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Research Article

Boundedness for a Class of Generalized Commutators of Fractional Hardy Operators with a Rough Kernel

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The authors consider the generalized commutator of fractional Hardy operator with a rough kernel as follows: $\mathscr{H}^m_{\Omega,A,\beta} f(x) = 1/(|x|^{n-\beta}) \int_{|y|<|x|} (\Omega(x-y)/|x-y|^{m-1}) R_m(A;x,y) f(y)dy$, where $\Omega \in L^r(\mathbb{S}^{n-1})$, $0 \leq \beta < n$, and $R_m(A;x,y) = A(x) - \sum_{|\gamma|<m}(1/\gamma!)D^\gamma A(y)(x-y)^\gamma$ with $m \in Z^+$. The authors prove that $\mathscr{H}^m_{\Omega,A,\beta}$ is bounded on Herz type space and λ-Central Morrey space with $m \geq 1$, which is an open problem for $m > 2$.

1. Introduction

It is well known that the C-Z singular integrals and their commutators have been studied a lot by many mathematicians; please see [1] or [2] for more details. For the generalizations of the commutators of singular integrals, Cohen [3] studied the following generalized commutator T^2_A which is defined by

$$T^2_A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+1}} \left(A(x) - A(y) - \nabla A(y)(x-y)\right) f(y)\, dy, \tag{1}$$

where $\Omega \in L^1(\mathbb{S}^{n-1})$ is homogeneous of degree zero and satisfies the moment condition

$$\int_{\mathbb{S}^{n-1}} \Omega(x) x^\gamma d\sigma(x) = 0 \tag{2}$$

with $|\gamma| = 1$. Cohen [3] proved that if $\Omega \in \mathrm{Lip}_1(\mathbb{S}^{n-1})$ and $\nabla A \in \mathrm{BMO}$, then T^2_A is bounded on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Later, Cohen and Gosselin [4] considered another type of generalized commutator as follows:

$$T^m_A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,y) f(y)\, dy, \tag{3}$$

where $R_m(A;x,y) (m \in Z^+)$ is defined by $R_m(A;x,y) = A(x) - \sum_{|\gamma|<m}(1/\gamma!)D^\gamma A(y)(x-y)^\gamma$, the mth remainder of Taylor series of the function A at y about x, and Ω satisfies the following moment condition:

$$\int_{\mathbb{S}^{n-1}} \Omega(x) x^\gamma d\sigma(x) = 0, \tag{4}$$

with $|\gamma| = m-1$. Obviously, if we choose $m = 1$, T^m_A becomes $[A,T]$, the commutator of T generalized by A and T. Furthermore, T^m_A becomes T^2_A if we choose $m = 2$.

Cohen and Gosselin proved that if $m \geq 2$, $\Omega \in \mathrm{Lip}_1(\mathbb{S}^{n-1})$, and the function A has derivatives of order $m-1$ in $\mathrm{BMO}(\mathbb{R}^n)$, then the operator T^m_A is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later, T^m_A was studied by many mathematicians; please see [5, 6] or [7] for more details. Recently, Wang and Zhang [8] gave a new proof of Wu's theorem in [9] by using the $W^{1,p}$ estimate for the elliptic equation of divergence form with partially BMO coefficients and the L^p boundedness of the Cohen-Gosselin type generalized commutators proved by Yan in [6]. Furthermore, the method used in [8] is much simpler than that in [9]. Recently, Yu and Tao [7] proved that T^m_A is bounded on λ-Central Morrey space.

Let f be a nonnegative integral on $\mathbb{R}^+$: then the Hardy operator is defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x \neq 0. \tag{5}$$

In 1920, Hardy [10] proved the following inequality:

$$\|Hf\|_{L^p(\mathbb{R}^+)} \leq \frac{p}{p-1}\|f\|_{L^p(\mathbb{R}^+)}, \tag{6}$$

where $1 < p < \infty$ and the constant $p/(p-1)$ is the best possible.

In 2007, Fu et al. [11] introduced the n-dimensional fractional type Hardy operator $\mathscr{H}_\beta$ as follows:

$$\mathscr{H}_\beta f(x) = \frac{1}{|x|^{n-\beta}}\int_{|t|<|x|} f(t)\,dt, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{7}$$

where $-n < \beta < n$ and f is a locally integrable function on $\mathbb{R}^n$.

Obviously, when $\beta = 0$, $\mathscr{H}_0$ is just the n-dimensional Hardy operator $\mathscr{H}$ which was proposed by Christ and Grafakos in [12].

In [11], the authors gave the characterization of the $C\dot{B}MO^q(\mathbb{R}^n)$ by the boundedness of the commutator of the fractional type Hardy operator $[\mathscr{H}_\beta, b]$ on Herz type spaces. Here the $C\dot{B}MO^q(\mathbb{R}^n)$ space is defined by the following.

Definition 1 (see [13]). Let $1 \leq q < \infty$. A function $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the homogeneous Central BMO space $C\dot{B}MO^q(\mathbb{R}^n)$ if

$$\|b\|_{C\dot{B}MO^q(\mathbb{R}^n)} := \sup_{r>0}\left(\frac{1}{|B(0,r)|}\int_{B(0,r)}|f(x)-f_B|^q dx\right)^{1/q} < \infty, \tag{8}$$

where $f_B = (1/|B(0,r)|)\int_{B(0,r)} f(x)dx$.

From [14], we know that $\mathrm{BMO}(\mathbb{R}^n) \subset C\dot{B}MO^q(\mathbb{R}^n)$ for $1 \leq q < \infty$.

The $C\dot{B}MO^q(\mathbb{R}^n)$ space can be regarded as the space of bounded mean oscillation, a local version of $\mathrm{BMO}(\mathbb{R}^n)$ at the origin. But the famous John-Nirenberg inequality no longer holds in $C\dot{B}MO^q(\mathbb{R}^n)$.

Now we are interested in the following generalized commutator of Hardy operator:

$$\mathscr{H}^m_A f(x) = \frac{1}{|x|^n}\int_{|y|<|x|}\frac{1}{|x-y|^{m-1}} f(y)\,R_m(A;x,y)\,dy, \tag{9}$$

where $R_m(A;x,y) = A(x) - \sum_{|\gamma|<m}(1/\gamma!)D^\gamma A(y)(x-y)^\gamma$ and $m \in Z^+$.

In 2010, Lu and Zhao [15] proved that when $m = 2$, $\mathscr{H}^2_A$ is bounded on Herz type space and Morrey-Herz type space. Later, Gao and Yu [16] proved that $\mathscr{H}^2_A$ is bounded on λ-Central Morrey spaces. However, we would like to point out that the method used in [15, 16] cannot apply to the case when $m > 2$. An interesting question is whether the boundedness of $\mathscr{H}^m_A$ on Herz type space or λ-Central Morrey space still holds with $m > 2$. In this paper, we will use a different method to answer this question. Furthermore, we will consider the generalized commutator of fractional Hardy operator with a rough kernel as follows:

$$\mathscr{H}^m_{\Omega,A,\beta} f(x) = \frac{1}{|x|^{n-\beta}}\int_{|y|<|x|}\frac{\Omega(x-y)}{|x-y|^{m-1}} f(y)\,R_m(A;x,y)\,dy, \tag{10}$$

where $m \in Z^+$, $0 \leq \beta < n$, and $\Omega \in L^r(\mathbb{S}^{n-1})$.

