Research Article

Existence of Solutions for a Modified Nonlinear Schrödinger System

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We are concerned with the following modified nonlinear Schrödinger system: $-\Delta u + u - (1/2)u\Delta(u^2) = (2\alpha/(\alpha+\beta))|u|^{\alpha-2}|v|^{\beta}u$, $x \in \Omega$, $-\Delta v + v - (1/2)v\Delta(v^2) = (2\beta/(\alpha+\beta))|u|^{\alpha}|v|^{\beta-2}v$, $x \in \Omega$, $u = 0$, $v = 0$, $x \in \partial\Omega$, where $\alpha > 2$, $\beta > 2$, $\alpha + \beta < 2 \cdot 2^*$, $2^* = 2N/(N-2)$ is the critical Sobolev exponent, and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain. By using the perturbation method, we establish the existence of both positive and negative solutions for this system.

1. Introduction

Let us consider the following modified nonlinear Schrödinger system:

$$-\Delta u + u - \frac{1}{2}u\Delta\left(u^2\right) = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^{\beta}u, \quad x \in \Omega,$$

$$-\Delta v + v - \frac{1}{2}v\Delta\left(v^2\right) = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, \quad x \in \Omega, \quad (1)$$

$$u = 0, \qquad v = 0, \quad x \in \partial\Omega,$$

where $\alpha > 2$, $\beta > 2$, $\alpha + \beta < 2 \cdot 2^*$, $2^* = 2N/(N-2)$ is the critical Sobolev exponent, and $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded smooth domain.

Solutions for the system (1) are related to the existence of the standing wave solutions of the following quasilinear Schrödinger equation:

$$i\partial_t z = -\Delta z + V(x)z - f\left(|z|^2\right)z - k\Delta h\left(|z|^2\right)h'\left(|z|^2\right)z,$$
$$x \in \mathbb{R}^N, \quad (2)$$

where $V(x)$ is a given potential, k is a real constant, and f, h are real functions. We would like to mention that (2) appears more naturally in mathematical physics and has been derived as models of several physical phenomena corresponding to various types of h. For instance, the case $h(s) = s$ was used for the superfluid film equation in plasma physics by Kurihara [1] (see also [2]); in the case of $h(s) = (1 + s)^{1/2}$, (2) was used as a model of the self-changing of a high-power ultrashort laser in matter (see [3–6] and references therein).

In recent years, much attention has been devoted to the quasilinear Schrödinger equation of the following form:

$$-\Delta u + \lambda V(x)u - k\Delta\left(u^2\right)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N. \quad (3)$$

See, for example, by using a constrained minimization argument, the existence of positive ground state solution was proved by Poppenberg et al. [7]. Using a change of variables, Liu et al. [8] used an Orlicz space to prove the existence of soliton solution for (3) via mountain pass theorem. Colin and Jeanjean [9] also made use of a change of variables but worked in the Sobolev space $H^1(R^N)$; they proved the existence of positive solution for (3) from the classical results given by Berestycki and Lions [10]. Liu et al. [11] established the existence of both one-sign and nodal ground states of soliton type solutions for (3) by the Nehari method. In particular, in [12], by using Nehari manifold method and concentration compactness principle (see [13]) in the Orlicz space, Guo

and Tang considered the following quasilinear Schrödinger system:

$$-\Delta u + (\lambda a(x) + 1) u - \frac{1}{2} \left(\Delta |u|^2\right) u = \frac{2\alpha}{\alpha + \beta} |u|^{\alpha-2} |v|^\beta u,$$

$$x \in R^N,$$

$$-\Delta u + (\lambda b(x) + 1) u - \frac{1}{2} \left(\Delta |u|^2\right) u = \frac{2\beta}{\alpha + \beta} |u|^\alpha |v|^{\beta-2} v,$$

$$x \in R^N,$$

$$u(x) \longrightarrow 0, \quad v(x) \longrightarrow 0, \quad |x| \longrightarrow \infty \tag{4}$$

with $a(x) \geq 0$, $b(x) \geq 0$ having a potential well and $\alpha > 2$, $\beta > 2$, $\alpha + \beta < 2 \cdot 2^*$, where $2^* = 2N/(N - 2)$ is the critical Sobolev exponent, and they proved the existence of a ground state solution for the system (4) which localizes near the potential well int $a^{-1}(0)$ for λ large enough. Guo and Tang [14] also considered ground state solutions of the single quasilinear Schrödinger equation corresponding to the system (4) by the same methods and obtained similar results.

It is worth pointing out that the existence of one-bump or multibump bound state solutions for the related semilinear Schrödinger equation (3) for $k = 0$ has been extensively studied. One can see Bartsch and Wang [15], Ambrosetti et al. [16], Ambrosetti et al. [17], Byeon and Wang [18], Cingolani and Lazzo [19], Cingolani and Nolasco [20], Del Pino and Felmer [21, 22], Floer and Weinstein [23], and Oh [24, 25] and the references therein.

