+|U_j^n|^4\right]=\frac{1}{4}\|N^{n+1}\|^2+\|U^n\|_4^4. \tag{32}$$

Thus, it follows from (26) that

$$\|N^n\|\le C. \tag{33}$$

This completes the proof. □

4. Convergence and Stability of the Difference Scheme

In this section, we will discuss the convergence and stability of the difference scheme (10)–(15). First, we define the truncation errors by

$$\begin{aligned} r_j^n={}&\left(U(x_j,t^n)\right)_{t\bar{t}}+\frac{h^2}{12}\left(U(x_j,t^n)\right)_{x\bar{x}t\bar{t}}-\frac{1}{2}\left(U(x_j,t^{n+1})+U(x_j,t^{n-1})\right)_{x\bar{x}}\\ &+\frac{1}{2}\left(U(x_j,t^{n+1})+U(x_j,t^{n-1})\right)+\frac{1}{2}N(x_j,t^n)\left(U(x_j,t^{n+1})+U(x_j,t^{n-1})\right)\\ &+\frac{1}{4}\left(|U(x_j,t^{n+1})|^2+|U(x_j,t^{n-1})|^2\right)\times\left(U(x_j,t^{n+1})+U(x_j,t^{n-1})\right), \end{aligned} \tag{34}$$

$$\sigma_j^n=\left(N(x_j,t^n)\right)_{t\bar{t}}+\frac{h^2}{12}\left(N(x_j,t^n)\right)_{x\bar{x}t\bar{t}}-\frac{1}{2}\left(N(x_j,t^{n+1})+N(x_j,t^{n-1})\right)_{x\bar{x}}-\left(|U(x_j,t^n)|^2\right)_{x\bar{x}}. \tag{35}$$

Lemma 6. *Assume that the conditions of Theorem 5 are satisfied and $U(x.t)\in C^{4,4}, N(x,t)\in C^{4,4}$; then the truncation errors of the difference scheme* (10)–(15) *satisfy $|r_j^n|+|\sigma_j^n|=O(\tau^2+h^2)$ as $\tau\to 0, h\to 0$.*

By Taylor's expansion, Lemma 6 can be proved directly. Besides, we note that the approximations of the initial conditions (13) have truncation errors of order $O(\tau^2)$, which are consistent with the scheme.

Now, we are going to analyze the convergence of the difference scheme (10)–(15).

Set the following:

$$\begin{gathered} e_j^n=U(x_j,t^n)-U_j^n,\qquad \eta_j^n=N(x_j,t^n)-N_j^n,\\ \frac{1}{h^2}\left(F_{j+1}^n-2F_j^n+F_{j-1}^n\right)=\frac{1}{\tau}\left(\eta_j^{n+1}-\eta_j^n\right),\\ j=1,2,\ldots,J-1,\\ F_0^n=F_J^n=0. \end{gathered} \tag{36}$$

Theorem 7. *Assume that the conditions of Lemma 6 are satisfied; then the solutions of the difference scheme* (10)–(15)

converge to the solutions of the problem stated in (1)–(4) *with order* $O(\tau^2+h^2)$ *in the* L_∞ *norm for* U^n *and in the* L_2 *norm for* N^n*.*

Proof. Subtracting (10) from (34), we obtain

$$\begin{aligned} r_j^n &= \left(e_j^n\right)_{t\bar{t}} + \frac{h^2}{12}\left(e_j^n\right)_{x\bar{x}t\bar{t}} - \frac{1}{2}\left(e_j^{n+1}+e_j^{n-1}\right)_{x\bar{x}} \\ &\quad + \frac{1}{2}\left(e_j^{n+1}+e_j^{n-1}\right) \\ &\quad + \frac{1}{2}\eta_j^n\left(U\left(x_j,t^{n+1}\right)+U\left(x_j,t^{n-1}\right)\right) \\ &\quad + \frac{1}{2}N_j^n\left(e_j^{n+1}+e_j^{n+1}\right) + L_1; \end{aligned} \tag{37}$$

that is,

$$\begin{aligned} &\left(e_j^n\right)_{t\bar{t}} + \frac{h^2}{12}\left(e_j^n\right)_{x\bar{x}t\bar{t}} - \frac{1}{2}\left(e_j^{n+1}+e_j^{n-1}\right)_{x\bar{x}} \\ &\quad + \frac{1}{2}\left(e_j^{n+1}+e_j^{n-1}\right) \\ &\quad = r_j^n - \frac{1}{2}\eta_j^n\left(U\left(x_j,t^{n+1}\right)+U\left(x_j,t^{n-1}\right)\right) \\ &\quad - \frac{1}{2}N_j^n\left(e_j^{n+1}+e_j^{n+1}\right) - L_1, \end{aligned} \tag{38}$$

where

$$\begin{aligned} L_1 &= \frac{1}{4}\left(\left|U\left(x_j,t^{n+1}\right)\right|^2+\left|U\left(x_j,t^{n-1}\right)\right|^2\right) \\ &\quad \times\left(U\left(x_j,t^{n+1}\right)+U\left(x_j,t^{n-1}\right)\right) \\ &\quad - \frac{1}{4}\left(\left|U_j^{n+1}\right|^2+\left|U_j^{n-1}\right|^2\right)\left(U_j^{n+1}+U_j^{n-1}\right) \\ &= \frac{1}{4}\left(\left|U\left(x_j,t^{n+1}\right)\right|^2+\left|U\left(x_j,t^{n-1}\right)\right|^2\right)\left(e_j^{n+1}+e_j^{n-1}\right) \\ &\quad + \frac{1}{4}\left[\left|U\left(x_j,t^{n+1}\right)\right|^2+\left|U\left(x_j,t^{n-1}\right)\right|^2\right. \\ &\quad \left. - \left|U_j^{n+1}\right|^2 - \left|U_j^{n-1}\right|^2\right]\cdot\left(U_j^{n+1}+U_j^{n-1}\right) \\ &= \frac{1}{4}\left(\left|U\left(x_j,t^{n+1}\right)\right|^2+\left|U\left(x_j,t^{n-1}\right)\right|^2\right)\left(e_j^{n+1}+e_j^{n-1}\right) \\ &\quad + \frac{1}{4}\left[U\left(x_j,t^{n+1}\right)\bar{e}_j^{n+1}\right. \\ &\quad + e_j^{n+1}\overline{U}_j^{n+1} + e_j^{n-1}\overline{U}_j^{n-1} \\ &\quad \left. + U\left(x_j,t^{n-1}\right)\bar{e}_j^{n-1}\right]\cdot\left(U_j^{n+1}+U_j^{n-1}\right). \end{aligned} \tag{39}$$