In [17], we prove that $\mathscr{H}^m_{\Omega,A,\beta}$ is bounded from L^p to L^q with $1/p - 1/q = \beta/n$. Furthermore, we study the endpoint estimates of $\mathscr{H}^m_{\Omega,A,\beta}$ on H^1 spaces with $\Omega \in \mathrm{Lip}_1(\mathbb{S}^{n-1})$ in [17]. In this paper, we will prove that $\mathscr{H}^m_{\Omega,A,\beta}$ is bounded on Herz type space and λ-Central Morrey space when $\Omega \in L^r(\mathbb{S}^{n-1})$.

2. Boundedness of $\mathscr{H}^m_{\Omega,A,\beta}$ on Herz Type Spaces

In this section, we will give the boundedness of $\mathscr{H}^m_{\Omega,A,\beta}$ on Herz type spaces. First we introduce some notations that will be used throughout this paper.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{C_k}$ for $k \in Z$: here χ_{C_k} is the characteristic function of the set C_k.

Definition 2 (see [18]). Let $\alpha \in R$, $0 < p, q \leq \infty$. Then the homogeneous Herz type space $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_q(\mathbb{R}^n) = \left\{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} < \infty\right\}, \tag{11}$$

where $\|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)}$ is defined as

$$\|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\right\}^{1/p} \tag{12}$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

Now we show our main results in this section.

Theorem 3. *Suppose $m \geq 2$, $\Omega \in L^r(\mathbb{S}^{n-1})$ with $1 < r < \infty$, and A has derivatives of order $m-1$ in $C\dot{B}MO^{p_2}$ with $n < p_2 < \infty$. Let $0 < s \leq p < \infty$, $1 < q, p_1 < \infty$, $1/q = 1/p_1 + 1/p_2 - \beta/n$ with $0 \leq \beta < n$. If $1/r' - 1/q - \beta/n > 0$, $r > p_1'$, $\alpha_2 = \alpha_1 + n/p_2$ and α_2 satisfies the following condition:*

$$\alpha_2 + n/q - n/r' - 1/r - n/p_2 + \beta < 0, \tag{13}$$

then there exists a constant C, such that

$$\left\|\mathscr{H}^m_{\Omega,A,\beta} f\right\|_{\dot{K}^{\alpha_1,s}_q} \leq C\sum_{|\gamma|=m-1}\left\|D^\gamma A\right\|_{C\dot{B}MO^{p_2}}\|f\|_{\dot{K}^{\alpha_2,p}_{p_1}}. \tag{14}$$

For $m = 1$, $\mathscr{H}^1_{\Omega,A,\beta}$ is just the commutator of Hardy operator; that is, $\mathscr{H}^1_{\Omega,A,\beta} = \mathscr{H}^A_{\Omega,\beta} f(x) = A(x)\mathscr{H}_{\Omega,\beta} f(x) - \mathscr{H}_{\Omega,\beta}(Af)(x)$. We have the following theorem of $\mathscr{H}^A_{\Omega,\beta}$ on Herz type space.

Theorem 4. *Suppose $\Omega \in L^r(\mathbb{S}^{n-1})$ with $1 < r < \infty$ and $A \in C\dot{B}MO^{p_2}$. Let $0 < s \le p < \infty, 1 < q, p_1, p_2 < \infty, 1/q = 1/p_1 + 1/p_2 - \beta/n$ with $0 \le \beta < n$. If $1/r' - 1/q - \beta/n > 0, r > p_1'$, $\alpha_2 = \alpha_1 + n/p_2$, and α_2 satisfies (13), then there exists a constant C, such that*

$$\left\|\mathscr{H}^A_{\Omega,\beta} f\right\|_{\dot{K}^{\alpha_1,s}_q} \le C\|A\|_{C\dot{B}MO^{p_2}} \|f\|_{\dot{K}^{\alpha_2,p}_{p_1}}. \tag{15}$$

Remark 5. Comparing Theorems 3 and 4, we find that the restrictions on α_1 and α_2 are more rigid in Theorem 4 than in Theorem 3, which indicates that $\mathscr{H}^m_{\Omega,A,\beta}$ with $m \ge 2$ has better properties than the commutators.

In order to prove Theorems 3 and 4, we need the following lemmas.

Lemma 6 (see [19]). *Let $1 < p_1, p_2 < \infty, 0 \le \beta < n$, and $\beta/n = 1/p_1 - 1/p_2$. If $\Omega \in L^r(\mathbb{S}^{n-1})$ with $r > p_1'$, then there exists a constant C independent of f, such that*

$$\left\|\mathscr{H}_{\Omega,\beta} f\right\|_{L^{p_2}} \le C\|f\|_{L^{p_1}}, \tag{16}$$

where $\mathscr{H}_{\Omega,\beta}$ is defined by

$$\mathscr{H}_{\Omega,\beta} f(x) = \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \Omega(x-y) f(y)\, dy. \tag{17}$$

By checking [19] carefully, one can draw the conclusion that if one replaces $\mathscr{H}_{\Omega,\beta} f(x)$ by $\mathscr{H}_{|\Omega|,\beta}|f|(x)$, then (16) still holds.

Lemma 7. *Let $m \ge 1, 1 < p_1 < p_2 < \infty$ and $0 \le \beta < n$. If A has derivatives of order $m-1$ in $L^r(\mathbb{R}^n)$ with $1/p_2 = 1/p_1 + 1/r - \beta/n$ and $r > p_1'$, then one has*

$$\left\|\mathscr{H}^m_{\Omega,A,\beta} f\right\|_{L^{p_2}} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{L^r} \|f\|_{L^{p_1}}, \tag{18}$$

where the constant C is independent of f and A.

Proof. From [20, p. 241], we have the following estimates:

$$\begin{aligned} \frac{R_m(A;x,y)}{|x-y|^{m-1}} &\le \frac{R_{m-1}(A;x,y)}{|x-y|^{m-1}} + C \sum_{|\gamma|=m-1} \left|D^\gamma A(y)\right| \\ &\le C \sum_{|\gamma|=m-1} \left((D^\gamma A)^*(x) + (D^\gamma A)^*(y)\right), \end{aligned} \tag{19}$$

where $m \ge 1$ and $(f)^*$ is the Hardy-Littlewood maximal function of f.