The system (1) is a kind of "limit" problem of the system (4) as $\lambda \to \infty$. The existence of solutions for the system (1) has important physical interest. The purpose of this paper is to study the existence of both positive and negative solutions for the system (1). We mainly follow the idea of Liu et al. [26] to perturb the functional and obtain our main results. We point out that the procedure to the system (1) is not trivial at all. Since the appearance of the quasilinear terms $u\Delta(u^2)$ and $v\Delta(v^2)$, we need more delicate estimates.

The paper is organized as follows. In Section 2, we introduce a perturbation of the functional and give our main results (Theorem 1 and 2). In Section 3, we verify the Palais-Smale condition for the perturbed functional. Section 4 is devoted to some asymptotic behavior of sequence $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ and $\{\mu_n\} \subset (0, 1]$ satisfying some conditions. Finally, our main results will be proved in Section 5.

Throughout this paper, we will use the same C to denote various generic positive constants, and we will use $o(1)$ to denote quantities that tend to 0.

2. Perturbation of the Functional and Main Results

In order to obtain the desired existence of solutions for the system (1), in this section, we introduce a perturbation of the functional and give our main results.

The weak form of the system (1) is

$$\int_\Omega \left(\left(1 + u^2\right) \nabla u \nabla \varphi + \left(1 + |\nabla u|^2\right) u\varphi\right)$$

$$+ \int_\Omega \left(\left(1 + v^2\right) \nabla v \nabla \psi + \left(1 + |\nabla v|^2\right) v\psi\right)$$

$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha-2} |v|^\beta u\varphi - \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta-2} v\psi = 0,$$

$$(\varphi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega), \tag{5}$$

which is formally the variational formulation of the following functional:

$$I_0(u, v) = \frac{1}{2} \int_\Omega \left(\left(1 + u^2\right) |\nabla u|^2 + u^2\right)$$

$$+ \frac{1}{2} \int_\Omega \left(\left(1 + v^2\right) |\nabla v|^2 + v^2\right) - \frac{2}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^\beta. \tag{6}$$

We may define the derivative of I_0 at (u, v) in the direction of $(\varphi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ as follows:

$$\left\langle I_0'(u, v), (\varphi, \psi) \right\rangle = \int_\Omega \left(1 + u^2\right) \nabla u \nabla \varphi$$

$$+ \int_\Omega \left(1 + |\nabla u|^2\right) u\varphi + \int_\Omega \left(1 + v^2\right) \nabla v \nabla \psi$$

$$+ \int_\Omega \left(1 + |\nabla v|^2\right) v\psi$$

$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha-2} |v|^\beta u\varphi$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta-2} v\psi. \tag{7}$$

We call that (u, v) is a critical point of I_0 if $(u, v) \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, $\int_\Omega u^2 |\nabla u|^2 < \infty$, $\int_\Omega v^2 |\nabla v|^2 < \infty$ and $\langle I_0'(u, v), (\varphi, \psi) \rangle = 0$ for all $(\varphi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. That is, (u, v) is a weak solution for the system (1).

When we consider the system (1) by using the classical critical point theory, we encounter the difficulties due to the lack of an appropriate working space. In general it seems there is no suitable space in which the variational functional I_0 possesses both smoothness and compactness properties. For smoothness one would need to work in a space smaller than $W_0^{1,2}(\Omega)$ to control the term involving the quasilinear term in the system (1), but it seems impossible to obtain bounds for $(PS)_c$ sequence in this setting. There have been several ideas and approaches used in recent years to overcome the difficulties such as by minimizations [7, 27], the Nehari

method [11], and change of variables [8, 9]. In this paper, we consider a perturbed functional

$$
\begin{aligned}
I_\mu (u, v) &= \frac{1}{4} \mu \int_\Omega \left(|\nabla u|^4 + |\nabla v|^4 \right) + I_0 (u, v) \\
&= \frac{1}{4} \mu \int_\Omega \left(|\nabla u|^4 + |\nabla v|^4 \right) + \frac{1}{2} \int_\Omega \left(\left(1 + u^2 \right) |\nabla u|^2 + u^2 \right) \\
&\quad + \frac{1}{2} \int_\Omega \left(\left(1 + v^2 \right) |\nabla v|^2 + v^2 \right) - \frac{2}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^\beta,
\end{aligned}
\tag{8}
$$

where $\mu \in (0, 1]$ is a parameter. Then it is easy to see that I_μ is a C^1-functional on $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. We also can define the derivative of I_μ at (u, v) in the direction of (φ, ψ) as follows:

$$
\begin{aligned}
\langle I_\mu' (u, v), (\varphi, \psi) \rangle &= \mu \int_\Omega |\nabla u|^2 \nabla u \nabla \varphi + \mu \int_\Omega |\nabla v|^2 \nabla v \nabla \psi \\
&\quad + \int_\Omega \left(1 + u^2 \right) \nabla u \nabla \varphi + \int_\Omega \left(1 + |\nabla u|^2 \right) u \varphi \\
&\quad + \int_\Omega \left(1 + v^2 \right) \nabla v \nabla \psi + \int_\Omega \left(1 + |\nabla v|^2 \right) v \psi \\
&\quad - \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha-2} |v|^\beta u \varphi \\
&\quad - \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta-2} v \psi
\end{aligned}
\tag{9}
$$

for all $(\varphi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. The idea is to obtain the existence of the critical points of I_μ for $\mu > 0$ small and to establish suitable estimates for the critical points as $\mu \to 0$ so that we may pass to the limit to get the solutions for the original system (1).

Our main results are as follows.

Theorem 1. *Assume that $\alpha > 2$, $\beta > 2$ and $\alpha + \beta < 2 \cdot 2^*$. Let $\mu_n \to 0$ and let $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ be a sequence of I_{μ_n} satisfying $I_{\mu_n}'(u_n, v_n) = 0$ and $I_{\mu_n}(u_n, v_n) \le C$ for some C independent of n. Then, up to a subsequence*

$$
\begin{aligned}
&u_n \longrightarrow u, \quad v_n \longrightarrow v \quad \text{in } W_0^{1,2}(\Omega), \\
&u_n \nabla u_n \longrightarrow u \nabla u, \quad v_n \nabla v_n \longrightarrow v \nabla v \quad \text{in } L^2(\Omega), \\
&\mu_n \int_\Omega \left(|\nabla u_n|^4 + |\nabla v_n|^4 \right) \longrightarrow 0, \\
&I_{\mu_n}'(u_n, v_n) \longrightarrow I_0'(u, v)
\end{aligned}
\tag{10}
$$

as $n \to \infty$ and (u, v) is a critical point of I_0.

Using Theorem 1, we have the following existence result.

Theorem 2. *Assume that $\alpha > 2$, $\beta > 2$ and $\alpha + \beta < 2 \cdot 2^*$. Then I_μ has a positive critical point (u_μ, v_μ) and a negative critical point $(\tilde{u}_\mu, \tilde{v}_\mu)$, and (u_μ, v_μ) (resp., $(\tilde{u}_\mu, \tilde{v}_\mu)$) converges to a positive (resp., negative) solution for the system (1) as $\mu \to 0$.*

Notation. We denote by $\| \cdot \|$ the norm of $W_0^{1,4}(\Omega)$ and by $| \cdot |_s$ the norm of $L^s(\Omega)$ $(1 \le s < +\infty)$.

3. Compactness of the Perturbed Functional

In this section, we verify the Palais-Smale condition $((PS)_c$ condition in short) for the perturbed functional $I_\mu(u, v)$. We have the following proposition.

Proposition 3. *For $\mu > 0$ fixed, the functional $I_\mu(u, v)$ satisfies $(PS)_c$ condition for all $c \in \mathbb{R}$. That is, any sequence $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ satisfying, for $c \in \mathbb{R}$,*

$$
I_\mu (u_n, v_n) \longrightarrow c,
$$

$$
I_\mu' (u_n, v_n) \longrightarrow 0 \quad \text{strongly in } \left(W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega) \right)^*
\tag{11}
$$

has a strongly convergent subsequence in $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$, where $\left(W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega) \right)^*$ is the dual space of $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$.

For giving the proof of Proposition 3, we need the following lemma firstly.

Lemma 4. *Suppose that a sequence $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ satisfies (11). Then*

$$
\limsup_{n \to \infty} \| (u_n, v_n) \|^4 \le \left(\frac{1}{4} - \frac{1}{\alpha + \beta} \right)^{-1} \mu^{-1} c.
\tag{12}
$$

Proof. It follows from (11) that

$$
\begin{aligned}
&c + o(1) - \frac{1}{\alpha + \beta} o(1) \| (u_n, v_n) \| \\
&= I_\mu (u_n, v_n) - \frac{1}{\alpha + \beta} \langle I_\mu' (u_n, v_n), (u_n, v_n) \rangle \\
&= \left(\frac{1}{4} - \frac{1}{\alpha + \beta} \right) \mu \int_\Omega \left(|\nabla u_n|^4 + |\nabla v_n|^4 \right) \\
&\quad + \left(\frac{1}{2} - \frac{1}{\alpha + \beta} \right) \int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) \\
&\quad + \left(\frac{1}{2} - \frac{1}{\alpha + \beta} \right) \int_\Omega \left(|u_n|^2 + |v_n|^2 \right) \\
&\quad + \left(\frac{1}{2} - \frac{2}{\alpha + \beta} \right) \int_\Omega \left(u_n^2 |\nabla u_n|^2 + v_n^2 |\nabla v_n|^2 \right) \\
&\ge \left(\frac{1}{4} - \frac{1}{\alpha + \beta} \right) \mu \int_\Omega \left(|\nabla u|^4 + |\nabla v|^4 \right).
\end{aligned}
\tag{13}
$$