Then computing the inner product of (38) with $e^{n+1} - e^{n-1}$ and taking the real part, we have

$$\begin{aligned} &\| e_t^n\|^2 - \| e_t^{n-1}\|^2 - \frac{h^2}{12}\left(\| e_{xt}^n\|^2 - \| e_{xt}^{n-1}\|^2\right) \\ &\quad + \frac{1}{2}\left(\| e_x^{n+1}\|^2 - \| e_x^{n-1}\|^2\right) \\ &\quad + \frac{1}{2}\left(\| e^{n+1}\|^2 - \| e^{n-1}\|^2\right) \\ &\quad \le C\tau\left(\| r^n\|^2 + \| \eta^n\|^2 + \| N^n\left(e^{n+1}+e^{n-1}\right)\|^2\right. \\ &\qquad \left. + \| e_t^n\|^2 + \| e_t^{n-1}\|^2 + \| e^{n+1}\|^2 + \| e^{n-1}\|^2\right). \end{aligned} \tag{40}$$

From Lemma 3 and Theorem 5 it follows that

$$\begin{aligned} \left\|N^n\left(e^{n+1}+e^{n-1}\right)\right\| &\le C\left\|N^n\right\|\cdot\left\|e^{n+1}+e^{n-1}\right\|_\infty \\ &\le C\left\|e_x^{n+1}+e_x^{n-1}\right\| + C\left\|e^{n+1}+e^{n-1}\right\| \\ &\le C\left(\left\|e_x^{n+1}\right\|+\left\|e_x^{n-1}\right\|+\left\|e^{n+1}\right\|+\left\|e^{n-1}\right\|\right). \end{aligned} \tag{41}$$

So, substituting (41) into (40), we have

$$\begin{aligned} &\left\|e_t^n\right\|^2 - \left\|e_t^{n-1}\right\|^2 - \frac{h^2}{12}\left(\left\|e_{xt}^n\right\|^2 - \left\|e_{xt}^{n-1}\right\|^2\right) \\ &\quad + \frac{1}{2}\left(\left\|e_x^{n+1}\right\|^2 - \left\|e_x^{n-1}\right\|^2\right) + \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) \\ &\quad \le C\tau\left(\left\|r^n\right\|^2 + \left\|\eta^n\right\|^2 + \left\|e_x^{n+1}\right\|^2\right. \\ &\qquad \left. + \left\|e_x^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2 + \left\|e_t^n\right\|^2 + \left\|e_t^{n-1}\right\|^2\right). \end{aligned} \tag{42}$$

Next, subtracting (11) from (35), we obtain

$$\begin{aligned} &\left(\eta_j^n\right)_{t\bar{t}} + \frac{h^2}{12}\left(\eta_j^n\right)_{x\bar{x}t\bar{t}} - \frac{1}{2}\left(\eta_j^{n+1}+\eta_j^{n-1}\right)_{x\bar{x}} \\ &\quad = \sigma_j^n + \left(\left|U\left(x_j,t^n\right)\right|^2 - \left|U_j^n\right|^2\right)_{x\bar{x}}. \end{aligned} \tag{43}$$

Computing the inner product of (43) with $(1/2)(F^n + F^{n-1})$, we get

$$\begin{aligned} &-\frac{1}{2\tau}\left(\left\|F_x^n\right\|^2 - \left\|F_x^{n-1}\right\|^2\right) + \frac{h^2}{24\tau}\left(\left\|\eta_t^n\right\|^2 - \left\|\eta_t^{n-1}\right\|^2\right) \\ &\quad - \frac{1}{4\tau}\left(\left\|\eta^{n+1}\right\|^2 - \left\|\eta^{n-1}\right\|^2\right) \\ &\quad = \left(\sigma_j^n, \frac{1}{2}\left(F^n+F^{n-1}\right)\right) \\ &\quad + \left(\left(\left|U\left(x_j,t^n\right)\right|^2 - \left|U_j^n\right|^2\right)_{x\bar{x}},\right. \\ &\qquad \left.\frac{1}{2}\left(F^n+F^{n-1}\right)\right). \end{aligned} \tag{44}$$

Note that

$$\begin{aligned}&\left(\left(\left|U\left(x_j,t^n\right)\right|^2-\left|U_j^n\right|^2\right)_{x\bar{x}},\frac{1}{2}\left(F^n+F^{n-1}\right)\right)\\&\quad\le C\left(\left\|F_x^n\right\|^2+\left\|F_x^{n-1}\right\|^2+\left\|e_x^n\right\|^2+\left\|e^n\right\|^2\right),\\&\left(\sigma^n,\frac{1}{2}\left(F^n+F^{n-1}\right)\right)=\frac{1}{2}h\sum_{j=1}^{J-1}\sigma_j^n\left(F_j^n+F_j^{n-1}\right)\\&\quad\le C\left(\left\|\sigma^n\right\|^2+\left\|F_x^n\right\|^2+\left\|F_x^{n-1}\right\|^2\right).\end{aligned}\tag{45}$$

Then substituting (45) into (44) we have

$$\begin{aligned}&\left\|F_x^n\right\|^2-\left\|F_x^{n-1}\right\|^2-\frac{h^2}{12}\left(\left\|\eta_t^n\right\|^2-\left\|\eta_t^{n-1}\right\|^2\right)\\&\quad+\frac{1}{2}\left(\left\|\eta^{n+1}\right\|^2-\left\|\eta^{n-1}\right\|^2\right)\\&\quad\le C\tau\left(\left\|\sigma^n\right\|^2+\left\|F_x^n\right\|^2\right.\\&\qquad\left.+\left\|F_x^{n-1}\right\|^2+\left\|e_x^n\right\|^2+\left\|e^n\right\|^2\right).\end{aligned}\tag{46}$$

Now, adding (42) to(46), we get

$$\begin{aligned}&\left\|e_t^n\right\|^2-\left\|e_t^{n-1}\right\|^2-\frac{h^2}{12}\left(\left\|e_{xt}^n\right\|^2-\left\|e_{xt}^{n-1}\right\|^2\right)\\&\quad+\frac{1}{2}\left(\left\|e_x^{n+1}\right\|^2-\left\|e_x^{n-1}\right\|^2\right)\\&\quad+\frac{1}{2}\left(\left\|e^{n+1}\right\|^2-\left\|e^{n-1}\right\|^2\right)+\left\|F_x^n\right\|^2-\left\|F_x^{n-1}\right\|^2\\&\quad-\frac{h^2}{12}\left(\left\|\eta_t^n\right\|^2-\left\|\eta_t^{n-1}\right\|^2\right)+\frac{1}{2}\left(\left\|\eta^{n+1}\right\|^2-\left\|\eta^{n-1}\right\|^2\right)\\&\le C\tau\left(\left\|r^n\right\|^2+\left\|\eta^n\right\|^2+\left\|\sigma^n\right\|^2\right.\\&\qquad+\left\|e_x^{n+1}\right\|^2+\left\|e_x^n\right\|^2+\left\|e_x^{n-1}\right\|^2+\left\|e^{n+1}\right\|^2\\&\qquad+\left\|e^n\right\|^2+\left\|e^{n-1}\right\|^2+\left\|e_t^n\right\|^2+\left\|e_t^{n-1}\right\|^2\\&\qquad\left.+\left\|F_x^n\right\|^2+\left\|F_x^{n-1}\right\|^2\right).\end{aligned}\tag{47}$$