Thus we obtain

$$\begin{aligned} &\left|\mathscr{H}^m_{\Omega,A,\beta} f(x)\right| \\ &\quad = \left|\frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} \frac{\Omega(x-y) f(y)}{|x-y|^{m-1}} R_m(A;x,y)\, dy\right| \\ &\quad \le C \frac{1}{|x|^{n-\beta}} \int_{|y|<|x|} |f(y)| \, |\Omega(x-y)| \\ &\qquad \times \sum_{|\gamma|=m-1} \left((D^\gamma A)^*(x) + (D^\gamma A)^*(y)\right) dy \\ &\quad \le C \sum_{|\gamma|=m-1} \Big[(D^\gamma A)^*(x)\, \mathscr{H}_{|\Omega|,\beta}|f|(x) \\ &\qquad\qquad + \mathscr{H}_{|\Omega|,\beta}\left((D^\gamma A)^* |f|\right)(x)\Big]. \end{aligned} \tag{20}$$

By the above estimates, we can get

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \left|\mathscr{H}^m_{\Omega,A,\beta} f(x)\right|^q dx\right)^{1/q} \\ &\quad \le C \sum_{|\gamma|=m-1} \left(\left(\int_{\mathbb{R}^n} \left|(D^\gamma A)^*(x)\, \mathscr{H}_{|\Omega|,\beta}|f|(x)\right|^q dx\right)^{1/q} \right. \\ &\qquad\qquad \left. + \left(\int_{\mathbb{R}^n} \left|\mathscr{H}_{|\Omega|,\beta}\left((D^\gamma A)^* |f|\right)(x)\right|^q\right)^{1/q}\right) \\ &\quad \le C \sum_{|\gamma|=m-1} (I + II). \end{aligned} \tag{21}$$

For the term I, let $1/q = 1/r + 1/l = 1/r + 1/p - \beta/n$; then by the Hölder inequality, Lemma 6, and the boundedness of Hardy-Littlewood maximal function on L^p spaces, we obtain

$$\begin{aligned} I &\le \left(\int_{\mathbb{R}^n} (D^\gamma A)^*(x)^r dx\right)^{1/r} \left(\int_{\mathbb{R}^n} \left|\mathscr{H}_{|\Omega|,\beta}|f|(x)\right|^l dx\right)^{1/l} \\ &\le C\left\|(D^\gamma A)^*\right\|_{L^r} \|f\|_{L^p} \\ &\le C\|D^\gamma A\|_{L^r} \|f\|_{L^p}. \end{aligned} \tag{22}$$

For the term II, let $1/q = 1/t - \beta/n = 1/r + 1/p - \beta/n$; then by the Hölder inequality and Lemma 6, we have

$$\begin{aligned} II &\le C\left(\int_{\mathbb{R}^n} \left|(D^\gamma A)^*(x) f(x)\right|^t dx\right)^{1/t} \\ &\le C\left\|(D^\gamma A)^*\right\|_{L^r} \|f\|_{L^p} \\ &\le C\|D^\gamma A\|_{L^r} \|f\|_{L^p}. \end{aligned} \tag{23}$$

Combining the estimates of I and II, we finish the proof of Lemma 7. □

Lemma 8 (see [4]). *Let b be a function on $\mathbb{R}^n$ with mth order derivatives in $L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for some $q > n$. Then*

$$\left|R_m(b;x,y)\right| \leq C_{m,n}|x-y|^m \times \sum_{|\gamma|=m}\left(\frac{1}{|\widetilde{Q}(x,y)|}\int_{\widetilde{Q}(x,y)}\left|D^\gamma b(z)\right|^q dz\right)^{1/q}, \tag{24}$$

where $\widetilde{Q}(x,y)$ is the cube centered at x having diameter $5\sqrt{n}|x-y|$.

Lemma 9 (see [5]). *Suppose that $f \in C\dot{B}MO^q(\mathbb{R}^n)$, $1 \leq q < \infty$, and $r_1, r_2 > 0$; then*

$$\begin{aligned}&\left(\frac{1}{|B(0,r_1)|}\int_{B(0,r_1)}\left|f(x)-f_{B(0,r_2)}\right|^q dx\right)^{1/q}\\ &\quad\leq C\left(1+\left|\log\left(\frac{r_1}{r_2}\right)\right|\right)\|f\|_{C\dot{B}MO^q}.\end{aligned} \tag{25}$$

Proof of Theorem 3. To prove Theorem 3, first we split each f as

$$f(x) = \sum_{i=-\infty}^{+\infty} f(x)\chi_i(x) = \sum_{i=-\infty}^{+\infty} f_i(x); \tag{26}$$

then we have

$$\begin{aligned}&\left\|\mathscr{H}^m_{\Omega,A,\beta}f\chi_k\right\|^q_{L^q}\\ &\quad= \int_{C_k}\left(\int_{B(0,|x|)}\frac{|f(y)||\Omega(x-y)|}{|x-y|^{m-1}}R_m(A;x,y)\,dy\right)^q \times |x|^{(\beta-n)q}dx\\ &\quad\leq \int_{C_k}\left(\sum_{i=-\infty}^{k}\int_{C_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}}R_m(A;x,y)\,dy\right)^q \times |x|^{(\beta-n)q}dx\\ &\quad\leq C\int_{C_k}\left(\sum_{i=-\infty}^{k-3}\int_{C_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}}R_m(A;x,y)\,dy\right)^q \times |x|^{(\beta-n)q}dx\\ &\qquad+ C\int_{C_k}\left(\sum_{i=k-2}^{k}\int_{C_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}}R_m(A;x,y)\,dy\right)^q \times |x|^{(\beta-n)q}dx\\ &\quad= C(I_1+I_2).\end{aligned} \tag{27}$$

For the term I_1, we denote $A_k(x) = A(x) - \sum_{|\gamma|=m-1}(1/\gamma!)m_{B_k}(D^\gamma A)x^\gamma$; then it is easy to check $R_m(A;x,y) = R_m(A_k;x,y)$. By the fact that $x \in C_k$, $y \in C_i$ with $i \leq k-3$, we have $|x-y| \sim |x| \sim 2^k$. As $p_2 > n$, then by Lemmas 8 and 9, we obtain

$$\begin{aligned}&\left|R_m(A_k;x,y)\right|\\ &\quad\leq \left|R_{m-1}(A_k;x,y)\right| + \sum_{|\gamma|=m-1}\frac{1}{\gamma!}\left|D^\gamma A_k(x)\right||x-y|^{m-1}\\ &\quad\leq C|x-y|^{m-1}\sum_{|\gamma|=m-1}\left\{\left(\frac{1}{|\widetilde{Q}(x,y)|} \times \int_{\widetilde{Q}(x,y)}\left|D^\gamma A_k(z)\right|^{p_2}dz\right)^{1/p_2} + \left|D^\gamma A_k(y)\right|\right\}\\ &\quad\leq C|x-y|^{m-1}\sum_{|\gamma|=m-1}\left(\left\|D^\gamma A\right\|_{C\dot{B}MO^{p_2}} + \left|D^\gamma A_k(y)\right|\right).\end{aligned} \tag{28}$$