Thus we have

$$
\limsup_{n \to \infty} \| (u_n, v_n) \|^4 \le \left(\frac{1}{4} - \frac{1}{\alpha + \beta} \right)^{-1} \mu^{-1} c.
\tag{14}
$$

This completes the proof of Lemma 4. □

Now we give the proof of Proposition 3.

Proof of Proposition 3. From Lemma 4, we know that $\{(u_n, v_n)\}$ is bounded in $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. So there exists a subsequence of $\{(u_n, v_n)\}$, still denoted $\{(u_n, v_n)\}$, such that

$$
\begin{aligned}
(u_n, v_n) &\rightharpoonup (u, v) \quad \text{weakly in } W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{as } n \longrightarrow \infty, \\
u_n &\longrightarrow u, \qquad v_n \longrightarrow v \quad \text{strongly in } L^s(\Omega) \\
&\text{as } n \longrightarrow \infty \quad \text{for any } 2 < s < 2 \cdot 2^*.
\end{aligned}
\tag{15}
$$

Now we prove that $(u_n, v_n) \to (u, v)$ in $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. In (9), choosing $(\varphi, \psi) = (u_n - u_m, v_n - v_m)$, we have

$$
\begin{aligned}
&o(1) \left\| (u_n - u_m, v_n - v_m) \right\| \\
&= \left\langle I'_\mu(u_n, v_n) - I'_\mu(u_m, v_m), (u_n - u_m, v_n - v_m) \right\rangle \\
&= \mu \int_\Omega \left(|\nabla u_n|^2 \nabla u_n - |\nabla u_m|^2 \nabla u_m \right)(\nabla u_n - \nabla u_m) \\
&\quad + \mu \int_\Omega \left(|\nabla v_n|^2 \nabla v_n - |\nabla v_m|^2 \nabla v_m \right)(\nabla v_n - \nabla v_m) \\
&\quad + \int_\Omega |u_n - u_m|^2 + \int_\Omega |\nabla u_n - \nabla u_m|^2 \\
&\quad + \int_\Omega \left(u_n^2 \nabla u_n - u_m^2 \nabla u_m \right)(\nabla u_n - \nabla u_m) + \int_\Omega |v_n - v_m|^2 \\
&\quad + \int_\Omega |\nabla v_n - \nabla v_m|^2 + \int_\Omega \left(v_n^2 \nabla v_n - v_m^2 \nabla v_m \right)(\nabla v_n - \nabla v_m) \\
&\quad + \int_\Omega \left(u_n |\nabla u_n|^2 - u_m |\nabla u_m|^2 \right)(u_n - u_m) \\
&\quad + \int_\Omega \left(v_n |\nabla v_n|^2 - v_m |\nabla v_m|^2 \right)(v_n - v_m) \\
&\quad - \frac{2\alpha}{\alpha + \beta} \int_\Omega \left(|u_n|^{\alpha-2} |v_n|^\beta u_n - |u_m|^{\alpha-2} |v_m|^\beta u_m \right)(u_n - u_m) \\
&\quad - \frac{2\beta}{\alpha + \beta} \int_\Omega \left(|u_n|^\alpha |v_n|^{\beta-2} v_n - |u_m|^\alpha |v_m|^{\beta-2} v_m \right)(v_n - v_m).
\end{aligned}
\tag{16}
$$

We may estimate the terms involved as follows:

$$
\mu \int_\Omega \left(|\nabla u_n|^2 \nabla u_n - |\nabla u_m|^2 \nabla u_m \right)(\nabla u_n - \nabla u_m)
$$
$$
\geq \mu \int_\Omega |\nabla u_n - \nabla u_m|^4,
$$
$$
\mu \int_\Omega \left(|\nabla v_n|^2 \nabla v_n - |\nabla v_m|^2 \nabla v_m \right)(\nabla v_n - \nabla v_m)
$$
$$
\geq \mu \int_\Omega |\nabla v_n - \nabla v_m|^4,
$$