Let

$$\begin{aligned}B^n&=\left\|e_t^n\right\|^2-\frac{h^2}{12}\left\|e_{xt}^n\right\|^2+\frac{1}{2}\left(\left\|e_x^{n+1}\right\|^2+\left\|e_x^n\right\|^2\right)\\&\quad+\frac{1}{2}\left(\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2\right)+\left\|F_x^n\right\|^2-\frac{h^2}{12}\left\|\eta_t^n\right\|^2\\&\quad+\frac{1}{2}\left(\left\|\eta^{n+1}\right\|^2+\left\|\eta^n\right\|^2\right).\end{aligned}\tag{48}$$

It is easy to see that

$$\begin{aligned}B^n&\ge\frac{2}{3}\left\|e_t^n\right\|^2+\frac{1}{2}\left(\left\|e_x^{n+1}\right\|^2+\left\|e_x^n\right\|^2\right)\\&\quad+\frac{1}{2}\left(\left\|e^{n+1}\right\|^2+\left\|e^n\right\|^2\right)+\frac{2}{3}\left\|F_x^n\right\|^2\\&\quad+\frac{1}{2}\left(\left\|\eta^{n+1}\right\|^2+\left\|\eta^n\right\|^2\right).\end{aligned}\tag{49}$$

Then by (47) and Lemma 6 we have

$$\begin{aligned}B^n-B^{n-1}&\le C\tau\left(\left\|r^n\right\|^2+\left\|\sigma^n\right\|^2\right)+C\tau\left(B^n+B^{n-1}\right)\\&\le C\tau\left(h^2+\tau^2\right)^2+C\tau\left(B^n+B^{n-1}\right).\end{aligned}\tag{50}$$

Summing (50) up for n and applying Lemma 4, we get

$$\begin{aligned}B^N&\le\left(B^0+C\left(h^2+\tau^2\right)^2\right)\exp\left(CN\tau\right)\\&\le C\left(B^0+\left(h^2+\tau^2\right)^2\right).\end{aligned}\tag{51}$$

Therefore, it follows from (49) that

$$\begin{aligned}&\frac{2}{3}\left\|e_t^N\right\|^2+\frac{1}{2}\left(\left\|e_x^{N+1}\right\|^2+\left\|e_x^N\right\|^2\right)\\&\quad+\frac{1}{2}\left(\left\|e^{N+1}\right\|^2+\left\|e^N\right\|^2\right)+\frac{2}{3}\left\|F_x^n\right\|^2\\&\quad+\frac{1}{2}\left(\left\|\eta^{N+1}\right\|^2+\left\|\eta^N\right\|^2\right)\\&\le C\left(B^0+\left(h^2+\tau^2\right)^2\right).\end{aligned}\tag{52}$$

Note that e^0, e^1, η^0, and η^1 are two-order precision and $\|F_x^0\| = O(h^2+\tau^2)$ (see [22]). Thus $B^0 = O(h^2+\tau^2)^2$. Hence, the following inequalities can be obtained by (52):

$$\left\|e_x^N\right\|\le O\left(h^2+\tau^2\right),\qquad\left\|e^N\right\|\le O\left(h^2+\tau^2\right),\tag{53}$$

$$\left\|e_t^N\right\|\le O\left(h^2+\tau^2\right),\qquad\left\|\eta^N\right\|\le O\left(h^2+\tau^2\right).\tag{54}$$

Then, applying Lemma 3, we get

$$\left\|e_x^N\right\|_\infty\le O\left(h^2+\tau^2\right).\tag{55}$$

So the proof of Theorem 7 is complete. □

In the same way, we can prove that the solutions of the difference schemes (10)–(15) are unconditionally stable for initial data.

5. Another Conservative Difference Scheme

In this section, we will propose another conservative difference scheme for the problem given in (1)–(4) and discuss the discrete conservative law of this scheme.

Now, we consider the finite difference simulations for (1) and (2) as follows.

Scheme B. We consider the following:

$$\begin{aligned}&\left(U_j^n\right)_{t\bar{t}}-\frac{1}{2}\left(U_j^{n+1}+U_j^{n-1}\right)_{x\bar{x}}+\frac{1}{2}\left(U_j^{n+1}+U_j^{n-1}\right)\\&\quad+\frac{1}{2}N_j^n\left(U_j^{n+1}+U_j^{n-1}\right)+\frac{1}{8}\left(\left|U_j^{n+1}\right|^2+2\left|U_j^n\right|^2+\left|U_j^{n-1}\right|^2\right)\\&\quad\times\left(U_j^{n+1}+U_j^{n-1}\right)=0,\end{aligned}\tag{56}$$

$$\begin{aligned}&\frac{1}{2}\left(N_j^n\right)_{t\bar{t}}+\frac{1}{4}\left(N_{j+1}^n+N_{j-1}^n\right)_{t\bar{t}}\\&\quad-\frac{1}{2}\left(N_j^{n+1}+N_j^{n-1}\right)_{x\bar{x}}=\left(\left|U_j^n\right|^2\right)_{x\bar{x}}.\end{aligned}\tag{57}$$

In addition, the initial and boundary conditions (3) and (4) are also, respectively, approximated as

$$\begin{aligned}U_j^0&=U_0\left(x_j\right),\qquad N_j^0=N_0\left(x_j\right),\\U_0^n&=U_J^n=0,\qquad N_0^n=N_J^n=0,\end{aligned}\tag{58}$$

$$U_j^1-U_j^{-1}=2\tau U_1\left(x_j\right),\qquad N_j^1-N_j^{-1}=2\tau N_1\left(x_j\right).\tag{59}$$

We also define the function $\{f_j^n\}$ by

$$\left(f_j^n\right)_{x\bar{x}}=\left(N_j^n\right)_t,\quad j=1,2,\ldots,J-1,\ f_0^n=f_J^n=0.\tag{60}$$

In (56) and (57), let $n = 0$. Then eliminating U^{-1} and N^{-1} from (58) and (59), we get

$$\begin{aligned}&\frac{2}{\tau^2}\left(U_j^1-U_j^0-\tau U_1\left(x_j\right)\right)-\left(U_j^1-\tau U_1\left(x_j\right)\right)_{x\bar{x}}\\&\quad+\left(U_j^1-\tau U_1\left(x_j\right)\right)+N_j^0\left(U_j^1-\tau U_1\left(x_j\right)\right)\\&\quad+\frac{1}{4}\left(\left|U_j^1\right|^2+2\left|U_j^0\right|^2+\left|U_j^1-2\tau U_1\left(x_j\right)\right|^2\right)\\&\quad\times\left(U_j^1-\tau U_1\left(x_j\right)\right)=0,\\&\frac{1}{\tau^2}\left(N_j^1-N_j^0-\tau N_1\left(x_j\right)\right)\\&\quad+\frac{1}{2\tau^2}\left(N_{j+1}^1-N_{j+1}^0-\tau N_1\left(x_{j+1}\right)\right.\\&\quad\left.+N_{j-1}^1-N_{j-1}^0-\tau N_1\left(x_{j-1}\right)\right)\\&\quad-\left(N_j^1-\tau N_1\left(x_j\right)\right)_{x\bar{x}}=\left(\left|U_j^0\right|^2\right)_{x\bar{x}}.\end{aligned}\tag{61}$$