As $|x-y| \sim |x| \sim 2^k$ and $1 - 1/p_1 - 1/p_2 - 1/r = 1/r' - 1/q - \beta/n > 0$, then by the Hölder inequality, we have

$$\begin{aligned}I_1 &\leq C\int_{C_k}\left(\sum_{i=-\infty}^{k-3}\int_{C_i}|\Omega(x-y)f(y)| \times \sum_{|\gamma|=m-1}\left(\left\|D^\gamma A\right\|_{C\dot{B}MO^{p_2}} + \left|D^\gamma A_k(y)\right|\right)dy\right)^q \times |x|^{(\beta-n)q}dx\\ &\leq C2^{-k(n-\beta)q}\int_{C_k}\left(\sum_{i=-\infty}^{k-3}\int_{C_i}|f(y)||\Omega(x-y)| \times \sum_{|\gamma|=m-1}\left(\left\|D^\gamma A\right\|_{C\dot{B}MO^{p_2}} + \left|D^\gamma A_k(y)\right|\right)dy\right)^q dx\\ &\leq C2^{-k(n-\beta)q}\int_{C_k}\sum_{i=-\infty}^{k-3}\left(\int_{C_i}|f(y)|^{p_1}dy\right)^{q/p_1} \times \left(\int_{C_i}|\Omega(x-y)|^r dy\right)^{q/r}\\ &\quad\times\left(\sum_{|\gamma|=m-1}\int_{C_i}\left(\left\|D^\gamma A\right\|_{C\dot{B}MO^{p_2}} + \left|D^\gamma A_k(y)\right|\right)^{p_2}dy\right)^{q/p_2} \times |C_i|^{q(1-1/p_1-1/p_2-1/r)}dx.\end{aligned} \tag{29}$$

As

$$\left(\int_{C_i}\left(\|D^\gamma A\|_{C\dot{B}MO^{p_2}}+|D^\gamma A_k(y)|\right)^{p_2}dy\right)^{1/p_2}$$
$$\leq\left(\int_{C_i}\left(\|D^\gamma A\|_{C\dot{B}MO^{p_2}}\right.\right.$$
$$\left.\left.+\left|D^\gamma A_k(y)-m_{B_k}(D^\gamma A)(y)\right|\right)^{p_2}dy\right)^{1/p_2}$$
$$\leq C\|D^\gamma A\|_{C\dot{B}MO^{p_2}}|C_i||k-i|, \tag{30}$$
$$\int_{C_i}|\Omega(x-y)|^r dy$$
$$\leq\int_{|x|-2^i}^{|x|+2^i}\int_{\mathbb{S}^{n-1}}|\Omega(y')|^r d\sigma(y')r^{n-1}dr\leq C2^{kn-k+i},$$

we obtain the following estimates:

$$I_1\leq C\sum_{|\gamma|=m-1}\|D^\gamma A\|^q_{C\dot{B}MO^{p_2}}|k-i|^q 2^{-knq+k\beta q}$$
$$\times 2^{in(1-1/p_1-1/p_2-1/r)q}$$
$$\times 2^{((k(n-1)+i)/r)q+(in/p_2)q}$$
$$\times\int_{C_k}\sum_{i=-\infty}^{k-3}\|f_i\|^q_{L^{p_1}}dx \tag{31}$$
$$\leq C\sum_{i=-\infty}^{k-3}\left(|k-i|2^{in(1-1/p_1-1/p_2-1/r)}\right.$$
$$\times 2^{-kn+k\beta+(k(n-1)+i)/r+in/p_2+kn/q}$$
$$\left.\times\|f_i\|_{L^{p_1}}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{C\dot{B}MO^{p_2}}\right)^q.$$

For the term I_2, we choose $\phi\in C_0^\infty(\mathbb{R}^n)$ satisfying supp$\phi\subset B(0,4)$ as well as $\phi\equiv 1$ in $B(0,2)$ and we set $L=\max\{\|D^\gamma\phi\|_{L^\infty},|\gamma|\leq m-1\}$. Let $y_0\in B_{k+4}$ and $A_k^\phi(x)=R_{m-1}(A_k;x,y_0)\phi(2^{-k}x)$. Then it is easy to see that $R_m(A;x,y)=R_m(A_k^\phi;x,y)=R_m(A_k;x,y)$ for $x\in B_k$ and $y\in B_i$ with $k-2\leq i\leq k$. Thus we get

$$\mathcal{H}^m_{\Omega,A,\beta}f_i(x)=\mathcal{H}^m_{\Omega,A_k^\phi,\beta}f_i(x)=\mathcal{H}^m_{\Omega,A_k,\beta}f_i(x). \tag{32}$$

Thus by Lemma 7, we have

$$I_2\leq C\sum_{i=k-2}^{k}\int_{C_k}\left(\frac{1}{|x|^{n-\beta}}\int_{|y|<|x|}\frac{|\Omega(x-y)f_i(y)|}{|x-y|^{m-1}}\right.$$
$$\left.\times R_m(A_k^\phi;x,y)dy\right)^q dx$$
$$\leq C\sum_{i=k-2}^{k}\left\|\mathcal{H}^m_{\Omega,A_k^\phi,\beta}f_i\right\|^q_{L^q} \tag{33}$$
$$\leq C\sum_{i=k-2}^{k}\|f_i\|^q_{L^{p_1}}\sum_{|\gamma|=m-1}\left\|D^\gamma A_k^\phi\right\|^q_{L^{p_2}}.$$

From [5, p.80], we have the following estimates:

$$\left\|D^\gamma A_k^\phi\right\|_{L^{p_2}}\leq C2^{kn/p_2}\sum_{|\gamma|=m-1}\|D^\gamma A\|_{C\dot{B}MO^{p_2}}, \tag{34}$$

where C is dependent on L. Thus we get

$$I_2\leq C\sum_{i=k-2}^{k}\|f_i\|^q_{L^{p_1}}2^{knq/p_2}\sum_{|\gamma|=m-1}\|D^\gamma A\|^q_{C\dot{B}MO^{p_2}}. \tag{35}$$

As $0<s\leq p<\infty$, we obtain the following estimates:

$$\left\{\sum_{k=-\infty}^{+\infty}2^{k\alpha_1 s}\|\mathcal{H}^m_{\Omega,A,\beta}f\chi_k\|^s_{L^q}\right\}^{1/s}$$
$$\leq\left\{\sum_{k=-\infty}^{+\infty}2^{k\alpha_1 p}\left\|\mathcal{H}^m_{\Omega,A,\beta}f\chi_k\right\|^p_{L^q}\right\}^{1/p}$$
$$\leq C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{C\dot{B}MO^{p_2}}$$
$$\times\left\{\sum_{k=-\infty}^{+\infty}2^{k\alpha_1 p}\right.$$
$$\times\sum_{i=-\infty}^{k-3}\left(|k-i|2^{in(1-1/p_1-1/p_2-1/r)}\|f_i\|_{L^{p_1}}\right.$$
$$\left.\left.\times 2^{-kn+k\beta+(k(n-1)+i)/r+in/p_2)+kn/q}\right)^p\right\}^{1/p}$$
$$+C\sum_{|\gamma|=m-1}\|D^\gamma A\|_{C\dot{B}MO^{p_2}}$$
$$\times\left\{\sum_{k=-\infty}^{+\infty}2^{k\alpha_1 p}\sum_{i=k-2}^{k}\|f_i\|^p_{L^{p_1}}2^{(kn/p_2)p}\right\}^{1/p}$$
$$=C(A+B). \tag{36}$$