$$
\int_\Omega \left(u_n^2 \nabla u_n - u_m^2 \nabla u_m \right)(\nabla u_n - \nabla u_m)
$$
$$
\geq \int_\Omega u_n^2 |\nabla u_n - \nabla u_m|^2 + \int_\Omega \left(u_n^2 - u_m^2 \right) \nabla u_m (\nabla u_n - \nabla u_m)
$$
$$
\geq -|u_n - u_m|_4 \left(|u_n|_4 + |u_m|_4 \right) \|u_m\| \left(\|u_n\| + \|u_m\| \right)
$$
$$
\longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty,
$$

$$
\int_\Omega \left(v_n^2 \nabla v_n - v_m^2 \nabla v_m \right)(\nabla v_n - \nabla v_m)
$$
$$
\geq \int_\Omega v_n^2 |\nabla v_n - \nabla v_m|^2 + \int_\Omega \left(v_n^2 - v_m^2 \right) \nabla v_m (\nabla v_n - \nabla v_m)
$$
$$
\geq -|v_n - v_m|_4 \left(|v_n|_4 + |v_m|_4 \right) \|v_m\| \left(\|v_n\| + \|v_m\| \right)
$$
$$
\longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty,
$$

$$
\left| \int_\Omega \left(u_n |\nabla u_n|^2 - u_m |\nabla u_m|^2 \right)(u_n - u_m) \right|
$$
$$
\leq \left(|u_n|_4 \|u_n\|^2 + |u_m|_4 \|u_m\|^2 \right) |u_n - u_m|_4
$$
$$
\longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty,
$$

$$
\left| \int_\Omega \left(v_n |\nabla v_n|^2 - v_m |\nabla v_m|^2 \right)(v_n - v_m) \right|
$$
$$
\leq \left(|v_n|_4 \|v_n\|^2 + |v_m|_4 \|v_m\|^2 \right) |v_n - v_m|_4
$$
$$
\longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty,
$$

$$
\frac{2\alpha}{\alpha + \beta} \left| \int_\Omega \left(|u_n|^{\alpha-2} |v_n|^\beta u_n - |u_m|^{\alpha-2} |v_m|^\beta u_m \right)(u_n - u_m) \right|
$$
$$
\leq \frac{2\alpha}{\alpha + \beta} \int_\Omega \left(|u_n|^{\alpha-1} |v_n|^\beta + |u_m|^{\alpha-1} |v_m|^\beta \right) |u_n - u_m|
$$
$$
\leq \frac{2\alpha}{\alpha + \beta} \left(|u_n|_{\alpha+\beta}^{\alpha-1} |v_n|_{\alpha+\beta}^\beta + |u_m|_{\alpha+\beta}^{\alpha-1} |v_m|_{\alpha+\beta}^\beta \right) |u_n - u_m|_{\alpha+\beta}
$$
$$
\longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty,
$$

$$
\frac{2\beta}{\alpha + \beta} \left| \int_\Omega \left(|u_n|^\alpha |v_n|^{\beta-2} v_n - |u_m|^\alpha |v_m|^{\beta-2} v_m \right)(v_n - v_m) \right|
$$
$$
\leq \frac{2\beta}{\alpha + \beta} \int_\Omega \left(|u_n|^\alpha |v_n|^{\beta-1} + |u_m|^\alpha |v_m|^{\beta-1} \right) |v_n - v_m|
$$
$$
\leq \frac{2\beta}{\alpha + \beta} \left(|u_n|_{\alpha+\beta}^\alpha |v_n|_{\alpha+\beta}^{\beta-1} + |u_m|_{\alpha+\beta}^\alpha |v_m|_{\alpha+\beta}^{\beta-1} \right) |v_n - v_m|_{\alpha+\beta}
$$
$$
\longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty.
\tag{17}
$$

Returning to (16), we have

$$
\mu \int_\Omega \left(|\nabla u_n - \nabla u_m|^4 + |\nabla v_n - \nabla v_m|^4 \right)
$$
$$
\leq o(1) \left\| (u_n - u_m, v_n - v_m) \right\| + o(1),
\tag{18}
$$

which implies that $\|(u_n - u_m, v_n - v_m)\| \to 0$, that is, $(u_n, u_m) \to (u, v)$ in $W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. This completes the proof of Proposition 3. □

4. Some Asymptotic Behavior

Proposition 3 enables us to apply minimax argument to the functional $I_\mu(u, v)$. In this section, we also study the behavior of sequence $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ and $\{\mu_n\} \subset (0, 1]$ satisfying

$$\mu_n \longrightarrow 0, \qquad I_{\mu_n}(u_n, v_n) \longrightarrow c,$$
$$\left\| I'_{\mu_n}(u_n, v_n) \right\|^* \longrightarrow 0. \tag{19}$$

The following proposition is the key of this section.