Theorem 8. *Scheme B admits the following invariant:*

$$\widetilde{E}^n=\widetilde{E}^{n-1}=\cdots=\widetilde{E}^0=const,\tag{62}$$

where

$$\begin{aligned}\widetilde{E}^n&=\left\|U_t^n\right\|^2+\frac{1}{2}\left(\left\|U_x^{n+1}\right\|^2+\left\|U_x^n\right\|^2\right)\\&\quad+\frac{1}{2}\left(\left\|U^{n+1}\right\|^2+\left\|U^n\right\|^2\right)+\frac{1}{4}\left\|f_x^n\right\|^2\\&\quad+\frac{1}{2}h\sum_{j=1}^{J-1}\left(N_j^{n+1}\left|U_j^n\right|^2+N_j^n\left|U_j^{n+1}\right|^2\right)\\&\quad+\frac{1}{4}h\sum_{j=1}^{J-1}\left(f_j^n\right)_x\left(f_{j-1}^n\right)_x+\frac{1}{4}\left(\left\|N^{n+1}\right\|^2+\left\|N^n\right\|^2\right)\\&\quad+\frac{1}{8}\left(\left\|U^{n+1}\right\|_4^4+\left\|U^n\right\|_4^4+2h\sum_{j=1}^{J-1}\left|U_j^{n+1}\right|^2\left|U_j^n\right|^2\right).\end{aligned}\tag{63}$$

Proof. Computing the inner product of (56) with $U^{n+1}-U^{n-1}$ and taking the real part, we have

$$\begin{aligned}&\left\|U_t^n\right\|^2-\left\|U_t^{n-1}\right\|^2+\frac{1}{2}\left(\left\|U_x^{n+1}\right\|^2-\left\|U_x^{n-1}\right\|^2\right)\\&\quad+\frac{1}{2}\left(\left\|U^{n+1}\right\|^2-\left\|U^{n-1}\right\|^2\right)\\&\quad+\frac{1}{2}h\sum_{j=1}^{J-1}N_j^n\left(\left|U_j^{n+1}\right|^2-\left|U_j^{n-1}\right|^2\right)+L_1=0,\end{aligned}\tag{64}$$

where

$$\begin{aligned}L_1&=\frac{1}{8}\operatorname{Re}\left\{h\sum_{j=1}^{J-1}\left(\left|U_j^{n+1}\right|^2+2\left|U_j^n\right|^2+\left|U_j^{n-1}\right|^2\right)\right.\\&\qquad\left.\times\left(U_j^{n+1}+U_j^{n-1}\right)\left(U_j^{n+1}-U_j^{n-1}\right)\right\}\\&=\frac{1}{8}h\sum_{j=1}^{J-1}\left(\left|U_j^{n+1}\right|^2+2\left|U_j^n\right|^2+\left|U_j^{n-1}\right|^2\right)\\&\qquad\times\left(\left|U_j^{n+1}\right|^2-\left|U_j^{n-1}\right|^2\right)\\&=\frac{1}{8}h\sum_{j=1}^{J-1}\left(\left|U_j^{n+1}\right|^4+2\left|U_j^{n+1}\right|^2\left|U_j^n\right|^2\right.\\&\qquad\left.-2\left|U_j^n\right|^2\left|U_j^{n-1}\right|^2-\left|U_j^{n-1}\right|^4\right)\\&=\frac{1}{8}\left(\left\|U^{n+1}\right\|_4^4+2h\sum_{j=1}^{J-1}\left|U_j^{n+1}\right|^2\left|U_j^n\right|^2\right.\\&\qquad\left.-2h\sum_{j=1}^{J-1}\left|U_j^n\right|^2\left|U_j^{n-1}\right|^2-\left\|U^{n-1}\right\|_4^4\right).\end{aligned}\tag{65}$$

Next, computing the inner product of (57) with $(1/2)(f^n + f^{n-1})$ and by (60), we obtain

$$\begin{aligned} &-\frac{1}{4\tau}\left(\|f_x^n\|^2 - \|f_x^{n-1}\|^2\right) \\ &\quad + L_2 - \frac{1}{4\tau}\left(\|N^{n+1}\|^2 - \|N^{n-1}\|^2\right) \\ &\quad = \frac{h}{2\tau}\sum_{j=1}^{J-1}\left(N_j^{n+1}|U_j^n|^2 - N_j^{n-1}|U_j^n|^2\right), \end{aligned} \tag{66}$$

where

$$\begin{aligned} L_2 &= \frac{1}{4}h\sum_{j=1}^{J-1}\left(N_{j+1}^n + N_{j-1}^n\right)_{t\bar{t}} \cdot \frac{1}{2}\left(f_j^n + f_j^{n-1}\right) \\ &= \frac{1}{4\tau}h\sum_{j=1}^{J-1}\left[\left(f_{j+1}^n + f_{j-1}^n\right)_{x\bar{x}} - \left(f_{j+1}^{n-1} + f_{j-1}^{n-1}\right)_{x\bar{x}}\right] \\ &\qquad \cdot \frac{1}{2}\left(f_j^n + f_j^{n-1}\right) \\ &= -\frac{1}{4\tau}h\sum_{j=1}^{J-1}\left[\left(f_{j+1}^n + f_{j-1}^n\right)_x - \left(f_{j+1}^{n-1} + f_{j-1}^{n-1}\right)_x\right] \\ &\qquad \cdot \frac{1}{2}\left(f_j^n + f_j^{n-1}\right)_x \\ &= -\frac{1}{4\tau}h\sum_{j=1}^{J-1}\left[\left(f_j^n\right)_x\left(f_{j-1}^n\right)_x - \left(f_j^{n-1}\right)_x\left(f_{j-1}^{n-1}\right)_x\right]. \end{aligned} \tag{67}$$

Then

$$\begin{aligned} &\frac{1}{4}\left(\|f_x^n\|^2 - \|f_x^{n-1}\|^2\right) \\ &\quad + \frac{1}{4}h\sum_{j=1}^{J-1}\left[\left(f_j^n\right)_x\left(f_{j-1}^n\right)_x - \left(f_j^{n-1}\right)_x\left(f_{j-1}^{n-1}\right)_x\right] \\ &\quad + \frac{1}{4}\left(\|N^{n+1}\|^2 - \|N^{n-1}\|^2\right) \\ &\quad + \frac{h}{2\tau}\sum_{j=1}^{J-1}\left(N_j^{n+1}|U_j^n|^2 - N_j^{n-1}|U_j^n|^2\right) = 0. \end{aligned} \tag{68}$$

Hence, result (62) is obtained by adding (68) to (64). This completes the proof. □

6. Convergence and Stability of the Scheme

Before we prove the convergence of Scheme B, we estimate the difference solutions of this scheme.