For the term A, we have

$$\begin{aligned}
&A\\
&= \sum_{|\gamma|=m-1} \|D^{\gamma}A\|_{C\dot{B}MO^{p_2}}\\
&\quad\times\left\{\sum_{k=-\infty}^{+\infty} 2^{k\alpha_1 p}\right.\\
&\qquad\times\sum_{i=-\infty}^{k-3}\Big(|k-i|\,2^{in(1-1/p_1-1/p_2-1/r)}2^{-kn+k\beta}\\
&\qquad\qquad\left.\times\, 2^{(k(n-1)+i)/r+in/p_2+kn/q}\,\|f_i\|_{L^{p_1}}\Big)^{p}\right\}^{1/p}\\
&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{C\dot{B}MO^{p_2}}\\
&\quad\times\left\{\sum_{k=-\infty}^{+\infty}\sum_{i=-\infty}^{k-3}\Big(|k-i|\,2^{(k-i)(n/q-n/r'+\beta-1/r-n/p_2)}\right.\\
&\qquad\times 2^{(k-i)(\alpha_1+n/p_2)}\\
&\qquad\left.\times 2^{i(\alpha_1+n/p_2)}\|f_i\|_{L^{p_1}}\Big)^{p}\right\}^{1/p}.
\end{aligned}\tag{37}$$

When $0<p\le 1$, by condition (13), we get

$$\begin{aligned}
A^{p}&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{C\dot{B}MO^{p_2}}\\
&\quad\times\sum_{k=-\infty}^{+\infty}\Bigg(\sum_{i=-\infty}^{k-3}2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta)}\\
&\qquad\times 2^{(k-i)(\alpha_1+n/p_2)}\\
&\qquad\times|k-i|\,2^{i(\alpha_1+n/p_2)}\|f_i\|_{L^{p_1}}\Bigg)^{p}\\
&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{C\dot{B}MO^{p_2}}\\
&\quad\times\sum_{k=-\infty}^{+\infty}2^{i\alpha_2 p}\Bigg(\sum_{i=-\infty}^{k-3}|k-i|\\
&\qquad\times 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)}\\
&\qquad\times\|f_i\|_{L^{p_1}}\Bigg)^{p}\\
&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{C\dot{B}MO^{p_2}}\|f\|^{p}_{\dot{K}^{\alpha_2,p}_{p_1}}.
\end{aligned}\tag{38}$$

When $p\ge 1$, by the Hölder inequality and condition (13), we have

$$\begin{aligned}
&A^{p}\\
&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{C\dot{B}MO^{p_2}}\\
&\quad\times\sum_{k=-\infty}^{+\infty}2^{k\alpha_1 p}\Bigg(\sum_{i=-\infty}^{k-3}|k-i|\\
&\qquad\times 2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta)}\\
&\qquad\times 2^{kn/p_2}\|f_i\|_{L^{p_1}}\Bigg)^{p}\\
&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{C\dot{B}MO^{p_2}}\\
&\quad\times\sum_{k=-\infty}^{+\infty}\Bigg(\sum_{i=-\infty}^{k-3}|k-i|\,2^{k\alpha_1}2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta)}\\
&\qquad\times 2^{kn/p_2}\,\|f_i\|_{L^{p_1}}\Bigg)^{p}\\
&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{C\dot{B}MO^{p_2}}\\
&\quad\times\sum_{k=-\infty}^{+\infty}\Bigg(\sum_{i=-\infty}^{k-3}|k-i|\,2^{i\alpha_2}2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)}\\
&\qquad\times\|f_i\|_{L^{p_1}}\Bigg)^{p}\\
&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{C\dot{B}MO^{p_2}}\\
&\quad\times\sum_{k=-\infty}^{+\infty}\sum_{i=-\infty}^{k-3}2^{i\alpha_2 p}\|f_i\|^{p}_{L^{p_1}}2^{p(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)/p}\\
&\quad\times\Bigg(\sum_{i=-\infty}^{k-3}|k-i|^{p'}2^{((k-i)/p')(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)p'}\Bigg)^{p/p'}\\
&\le C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{C\dot{B}MO^{p_2}}\\
&\quad\times\sum_{i=-\infty}^{+\infty}\sum_{k=i+3}^{\infty}2^{ip(\alpha_1+n/p_2)}2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)}\|f_i\|_{L^{p_1}}\\
&\quad\times\Bigg(\sum_{i=-\infty}^{k-3}|k-i|^{p'}2^{(k-i)(n/q-n/r'-1/r-n/p_2+\beta+\alpha_2)}\Bigg)^{p/p'}
\end{aligned}$$

$$\begin{aligned}&\leq C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{\dot{C}BMO^{p_2}}\sum_{i=-\infty}^{+\infty}2^{i\alpha_2 p}\|f_i\|^{p}_{L^{p_1}}\\&\leq C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|^{p}_{\dot{C}BMO^{p_2}}\|f\|^{p}_{\dot{K}^{\alpha_2,p}_{p_1}}.\end{aligned} \tag{39}$$

For the term B, we have the following estimates:

$$\begin{aligned}B&\leq C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{C}BMO^{p_2}}\left\{\sum_{k=-\infty}^{+\infty}2^{k\alpha_1 p}\sum_{i=k-2}^{k}2^{knp/p_2}\|f_i\|^{p}_{L^{p_1}}\right\}^{1/p}\\&\leq C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{C}BMO^{p_2}}\left(\sum_{i=-\infty}^{+\infty}2^{i\alpha_2 p}\|f_i\|^{p}_{L^{p_1}}\right)^{1/p}\\&\leq C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{C}BMO^{p_2}}\|f\|_{\dot{K}^{\alpha_2,p}_{p_1}}.\end{aligned} \tag{40}$$

Combining the estimates of A and B, we finish the proof of Theorem 3. □

Proof of Theorem 4. The proof of Theorem 4 is quite similar and much easier than Theorem 3 and we omit the details here. □

3. Boundedness of $\mathscr{H}^{m}_{\Omega,A,\beta}$ on λ-Central Morrey Spaces

In [21], Wiener gave a way to describe the behavior of a function at the infinity. Later, Beurling [22] extended Weiner's idea and introduced a pair of dual Banach spaces, A^q and $B^{q'}$, with $1/q + 1/q' = 1$. In [23], Feichtinger proved that B^q can be described as

$$\|f\|_{B^q}=\sup_{k\geq 0}\left(2^{-kn/q}\|f\chi_k\|_{L^q}\right)<\infty, \tag{41}$$

where χ_0 is the characterization of the unit ball $\{x\in\mathbb{R}^n : |x|\leq 1\}$ and χ_k is defined as in Section 2.