Proposition 5. *Assume sequence* $\{(u_n, v_n)\} \subset W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ *and* $\{\mu_n\} \subset (0, 1]$ *satisfy* (19). *Then after extracting a sequence, still denoted by* n, *one has*

$$(u_n, v_n) \rightharpoonup (u, v) \quad in \ W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega),$$

$$(u_n \nabla u_n, v_n \nabla v_n) \rightharpoonup (u \nabla u, v \nabla v) \quad in \ L^2(\Omega) \times L^2(\Omega), \tag{20}$$

$$(u_n(x), v_n(x)) \longrightarrow (u(x), v(x)) \quad a.e. \ x \in \Omega$$

as $n \to \infty$.

Proof. Similar to the proof of Lemma 4, by (19), we have

$$C \geq I_{\mu_n}(u_n, v_n) - \frac{1}{\alpha + \beta} \left\langle I'_{\mu_n}(u_n, v_n), (u_n, v_n) \right\rangle$$

$$\geq \left(\frac{1}{4} - \frac{1}{\alpha + \beta} \right) \mu_n \int_\Omega \left(|\nabla u_n|^4 + |\nabla v_n|^4 \right)$$

$$+ \left(\frac{1}{2} - \frac{1}{\alpha + \beta} \right) \int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right) \tag{21}$$

$$+ \left(\frac{1}{2} - \frac{1}{\alpha + \beta} \right) \int_\Omega \left(|u_n|^2 + |v_n|^2 \right)$$

$$+ \left(\frac{1}{2} - \frac{2}{\alpha + \beta} \right) \int_\Omega \left(u_n^2 |\nabla u_n|^2 + v_n^2 |\nabla v_n|^2 \right).$$

Thus

$$\mu_n \int_\Omega \left(|\nabla u_n|^4 + |\nabla v_n|^4 \right) + \int_\Omega \left(|\nabla u_n|^2 + |\nabla v_n|^2 \right)$$

$$+ \int_\Omega \left(|u_n|^2 + |v_n|^2 \right) + \int_\Omega \left(u_n^2 |\nabla u_n|^2 + v_n^2 |\nabla v_n|^2 \right) \leq C \tag{22}$$

for some C independent of n. Then, up to a subsequence, we have

$$(u_n, v_n) \rightharpoonup (u, v) \quad in \ W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega),$$

$$(u_n \nabla u_n, v_n \nabla v_n) \rightharpoonup (u \nabla u, v \nabla v) \quad in \ L^2(\Omega) \times L^2(\Omega), \tag{23}$$

$$(u_n(x), v_n(x)) \longrightarrow (u(x), v(x)) \quad a.e. \ x \in \Omega$$

as $n \to \infty$. This completes the proof of Proposition 5. □

5. Proof of Main Results

In this section, we give the proof of our main results. Firstly, we prove Theorem 1.

Proof of Theorem 1. Note that (u_n, v_n) satisfies the following equation:

$$\mu_n \int_\Omega |\nabla u_n|^2 \nabla u_n \nabla \varphi + \mu_n \int_\Omega |\nabla v_n|^2 \nabla v_n \nabla \psi$$

$$+ \int_\Omega \left(\left(1 + u_n^2 \right) \nabla u_n \nabla \varphi + \left(1 + |\nabla u_n|^2 \right) u_n \varphi \right)$$

$$+ \int_\Omega \left(\left(1 + v_n^2 \right) \nabla v_n \nabla \psi + \left(1 + |\nabla v_n|^2 \right) v_n \psi \right) \tag{24}$$

$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u_n|^{\alpha-2} |v_n|^\beta u_n \varphi$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u_n|^\alpha |v_n|^{\beta-2} v_n \psi = 0$$

for all $(\varphi, \psi) \in W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$. Since

$$\left(\int_\Omega |u_n|^{4N/(N-2)} \right)^{(N-2)/N} \leq C \int_\Omega u_n^2 |\nabla u_n|^2 \leq C,$$

$$\left(\int_\Omega |v_n|^{4N/(N-2)} \right)^{(N-2)/N} \leq C \int_\Omega v_n^2 |\nabla v_n|^2 \leq C. \tag{25}$$

By Moser's iteration, we have

$$|u_n|_{L^\infty} \leq C, \qquad |v_n|_{L^\infty} \leq C. \tag{26}$$

Hence,

$$|u|_{L^\infty} \leq C, \qquad |v|_{L^\infty} \leq C \tag{27}$$

for some C independent of n. To show that (u, v) is a critical point of I_0 we use some arguments in [28, 29] (see more references therein). In (24) we choose $\varphi = \xi \exp(-u_n), \psi = \eta \exp(-v_n)$, where $\xi \in C_0^\infty(\Omega), \xi \geq 0, \eta \in C_0^\infty(\Omega), \eta \geq 0$. Substituting (φ, ψ) into (24), we have

$$0 = \mu_n \int_\Omega |\nabla u_n|^2 \nabla u_n \left(\nabla \xi \exp(-u_n) - \xi \nabla u_n \exp(-u_n) \right)$$