Theorem 9. *Assume that the conditions of Theorem 5 are satisfied; then the following estimates hold:*

$$\begin{aligned} &\|U_t^n\| \le C, \qquad \|U_x^n\| \le C, \qquad \|U^n\| \le C, \\ &\|U^n\|_\infty \le C, \qquad \|N^n\| \le C, \qquad \|U^n\|_4 \le C. \end{aligned} \tag{69}$$

Proof (by induction). First, because of the inequality

$$a \cdot b \le \frac{1}{p}(\varepsilon a)^p + \frac{1}{q}\left(\frac{1}{\varepsilon}b\right)^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q, \varepsilon > 0, \tag{70}$$

we have

$$\begin{aligned} &\frac{1}{2}\left|h\sum_{j=1}^{J-1}N_j^n|U_j^{n+1}|^2\right| \le \frac{\varepsilon_1}{2}\|N^n\|^2 + \frac{1}{8\varepsilon_1}\|U^{n+1}\|_4^4, \\ &\frac{1}{2}\left|h\sum_{j=1}^{J-1}N_j^{n+1}|U_j^n|^2\right| \le \frac{1}{8}\|N^{n+1}\|^2 + \frac{1}{2}\|U^n\|_4^4. \end{aligned} \tag{71}$$

Then, substituting (71) into (62) and choosing $\varepsilon_1 = \sqrt{2}\varepsilon > 1$, we get the following inequality:

$$\begin{aligned} &\|U_t^n\|^2 + \frac{1}{2}\left(\|U_x^{n+1}\|^2 + \|U_x^n\|^2\right) + \frac{1}{2}\left(\|U^{n+1}\|^2 + \|U^n\|^2\right) \\ &\quad + \frac{1}{4}\|f_x^n\|^2 + \frac{1}{4}h\sum_{j=1}^{J-1}\left(f_j^n\right)_x\left(f_{j-1}^n\right)_x \\ &\quad + \frac{1}{4}\left(\|N^{n+1}\|^2 + \|N^n\|^2\right) \\ &\quad + \frac{1}{8}\left(\|U^{n+1}\|_4^4 + \|U^n\|_4^4 + 2h\sum_{j=1}^{J-1}|U_j^{n+1}|^2|U_j^n|^2\right) \\ &\le C + \frac{1}{8}\|N^{n+1}\|^2 + \frac{1}{2}\|U^n\|_4^4 + \frac{\varepsilon_1}{2}\|N^n\|^2 + \frac{1}{8\varepsilon_1}\|U^{n+1}\|_4^4. \end{aligned} \tag{72}$$

That is,

$$\begin{aligned} &\|U_t^n\|^2 + \frac{1}{2}\left(\|U_x^{n+1}\|^2 + \|U_x^n\|^2\right) \\ &\quad + \frac{1}{2}\left(\|U^{n+1}\|^2 + \|U^n\|^2\right) \\ &\quad + \frac{1}{4}\|f_x^n\|^2 + \frac{1}{4}h\sum_{j=1}^{J-1}\left(f_j^n\right)_x\left(f_{j-1}^n\right)_x \\ &\quad + \frac{1}{8}\|N^{n+1}\|^2 + \frac{1}{4}\|N^n\|^2 + \left(\frac{1}{8} - \frac{1}{8\varepsilon_1}\right)\|U^{n+1}\|_4^4 \\ &\quad + \frac{1}{8}\|U^n\|_4^4 + \frac{2h}{8}\sum_{j=1}^{J-1}|U_j^{n+1}|^2|U_j^n|^2 \\ &\le C + \frac{1}{2}\|U^n\|_4^4 + \frac{\varepsilon_1}{2}\|N^n\|^2. \end{aligned} \tag{73}$$

Note that

$$\begin{aligned} &\left|\frac{1}{4}h\sum_{j=1}^{J-1}\left(f_j^n\right)_x\left(f_{j-1}^n\right)_x\right| \\ &\quad \le \frac{1}{4}h\sum_{j=1}^{J-1}\frac{1}{2}\left[\left(f_j^n\right)_x^2 + \left(f_{j-1}^n\right)_x^2\right] = \frac{1}{4}\|f_x^n\|^2, \end{aligned} \tag{74}$$

so we have

$$\| U_t^n\|^2 + \frac{1}{2}\left(\left\|U_x^{n+1}\right\|^2 + \|U_x^n\|^2\right) + \frac{1}{2}\left(\left\|U^{n+1}\right\|^2 + \|U^n\|^2\right) + \frac{1}{8}\left\|N^{n+1}\right\|^2 + \frac{1}{4}\|N^n\|^2 + \left(\frac{1}{8} - \frac{1}{8\varepsilon_1}\right)\left\|U^{n+1}\right\|_4^4 + \frac{1}{8}\|U^n\|_4^4 + \frac{2h}{8}\sum_{j=1}^{J-1}\left|U_j^{n+1}\right|^2\left|U_j^n\right|^2 \le C + \frac{1}{2}\|U^n\|_4^4 + \frac{\varepsilon_1}{2}\|N^n\|^2. \quad (75)$$

Obviously, by (58), (59), and the conditions of Theorem 5, the following inequalities hold:

$$\left\|U_t^{-1}\right\| \le C, \quad \left\|U_t^0\right\| \le C, \quad \left\|U_x^0\right\| \le C, \quad \left\|U^0\right\| \le C, \quad \left\|U^0\right\|_\infty \le C, \quad \left\|N^0\right\| \le C, \quad \left\|U^0\right\|_4 \le C. \quad (76)$$

Assume that Theorem 9 holds when $n = k$; that is,

$$\left\|U_t^{k-1}\right\| \le C, \quad \left\|U_x^k\right\| \le C, \quad \left\|U^k\right\| \le C, \quad \left\|U^k\right\|_\infty \le C, \quad \left\|N^k\right\| \le C, \quad \left\|U^k\right\|_4 \le C. \quad (77)$$

By (75) and (77), we get

$$\left\|U_t^k\right\|^2 + \frac{1}{2}\left(\left\|U_x^{k+1}\right\|^2 + \left\|U_x^k\right\|^2\right) + \frac{1}{2}\left(\left\|U^{k+1}\right\|^2 + \left\|U^k\right\|^2\right) + \frac{1}{8}\left\|N^{k+1}\right\|^2 + \frac{1}{4}\left\|N^k\right\|^2 + \left(\frac{1}{8} - \frac{1}{8\varepsilon_1}\right)\left\|U^{k+1}\right\|_4^4 + \frac{1}{8}\left\|U^k\right\|_4^4 + \frac{2h}{8}\sum_{j=1}^{J-1}\left|U_j^{k+1}\right|^2\left|U_j^k\right|^2 \le C + \frac{1}{2}\left\|U^k\right\|_4^4 + \frac{\varepsilon_1}{2}\left\|N^k\right\|^2 \le C, \quad (78)$$

from which the following inequalities are obtained,

$$\left\|U_t^k\right\| \le C, \quad \left\|U_x^{k+1}\right\| \le C, \quad \left\|U^{k+1}\right\| \le C, \quad \left\|N^{k+1}\right\| \le C, \quad \left\|U^{k+1}\right\|_4 \le C, \quad (79)$$

and applying Lemma 3, we have

$$\left\|U^{k+1}\right\|_\infty \le C. \quad (80)$$

Then, for any $n \in \{0, 1, 2, \ldots, N\}$, the following estimates are obtained:

$$\|U_t^n\| \le C, \quad \|U_x^n\| \le C, \quad \|U^n\| \le C, \quad \|U^n\|_\infty \le C, \quad \|N^n\| \le C, \quad \|U^n\|_4 \le C. \quad (81)$$