Now by duality, the space A^q, which is called the Beurling algebra, can be described by

$$\|f\|_{A^q}=\sum_{k=0}^{\infty}2^{kn/q'}\|f\chi_k\|_{L^q}<\infty. \tag{42}$$

Later, Chen and Lau [24] as well as García-Cuerva [25] introduced atomic spaces $HA^q(\mathbb{R}^n)$ associated with the Buerling algebra A^q and the dual space of $HA^q(\mathbb{R}^n)$ can be described by

$$\begin{aligned}&\|f\|_{CBMO^q}\\&:=\sup_{R\geq 1}\left(\frac{1}{|B(0,R)|}\int_{B(0,R)}\left|f(x)-f_{B(0,R)}\right|^q dx\right)^{1/q}<\infty;\end{aligned} \tag{43}$$

here the CBMO^q can be regarded as the inhomogeneous central BMO spaces.

In 2000, Alvarez et al. [26] introduced the λ-Central bounded mean oscillation space and λ-Central Morrey space, respectively.

Definition 10 (see [26]). Given $\lambda < 1/n, 1 < q < \infty$, then a function $f\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the λ-Central bounded mean oscillation space $\dot{C}BMO^{q,\lambda}(\mathbb{R}^n)$ if

$$\begin{aligned}&\|f\|_{\dot{C}BMO^{q,\lambda}}\\&:=\sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}\left|f(x)-f_{B(0,R)}\right|^q dx\right)^{1/q}<\infty.\end{aligned} \tag{44}$$

Definition 11 (see [26]). Let $\lambda\in\mathbb{R}$ and $1<q<\infty$. Then the λ-Central Morrey space $\dot{E}^{q,\lambda}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\dot{E}^{q,\lambda}}=\sup_{R>0}\left(\frac{1}{|B(0,R)|^{1+\lambda q}}\int_{B(0,R)}|f(x)|^q dx\right)^{1/q}<\infty. \tag{45}$$

From [27], we know that if $1<q_1<q_2<\infty$, we obtain $\dot{E}^{q_2,\lambda}\subset\dot{E}^{q_1,\lambda}$ for $\lambda\in R$ and $\dot{C}BMO^{q_2,\lambda}\subset\dot{C}BMO^{q_1,\lambda}$ for $\lambda < 1/n$. Furthermore, when $\lambda < -1/q$, $\dot{E}^{q,\lambda}$ reduces to $\{0\}$ and $\dot{C}BMO^{q,\lambda}$ reduces to the space of constant functions. When $\lambda=-1/q$, $\dot{C}BMO^{q,\lambda}$ coincides with $L^q(\mathbb{R}^n)$ modulo constant and $\dot{E}^{q,\lambda}=L^q$.

In 2011, Fu et al. [19] proved the boundedness of the commutator of fractional Hardy operator with a rough kernel on λ-Central Morrey space. Later, Fu et al. [28] proved the boundness of the weighted Hardy operator and its commutator on λ-Central Morrey space. In this paper, we will give the boundedness of $\mathscr{H}^{m}_{\Omega,A,\beta}$ on λ-Central Morrey space with $m\geq 1$.

Our results can be stated as follows.

Theorem 12. *Suppose $m\geq 2$, $n<p_2<\infty, 1<p_1<\infty$, $1/q=1/p_1+1/p_2-\beta/n$ with $0\leq\beta<n$ and $\lambda=\lambda_1+\lambda_2+\beta/n$. Let $\Omega\in L^r(\mathbb{S}^{n-1})$ with $1/r'>\beta/n+1/q$, and A has derivatives of order $m-1$ in $\dot{C}BMO^{p_2,\lambda_2}$. If $r>p_1'$, $\lambda_1>-1/p_1$, and $q\lambda+1>0$, then one has*

$$\left\|\mathscr{H}^{m}_{\Omega,A,\beta}f\right\|_{\dot{E}^{q,\lambda}}\leq C\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{\dot{C}BMO^{p_2,\lambda_2}}\|f\|_{\dot{E}^{p_1,\lambda_1}}, \tag{46}$$

where the constant C is independent of f and A.

For the case $m=1$, we have the following theorem.

Theorem 13. *Suppose $1<p_1,p_2<\infty$, $1/q=1/p_1+1/p_2-\beta/n$ with $0\leq\beta<n$ and $\lambda=\lambda_1+\lambda_2+\beta/n$. Let $\Omega\in L^r(\mathbb{S}^{n-1})$ with $1/r'>\beta/n+1/q$ and $A\in\dot{C}BMO^{p_2,\lambda_2}$. If $\lambda_1>-1/p_1, r>p_1'$, and $q\lambda+1>0$, then one has*

$$\left\|\mathscr{H}^{A}_{\Omega,\beta}f\right\|_{\dot{E}^{q,\lambda}}\leq C\|A\|_{\dot{C}BMO^{p_2,\lambda_2}}\|f\|_{\dot{E}^{p_1,\lambda_1}}, \tag{47}$$

where the constant C is independent of f and A.

In order to prove Theorems 12 and 13, by a standard argument, we have the following lemma.

Lemma 14 (see [16]). *Suppose $f \in C\dot{B}MO^{p,\lambda}$, $1 \le p < \infty$, $\lambda < 1/n$, and $r_1, r_2 \in R^+$; then*

$$\begin{aligned}&\left(\frac{1}{|B(0,r_1)|^{1+p\lambda}}\int_{B(0,r_1)}\left|f(x)-f_{B(0,r_2)}\right|^p dx\right)^{1/p}\\&\quad\le C\left(1+\left|\log\frac{r_1}{r_2}\right|\right)\|f\|_{C\dot{B}MO^{p,\lambda}}.\end{aligned}\tag{48}$$

Proof of Theorem 12. For any $R > 0$, we denote $B(0,R)$ by B and $B(0,kR)$ by kB for any $k \in Z^+$. Thus we have the following estimates:

$$\begin{aligned}&\frac{1}{|B|}\int_B\left|\mathscr{H}^m_{\Omega,A,\beta}f(x)\right|^q dx\\&\le\frac{1}{|B|}\int_B\left|\frac{1}{|x|^{n-\beta}}\int_{B(0,|x|)}\frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}}\right.\\&\qquad\left.\times|R_m(A;x,y)|\,dy\right|^q dx\\&=\frac{1}{|B|}\sum_{k=-\infty}^{0}\int_{2^kB\setminus 2^{k-1}B}\left|\frac{1}{|x|^{n-\beta}}\sum_{i=-\infty}^{k}\int_{2^iB\setminus 2^{i-1}B}\frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}}\right.\\&\qquad\left.\times|R_m(A;x,y)|\,dy\right|^q dx\\&\le\frac{1}{|B|}\sum_{k=-\infty}^{0}\int_{2^kB\setminus 2^{k-1}B}\left|\frac{1}{|x|^{n-\beta}}\sum_{i=-\infty}^{k-3}\int_{2^iB\setminus 2^{i-1}B}\frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}}\right.\\&\qquad\left.\times|R_m(A;x,y)|\,dy\right|^q dx\\&\quad+\frac{1}{|B|}\\&\quad\times\sum_{k=-\infty}^{0}\int_{2^kB\setminus 2^{k-1}B}\left|\frac{1}{|x|^{n-\beta}}\sum_{i=k-2}^{k}\int_{2^iB\setminus 2^{i-1}B}\frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}}\right.\\&\qquad\left.\times|R_m(A;x,y)|\,dy\right|^q dx\\&=I+II.\end{aligned}\tag{49}$$