$$+ \mu_n \int_\Omega |\nabla v_n|^2 \nabla v_n \left(\nabla \eta \exp(-v_n) - \eta \nabla v_n \exp(-v_n) \right)$$

$$+ \int_\Omega \left(1 + u_n^2 \right) \nabla u_n \left(\nabla \xi \exp(-u_n) - \xi \nabla u_n \exp(-u_n) \right)$$

$$+ \int_\Omega \left(1 + v_n^2 \right) \nabla v_n \left(\nabla \eta \exp(-v_n) - \eta \nabla v_n \exp(-v_n) \right)$$

$$+ \int_\Omega \left(1 + |\nabla u_n|^2 \right) u_n \xi \exp(-u_n)$$

$$+ \int_\Omega \left(1 + |\nabla v_n|^2\right) v_n \eta \exp\left(-v_n\right)$$

$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u_n|^{\alpha - 2} |v_n|^\beta u_n \xi \exp\left(-u_n\right)$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u_n|^\alpha |v_n|^{\beta - 2} v_n \eta \exp\left(-v_n\right)$$

$$\leq \mu_n \int_\Omega |\nabla u_n|^2 \nabla u_n \nabla \xi \exp\left(-u_n\right)$$

$$+ \mu_n \int_\Omega |\nabla v_n|^2 \nabla v_n \nabla \eta \exp\left(-v_n\right)$$

$$+ \int_\Omega \left(1 + u_n^2\right) \nabla u_n \nabla \xi \exp\left(-u_n\right)$$

$$+ \int_\Omega \left(1 + v_n^2\right) \nabla v_n \nabla \eta \exp\left(-v_n\right)$$

$$+ \int_\Omega u_n \xi \exp\left(-u_n\right) + \int_\Omega v_n \eta \exp\left(-v_n\right)$$

$$- \int_\Omega \left(1 + u_n^2 - u_n\right) |\nabla u_n|^2 \xi \exp\left(-u_n\right)$$

$$- \int_\Omega \left(1 + v_n^2 - v_n\right) |\nabla v_n|^2 \eta \exp\left(-v_n\right)$$

$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u_n|^{\alpha - 2} |v_n|^\beta u_n \xi \exp\left(-u_n\right)$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u_n|^\alpha |v_n|^{\beta - 2} v_n \eta \exp\left(-v_n\right). \tag{28}$$

Note that $1 + u_n^2 - u_n \geq 0$, $1 + v_n^2 - v_n \geq 0$. By Fatou's Lemma, the weak convergence of $\{(u_n, v_n)\}$ and the fact that $\mu_n \int_\Omega (|\nabla u_n|^4 + |\nabla v_n|^4)$ is bounded, we have

$$0 \leq \int_\Omega \left(1 + u^2\right) \nabla u \nabla \xi \exp\left(-u\right) + \int_\Omega \left(1 + v^2\right) \nabla v \nabla \eta \exp\left(-v\right)$$

$$+ \int_\Omega u \xi \exp\left(-u\right) + \int_\Omega v \eta \exp\left(-v\right)$$

$$- \int_\Omega \left(1 + u^2 - u\right) |\nabla u|^2 \xi \exp\left(-u\right)$$

$$- \int_\Omega \left(1 + v^2 - v\right) |\nabla v|^2 \eta \exp\left(-v\right)$$

$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha - 2} |v|^\beta u \xi \exp\left(-u\right)$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta - 2} v \eta \exp\left(-v\right)$$

$$= \int_\Omega \left(1 + u^2\right) \nabla u \nabla \left(\xi \exp\left(-u\right)\right)$$

$$+ \int_\Omega \left(1 + v^2\right) \nabla v \nabla \left(\eta \exp\left(-v\right)\right)$$

$$+ \int_\Omega \left(1 + |\nabla u|^2\right) u \xi \exp\left(-u\right) + \int_\Omega \left(1 + |\nabla v|^2\right) v \eta \exp\left(-v\right)$$

$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha - 2} |v|^\beta u \xi \exp\left(-u\right)$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta - 2} v \eta \exp\left(-v\right). \tag{29}$$

Let $(\chi, \omega) \geq (0, 0)$, $(\chi, \omega) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. We may choose a sequence of nonnegative functions $\{(\xi_n, \eta_n)\} \subset C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ such that $(\xi_n, \eta_n) \to (\chi \exp u, \omega \exp v)$ in $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, $(\xi_n, \eta_n) \to (\chi \exp u, \omega \exp v)$ a.e. $x \in \Omega$ and $\{(\xi_n, \eta_n)\}$ is uniformly bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$. Then by approximations in (29) we may obtain

$$\int_\Omega \left(1 + u^2\right) \nabla u \nabla \chi + \int_\Omega \left(1 + v^2\right) \nabla v \nabla \omega + \int_\Omega \left(1 + |\nabla u|^2\right) u \chi$$

$$+ \int_\Omega \left(1 + |\nabla v|^2\right) v \omega - \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha - 2} |v|^\beta u \chi$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta - 2} v \omega \geq 0 \tag{30}$$

for all $(\chi, \omega) \geq (0, 0)$, $(\chi, \omega) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$.