This completes the proof. □

Theorem 10. *Assume that the conditions of Lemma 6 are satisfied; then the solutions of the difference scheme* (56)–(61) *converge to the solutions of the problem given in* (1)–(4) *with order* $O(\tau^2 + h^2)$ *in the* L_∞ *norm for* U^n *and in the* L_2 *norm for* N^n.

Here, we omit details of the proof of this theorem because it can be proved in the same way as that used to prove Theorem 7.

7. Numerical Experiments

In this section, we compute the following numerical example to demonstrate the effectiveness of our two difference schemes:

$$\begin{gathered} U_{tt} - U_{xx} + U + NU + |U|^2U = 0, \\ -20 < x < 20, \quad 0 \le t \le T, \\ N_{tt} - N_{xx} = \left(|U|^2\right)_{xx}, \quad -20 < x < 20, \quad 0 \le t \le T, \\ U(-20, t) = U(20, t) = 0, \quad N(-20, t) = N(20, t) = 0, \quad 0 \le t \le T, \\ U(x, 0) = \frac{\sqrt{10} - \sqrt{2}}{2}\operatorname{sech}\left(\sqrt{\frac{1 + \sqrt{5}}{2}}x\right) \cdot \exp\left[i\left(\sqrt{\frac{2}{1 + \sqrt{5}}}x\right)\right], \quad -20 \le x \le 20, \\ N(x, 0) = -2\operatorname{sech}^2\left(\sqrt{\frac{1 + \sqrt{5}}{2}}x\right), \quad -20 \le x \le 20. \end{gathered} \quad (82)$$

The analytic solution of KGZ equations, which is derived in [27], will be used in our computation for comparison. The solution can be written as

$$U(x, t) = \frac{\sqrt{10} - \sqrt{2}}{2}\operatorname{sech}\left(\sqrt{\frac{1 + \sqrt{5}}{2}}x - t\right) \cdot \exp\left[i\left(\sqrt{\frac{2}{1 + \sqrt{5}}}x - t\right)\right], \quad (83)$$

$$N(x, t) = -2\operatorname{sech}^2\left(\sqrt{\frac{1 + \sqrt{5}}{2}}x - t\right). \quad (84)$$

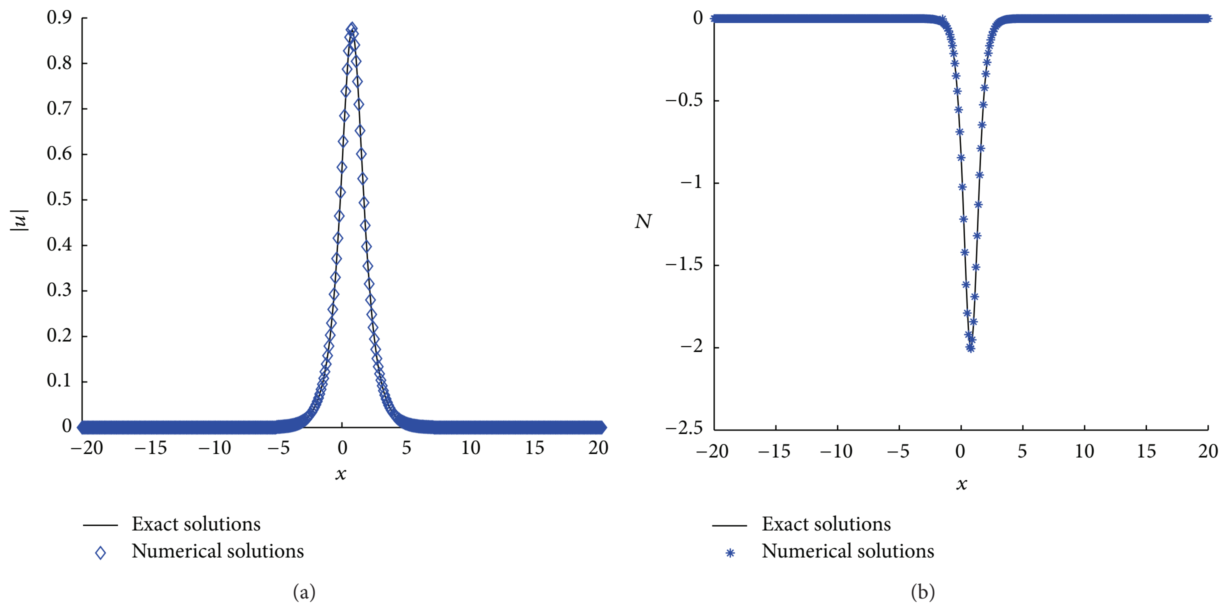

FIGURE 1: $|U|$ and N computed by Scheme A with $h = \tau = 0.1$. Comparison between analytic solution and numerical solution at time $T = 1$.

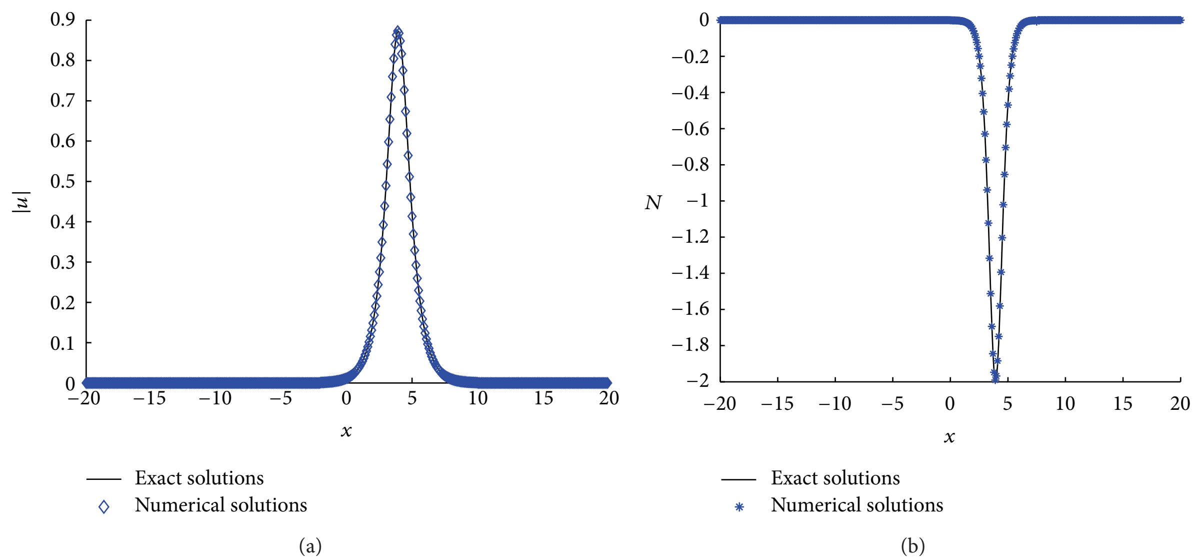

FIGURE 2: $|U|$ and N computed by Scheme B with $h = 0.1, \tau = 0.05$. Comparison between analytic solution and numerical solution at time $T = 5$.