For the term I, let $\widetilde{A}(x) = A(x) - \sum_{|\gamma|=m-1}(1/\gamma!)m_{2^kB}(D^\gamma A)x^\gamma$. Then it is easy to see $R_m(A;x,y) = R_m(\widetilde{A};x,y)$. As $p_2 > n$, then by Lemmas 8 and 9, we have

$$\begin{aligned}\left|R_m(\widetilde{A};x,y)\right|&\le\left|R_{m-1}(\widetilde{A};x,y)\right|\\&\quad+\sum_{|\gamma|=m-1}\frac{1}{\gamma!}\left|D^\gamma\widetilde{A}(y)\right||x-y|^{m-1}\\&\le C|x-y|^{m-1}\sum_{|\gamma|=m-1}\left[\left|\widetilde{Q}\right|^{\lambda_2}\|D^\gamma A\|_{C\dot{B}MO^{p_2,\lambda_2}}\right.\\&\qquad\left.+\left|D^\gamma(\widetilde{A})(y)\right|\right],\end{aligned}\tag{50}$$

where $\widetilde{Q}(x,y)$ is the cube centered at x and having diameter $5\sqrt{n}|x-y|$.

As $x \in 2^kB\setminus 2^{k-1}B$ and $y \in 2^iB\setminus 2^{i-1}B$ with $i \le k-3$, we have $|x-y| \sim |x| \sim |2^kB|^{1/n}$ and $|x-y| \ge C_1|2^iB|^{1/n}$.

Thus by the Hölder inequality and the condition $1-1/p_1-1/p_2-1/r = 1/r' - \beta/n - 1/q > 0$, we have

$$\begin{aligned}I&\le C|B|^{-1}\\&\quad\times\sum_{k=-\infty}^{0}\int_{2^kB\setminus 2^{k-1}B}\left|\frac{1}{|x|^{n-\beta}}\right.\\&\qquad\times\sum_{i=-\infty}^{k-3}\int_{2^iB\setminus 2^{i-1}B}\frac{|\Omega(x-y)f(y)|}{|x-y|^{m-1}}\\&\qquad\times|x-y|^{m-1}\\&\qquad\times\sum_{|\gamma|=m-1}\left(\left|\widetilde{Q}\right|^{\lambda_2}\|D^\gamma A\|_{C\dot{B}MO^{p_2,\lambda_2}}\right.\\&\qquad\left.\left.+\left|D^\gamma\widetilde{A}(y)\right|\right)dy\right|^q dx\\&\le C|B|^{-1}\\&\quad\times\sum_{k=-\infty}^{0}\int_{2^kB\setminus 2^{k-1}B}\left|\frac{1}{|x|^{n-\beta}}\right.\\&\qquad\times\sum_{i=-\infty}^{k-3}\int_{2^iB\setminus 2^{i-1}B}|\Omega(x-y)f(y)|\\&\qquad\times\sum_{|\gamma|=m-1}\left(\left|\widetilde{Q}\right|^{\lambda_2}\right.\\&\qquad\times\|D^\gamma A\|_{C\dot{B}MO^{p_2,\lambda_2}}\\&\qquad\left.\left.+\left|D^\gamma\widetilde{A}(y)\right|\right)dy\right|^q dx\end{aligned}$$

$$\leq C|B|^{-1} \times \sum_{k=-\infty}^{0} \int_{2^kB\backslash 2^{k-1}B} \frac{1}{|x|^{(n-\beta)q}} \times \left| \sum_{i=-\infty}^{k-3} \left(\int_{2^iB\backslash 2^{i-1}B} |\Omega(x-y)|^r dy \right)^{1/r} \times \left(\int_{2^iB\backslash 2^{i-1}B} \sum_{|\gamma|=m-1} \left(|x-y|^{\lambda_2 n} \times \|D^\gamma A\|_{C\dot{B}MO^{p_2,\lambda_2}} + \left| D^\gamma \widetilde{A}(y) \right| \right)^{p_2} dy \right)^{1/p_2} \times \left(\int_{2^iB\backslash 2^{i-1}B} |f(y)|^{p_1} dy \right)^{1/p_1} \times \left| 2^iB \right|^{1-1/p_1-1/p_2-1/r} \right|^q dx. \tag{51}$$

Note the following fact:

$$\begin{aligned}&\left(\int_{2^iB\backslash 2^{i-1}B} \sum_{|\gamma|=m-1} \left(|x-y|^{\lambda_2 n} \|D^\gamma A\|_{C\dot{B}MO^{p_2,\lambda_2}} + \left| D^\gamma \widetilde{A}(y) \right| \right)^{p_2} dy \right)^{1/p_2} \\ &\leq C \sum_{|\gamma|=m-1} \left[\|D^\gamma A\|_{C\dot{B}MO^{p_2,\lambda_2}} \left|2^kB\right|^{\lambda_2} \left|2^iB\right|^{1/p_2} + \left(\int_{2^iB} \left| D^\gamma A(y) - m_{\widetilde{Q}}(D^\gamma A) \right|^{p_2} dy \right)^{1/p_2} \right] \\ &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{C\dot{B}MO^{p_2,\lambda_2}} \left|2^iB\right|^{1/p_2} \left(\left|2^kB\right|^{\lambda_2} + \left|2^iB\right|^{\lambda_2} \right),\end{aligned} \tag{52}$$

where the last inequality follows from Lemma 14 and the fact $|x-y| \geq C_1|2^iB|^{1/n}$. Thus by the condition $1+q\lambda > 0$ and $\lambda_1 + 1 > \lambda_1 + 1/p_1 > 0$, we have