In order to quantify the numerical results, we define the "error" functions and "rate of convergence" as

$$e(h,\tau) = \|e^n\|_\infty = \sup_{0\le j\le J} \left|U\left(x_j, t^n\right) - U_j^n\right|,$$

$$\text{rate}_1 = \log_2\left(\frac{e(h,\tau)}{e(h/2,\tau/2)}\right), \tag{85}$$

$$\eta(h,\tau) = \|\eta^n\| = \sqrt{h\sum_{j=0}^{J}\left|U\left(x_j, t^n\right) - N_j^n\right|^2},$$

$$\text{rate}_2 = \log_2\left(\frac{\eta(h,\tau)}{\eta(h/2,\tau/2)}\right). \tag{86}$$

For the two iterative schemes, we use an error restrictor $\varepsilon \le 0.000001$ to control the iterative procedures.

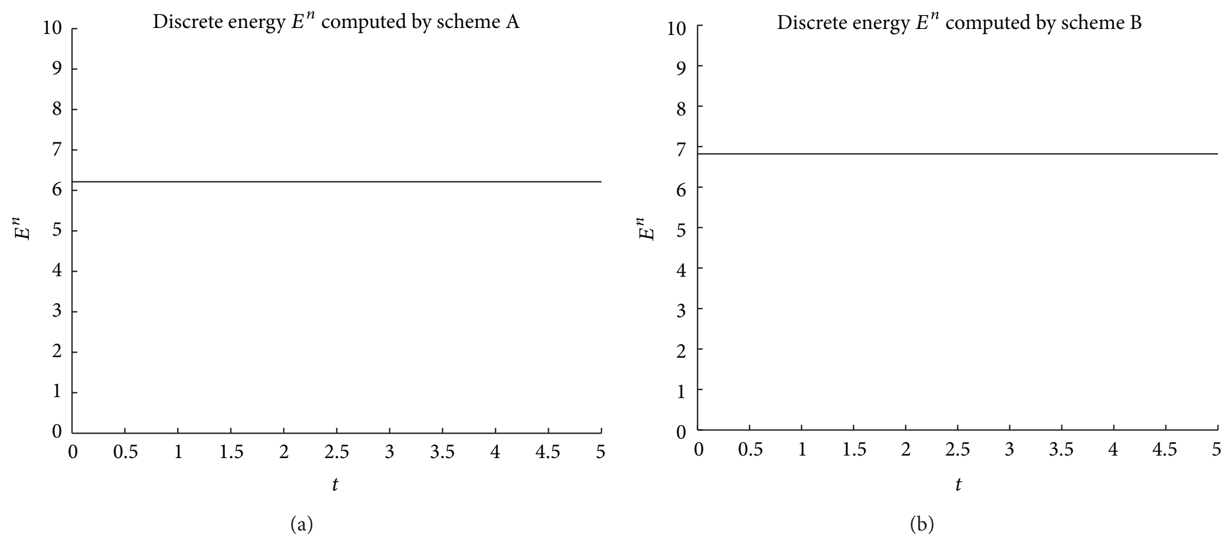

FIGURE 3: Discrete energy E^n computed by Scheme A and Scheme B with $h = 0.05, \tau = 0.05$.

TABLE 1: Errors and rates of convergence for Scheme A with different h and τ at time $T = 1$.

(h, τ)	$e(h, \tau)$	rate_1	$\eta(h, \tau)$	rate_2
(0.2, 0.1)	$1.1043e-002$	—	$2.8775e-002$	—
(0.1, 0.05)	$2.8652e-003$	1.95	$7.4419e-003$	1.95
(0.05, 0.025)	$7.2560e-004$	1.98	$1.8886e-003$	1.98
(0.025, 0.0125)	$1.8673e-004$	1.96	$4.7767e-004$	1.98

TABLE 2: Errors and rates of convergence for Scheme B with different h and τ at time $T = 5$.

(h, τ)	$e(h, \tau)$	rate_1	$\eta(h, \tau)$	rate_2
(0.1, 0.1)	$5.4233e-002$	—	0.1237	—
(0.05, 0.05)	$1.4050e-002$	1.95	$3.1279e-002$	1.98
(0.025, 0.025)	$3.4925e-003$	2.00	$7.7549e-003$	2.01
(0.0125, 0.0125)	$8.0729e-004$	2.11	$1.8386e-003$	2.08

TABLE 3: Error comparison for Scheme A and scheme in [6] with $\theta = 0.5$ at time $T = 5$.

Spatial step size	Time step size	Scheme A		Scheme in [6]	
		$e(h, \tau) \times 10^2$	$\eta(h, \tau) \times 10^2$	$e(h, \tau) \times 10^2$	$\eta(h, \tau) \times 10^2$
$h = 0.1$	$\tau = 0.05$	1.3527	3.8829	1.6527	5.5376
	$\tau = 0.025$	0.4430	1.2845	1.1721	3.6144
$h = 0.05$	$\tau = 0.025$	0.3390	0.9655	0.4148	1.3785
	$\tau = 0.0125$	0.1096	0.3189	0.2960	0.9170
$h = 0.025$	$\tau = 0.0125$	0.0849	0.2428	0.1081	0.3728

TABLE 4: Error comparison for Scheme B and Scheme I in [5] at time $T = 10$.

Spatial step size	Time step size	Scheme A		Scheme I in [5]	
		$e(h, \tau) \times 10^2$	$\eta(h, \tau) \times 10^2$	$e(h, \tau) \times 10^2$	$\eta(h, \tau) \times 10^2$
$h = 0.1$	$\tau = 0.01$	0.9742	2.4407	2.0623	6.3507
$h = 0.05$	$\tau = 0.025$	0.5894	1.8270	4.2819	8.0020
	$\tau = 0.0125$	0.3483	0.5278	2.1097	4.1693
$h = 0.025$	$\tau = 0.025$	0.4753	1.4006	4.3735	7.7397
	$\tau = 0.0125$	0.1839	0.3526	2.1330	3.8573

Firstly, in Figures 1 and 2, the solitary waves computed by Scheme A and Scheme B are compared with the waves of analytic solution, respectively. From (12) and (58), we will see that the the boundary conditions discretization produces no error in computation, so it is harmless to discrete energy E^n. The curves of discrete energy E^n obtained by the two schemes are plotted in Figure 3. Secondly, Tables 1 and 2 give the errors and the rates of convergence for Scheme A and Scheme B with various h and τ. Finally, errors produced by our two schemes and the schemes in [5, 6] are compared in Tables 3 and 4.