$$\begin{aligned}I &\leq C|B|^{-1} \times \sum_{k=-\infty}^{0} \left|2^kB\right|^{(\beta/n-1)q} \times \int_{2^kB\backslash 2^{k-1}B} \left(\sum_{i=-\infty}^{k-3} \|f\|_{\dot{E}^{p_1,\lambda_1}} \left|2^iB\right|^{1/p_1+\lambda_1} \left|2^iB\right|^{1/r} \times \sum_{|\gamma|=m-1} \|D^\gamma A\|_{C\dot{B}MO^{p_2,\lambda_2}} \left|2^iB\right|^{1/p_2} \times \left(\left|2^kB\right|^{\lambda_2} + \left|2^iB\right|^{\lambda_2} \right) \times \left|2^iB\right|^{1-1/p_1-1/p_2-1/r} \right)^q dx \\ &\leq C\|f\|^q_{\dot{E}^{p_1,\lambda_1}} \sum_{|\gamma|=m-1} \|D^\gamma A\|^q_{C\dot{B}MO^{p_2,\lambda_2}} |B|^{-1} \times \sum_{k=-\infty}^{0} \left|2^kB\right|^{(\beta/n-1)q} \left|2^kB\right| \left|2^kB\right|^{\lambda_2 q} \times \left(\sum_{i=-\infty}^{k-3} \left|2^iB\right|^{1/p_1+\lambda_1} \times \left|2^iB\right|^{1/p_2} \left|2^iB\right|^{1/r} \left|2^iB\right|^{1-1/p_1-1/p_2-1/r} \right)^q \\ &\leq C\|f\|^q_{\dot{E}^{p_1,\lambda_1}} \sum_{|\gamma|=m-1} \|D^\gamma A\|^q_{C\dot{B}MO^{p_2,\lambda_2}} \times |B|^{-1+\beta q/n-q+\lambda_2 q+1+\lambda_1 q+q} \times \sum_{k=-\infty}^{0} 2^{nk\beta q/n-knq+kn+k\lambda_2 nq+k\lambda_1 nq+knq} \\ &\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|^q_{C\dot{B}MO^{p_2,\lambda_2}} \|f\|^q_{\dot{E}^{p_1,\lambda_1}} |B|^{q\lambda}.\end{aligned} \tag{53}$$

To estimate the term II, we adopt some basic ideas from the estimates of the term I_2 in Theorem 3. First, we denote $R_m(A;x,y) = R_m(A_k^\phi;x,y) = R_m(A_k;x,y)$ for $x \in 2^kB$ and $y \in 2^iB$ with $k-2 \leq i \leq k$, where $A_k^\phi(x) = R_{m-1}(A_k;,x,y_0)\phi(|x-y_0|^{-1}x)$ with $y_0 \in 2^iB \setminus 2^{i-1}B$. Here $A_k(x) = A(x) - \sum_{|\gamma|=m-1}(1/\gamma!)m_{2^kB}(D^\gamma A)x^\gamma$ and ϕ is defined as in Section 2.

As $\Omega \in L^r(\mathbb{S}^{n-1})$ with $r > p_1'$, by Lemma 7, we get

$$\begin{aligned}II &\leq \frac{1}{|B|} \sum_{k=-\infty}^{0} \sum_{i=k-2}^{k} \left\| \mathscr{H}^m_{\Omega,A_k^\phi,\beta} f_i \right\|_q^q \\ &\leq \frac{C}{|B|} \sum_{k=-\infty}^{0} \sum_{i=k-2}^{k} \|f_i\|^q_{L^{p_1}} \sum_{|\gamma|=m-1} \left\|D^\gamma A_k^\phi\right\|^q_{L^{p_2}},\end{aligned} \tag{54}$$

where f_i is defined by $f_i(x) = f(x)\chi_{2^iB\backslash 2^{i-1}B}(x)$.

For $D^{\gamma}A_k^{\phi}$, as $|x-y_0|\le C|y_0|$, then by Lemmas 8 and 14, we have the following estimates:

$$\begin{aligned}
&\left|D^{\gamma}A_k^{\phi}(x)\right|\\
&\le \sum_{|\mu|+|\nu|=m-1} C_{\mu\nu}\left|R_{m-1-|\mu|}\left(D^{\mu}A_k;x,y_0\right)\right|\\
&\qquad\times\left|D^{\nu}\phi\left(\left|x-y_0\right|^{-1}x\right)\right|\left|x-y_0\right|^{-|\nu|}\\
&\le C\sum_{|\mu|+|\nu|=m-1}\left|x-y_0\right|^{m-1-|\mu|-|\nu|}\\
&\quad\times\sum_{|\gamma'|=m-1-|\mu|}\left(\frac{1}{\left|\widetilde{Q}(x,y_0)\right|}\int_{\widetilde{Q}(x,y_0)}\left|D^{\gamma'}\left(D^{\mu}A_k\right)(z)\right|^{p_2}dz\right)^{1/p_2}\\
&\qquad\times\chi_{|x|\le|x-y_0|}(x)\\
&\le C\sum_{|\gamma|=m-1}\left\|D^{\gamma}A\right\|_{C\dot{B}MO^{p_2,\lambda_2}}\left|x-y_0\right|^{\lambda_2 n}\chi_{|x|\le C|y_0|}(x).
\end{aligned}\tag{55}$$

Thus we have

$$\begin{aligned}
&\sum_{|\gamma|=m-1}\left\|D^{\gamma}A_k^{\phi}\right\|_{L^{p_2}}^q\\
&\qquad\le C\sum_{|\gamma|=m-1}\left\|D^{\gamma}A\right\|_{C\dot{B}MO^{p_2,\lambda_2}}^q\left|2^iB\right|^{q\lambda_2}\left|2^iB\right|^{q/p_2}.
\end{aligned}\tag{56}$$

By the above estimates, we obtain

$$\begin{aligned}
II&\le\frac{C}{|B|}\sum_{k=-\infty}^{0}\sum_{i=k-2}^{k}\left(\frac{1}{|2^iB|^{1+p_1\lambda_1}}\int_{2^iB}\left|f(x)\right|^{p_1}dx\right)^{q/p_1}\\
&\qquad\qquad\times\left|2^iB\right|^{q/p_1+q\lambda_1}\\
&\quad\times\sum_{|\gamma|=m-1}\left\|D^{\gamma}A\right\|_{C\dot{B}MO^{p_2,\lambda_2}}^q\left|2^iB\right|^{q\lambda_2+q/p_2}\\
&\le C\sum_{k=-\infty}^{0}\sum_{i=k-2}^{k}2^{in(1+\lambda q)}\\
&\quad\times\sum_{|\gamma|=m-1}\left\|D^{\gamma}A\right\|_{C\dot{B}MO^{p_2,\lambda_2}}^q\left\|f\right\|_{\dot{E}^{p_1,\lambda_1}}^q|B|^{q\lambda}\\
&\le C\sum_{|\gamma|=m-1}\left\|D^{\gamma}A\right\|_{C\dot{B}MO^{p_2,\lambda_2}}^q\left\|f\right\|_{\dot{E}^{p_1,\lambda_1}}^q|B|^{q\lambda}.
\end{aligned}\tag{57}$$

Combining the estimates of I and II and by the definition of $\dot{E}^{q,\lambda}$, we finish the proof of Theorem 12. □

Proof of Theorem 13. The proof of Theorem 13 is quite similar but much simpler than Theorem 12 and we omit the details here. □

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