In Figures 1 and 2, it is obvious that the solitary waves computed by Scheme A and Scheme B agree with the ones computed by exact solutions quite well. Figure 3 shows that both Schemes A and B possess satisfactory conservative property. Tables 1 and 2 verify the second-order convergence and good stability for the two schemes. Furthermore, Tables 3 and 4 show that our two schemes are more accurate than schemes in [5, 6]. Therefore, it is clear that our two new difference schemes are efficient and accurate for the studied problem.

8. Conclusion

In this paper, we study the finite difference method for the KGZ equations. We propose two difference schemes, both of them are conservative on discrete energy law. The two schemes are shown to possess second-order accuracy for U in L_∞ norm and for N in L_2 norm. Numerical results demonstrate that the two schemes are accurate and efficient. It is worth mentioning that our methods can be directly extended to two dimensions and/or three dimensions, but some new techniques are required to be used to deal with the prior estimates.

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References

[1] R. O. Dendy, *Plama Dynamics*, Oxford University Press, Oxford, UK, 1990.

[2] T. Ozawa, K. Tsutaya, and Y. Tsutsumi, "Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions," *Mathematische Annalen*, vol. 313, no. 1, pp. 127–140, 1999.

[3] G. Adomian, "Non-perturbative solution of the Klein-Gordon-Zakharov equation," *Applied Mathematics and Computation*, vol. 81, no. 1, pp. 89–92, 1997.

[4] B. Guo and G. Yuan, "Global smooth solution for the Klein-Gordon-Zakharov equations," *Journal of Mathematical Physics*, vol. 36, no. 8, pp. 4119–4124, 1995.

[5] T. Wang, J. Chen, and L. Zhang, "Conservative difference methods for the Klein-Gordon-Zakharov equations," *Journal of Computational and Applied Mathematics*, vol. 205, no. 1, pp. 430–452, 2007.

[6] J. Chen and L.-m. Zhang, "Numerical simulation for the initial-boundary value problem of the Klein-Gordon-Zakharov equations," *Acta Mathematicae Applicatae Sinica*, vol. 28, no. 2, pp. 325–336, 2012.

[7] F. Zhang, V. M. Pérez-García, and L. Vázquez, "Numerical simulation of nonlinear Schrödinger systems: a new conservative scheme," *Applied Mathematics and Computation*, vol. 71, no. 2-3, pp. 165–177, 1995.

[8] S. Li and L. Vu-Quoc, "Finite difference calculus invariant structure of a class of algorithms for the nonlinear Klein-Gordon equation," *SIAM Journal on Numerical Analysis*, vol. 32, no. 6, pp. 1839–1875, 1995.

[9] Q. S. Chang and L. B. Xu, "A numerical method for a system of generalized nonlinear Schrödinger equations," *Journal of Computational Mathematics*, vol. 4, no. 3, pp. 191–199, 1986.

[10] Q. Chang, E. Jia, and W. Sun, "Difference schemes for solving the generalized nonlinear Schrödinger equation," *Journal of Computational Physics*, vol. 148, no. 2, pp. 397–415, 1999.

[11] T. Wang and L. Zhang, "Analysis of some new conservative schemes for nonlinear Schrödinger equation with wave operator," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1780–1794, 2006.

[12] T. Wang, T. Nie, L. Zhang, and F. Chen, "Numerical simulation of a nonlinearly coupled Schrödinger system: a linearly uncoupled finite difference scheme," *Mathematics and Computers in Simulation*, vol. 79, no. 3, pp. 607–621, 2008.

[13] L. Zhang and Q. Chang, "A conservative numerical scheme for a class of nonlinear Schrödinger equation with wave operator," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 603–612, 2003.

[14] D. Furihata, "Finite-difference schemes for nonlinear wave equation that inherit energy conservation property," *Journal of Computational and Applied Mathematics*, vol. 134, no. 1-2, pp. 37–57, 2001.

[15] Y. S. Wong, Q. Chang, and L. Gong, "An initial-boundary value problem of a nonlinear Klein-Gordon equation," *Applied Mathematics and Computation*, vol. 84, no. 1, pp. 77–93, 1997.

[16] S. Jiménez and L. Vázquez, "Analysis of four numerical schemes for a nonlinear Klein-Gordon equation," *Applied Mathematics and Computation*, vol. 35, no. 1, pp. 61–94, 1990.

[17] L. Zhang, "Convergence of a conservative difference scheme for a class of Klein-Gordon-Schrödinger equations in one space dimension," *Applied Mathematics and Computation*, vol. 163, no. 1, pp. 343–355, 2005.

[18] L. M. Zhang and Q. S. Chang, "Convergence and stability of a conservative finite difference scheme for a class of equation system in interaction of complex Schrödinger field and real Klein-Gordon field," *Numerical Mathematics*, vol. 22, no. 3, pp. 362–370, 2000.

[19] W. Bao and L. Yang, "Efficient and accurate numerical methods for the Klein-Gordon-Schrödinger equations," *Journal of Computational Physics*, vol. 225, no. 2, pp. 1863–1893, 2007.

[20] F. Zhang and L. Vázquez, "Two energy conserving numerical schemes for the Sine-Gordon equation," *Applied Mathematics and Computation*, vol. 45, no. 1, pp. 17–30, 1991.

[21] G. L. Payne, D. R. Nicholson, and R. M. Downie, "Numerical solution of the Zakharov equations," *Journal of Computational Physics*, vol. 50, no. 3, pp. 482–498, 1983.

[22] R. T. Glassey, "Convergence of an energy-preserving scheme for the Zakharov equations in one space dimension," *Mathematics of Computation*, vol. 58, no. 197, pp. 83–102, 1992.

[23] R. T. Glassey, "Approximate solutions to the Zakharov equations via finite differences," *Journal of Computational Physics*, vol. 100, no. 2, pp. 377–383, 1992.

[24] Q. S. Chang and H. Jiang, "A conservative difference scheme for the Zakharov equations," *Journal of Computational Physics*, vol. 113, no. 2, pp. 309–319, 1994.

[25] Q. S. Chang, B. L. Guo, and H. Jiang, "Finite difference method for generalized Zakharov equations," *Mathematics of Computation*, vol. 64, no. 210, pp. 537–553, 1995.

[26] Y. L. Zhou, *Applications of Discrete Functional Analysis to the Finite Difference Method*, International Academic Publishers, 1991.

[27] S. Liu, Z. Fu, S. Liu, and Z. Wang, "The periodic solutions for a class of coupled nonlinear Klein-Gordon equations," *Physics Letters A*, vol. 323, no. 5-6, pp. 415–420, 2004.