Similarly, we may obtain an opposite inequality. Thus we have

$$\int_\Omega \left(1 + u^2\right) \nabla u \nabla \chi + \int_\Omega \left(1 + v^2\right) \nabla v \nabla \omega + \int_\Omega \left(1 + |\nabla u|^2\right) u \chi$$

$$+ \int_\Omega \left(1 + |\nabla v|^2\right) v \omega - \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha - 2} |v|^\beta u \chi$$

$$- \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta - 2} v \omega = 0 \tag{31}$$

for all $(\chi, \omega) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. That is, (u, v) is a critical point of I_0 and a solution for the system (1). By doing approximations again, we have (u, v) in the place of (χ, ω) of (31)

$$\int_\Omega \left(\left(1 + 2u^2\right) |\nabla u|^2 + u^2\right) + \int_\Omega \left(\left(1 + 2v^2\right) |\nabla v|^2 + v^2\right)$$

$$- 2 \int_\Omega |u|^\alpha |v|^\beta = 0. \tag{32}$$

Setting $(\varphi, \psi) = (u_n, v_n)$ in (24), we have

$$
\mu_n \int_\Omega \left(|\nabla u_n|^4 + |\nabla v_n|^4 \right) + \int_\Omega \left(\left(1 + 2u_n^2\right) |\nabla u_n|^2 + u_n^2 \right)
$$

$$
+ \int_\Omega \left(\left(1 + 2v_n^2\right) |\nabla v_n|^2 + v_n^2 \right) - 2 \int_\Omega |u_n|^\alpha |v_n|^\beta = 0. \tag{33}
$$

Using $\int_\Omega |u_n|^\alpha |v_n|^\beta \to \int_\Omega |u|^\alpha |v|^\beta$ as $n \to \infty$, (32), (33), and lower semicontinuity, we obtain

$$
\int_\Omega |\nabla u_n|^2 \longrightarrow \int_\Omega |\nabla u|^2, \qquad \int_\Omega |\nabla v_n|^2 \longrightarrow \int_\Omega |\nabla v|^2,
$$

$$
\int_\Omega u_n^2 |\nabla u_n|^2 \longrightarrow \int_\Omega u^2 |\nabla u|^2, \qquad \int_\Omega v_n^2 |\nabla v_n|^2 \longrightarrow \int_\Omega v^2 |\nabla v|^2 \tag{34}
$$

as $n \to \infty$.

In particular, we have

$$
u_n \longrightarrow u, \qquad v_n \longrightarrow v \quad \text{in } W_0^{1,2}(\Omega),
$$

$$
u_n \nabla u_n \longrightarrow u \nabla u, \qquad v_n \nabla v_n \longrightarrow v \nabla v \quad \text{in } L^2(\Omega),
$$

$$
\mu_n \int_\Omega \left(|\nabla u_n|^4 + |\nabla v_n|^4 \right) \longrightarrow 0, \tag{35}
$$

$$
I'_{\mu_n}(u_n, v_n) \longrightarrow I'_0(u, v)
$$

as $n \to \infty$. This completes the proof of Theorem 1. $\qquad \square$

Next, we apply the mountain pass theorem to obtain existence of critical points of I_μ. Set

$$
\Sigma_\rho = \Big\{ (u, v) \in W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega) \mid \int_\Omega \left(\left(1 + u^2\right) |\nabla u|^2 + u^2 \right)
$$

$$
+ \int_\Omega \left(\left(1 + v^2\right) |\nabla v|^2 + v^2 \right) \le \rho^2 \Big\} \tag{36}
$$

for $\rho > 0$.

Let us consider the functional

$$
I_\mu^+(u, v) = \frac{1}{4} \mu \int_\Omega \left(|\nabla u|^4 + |\nabla v|^4 \right)
$$

$$
+ \frac{1}{2} \int_\Omega \left(\left(1 + u^2\right) |\nabla u|^2 + u^2 \right)
$$

$$
+ \frac{1}{2} \int_\Omega \left(\left(1 + v^2\right) |\nabla v|^2 + v^2 \right) \tag{37}
$$

$$
- \frac{2}{\alpha + \beta} \int_\Omega (u^+)^\alpha (v^+)^\beta.
$$

Here and in the following we denote $u^+ = \max\{u, 0\}$. The functional I_μ satisfies $(PS)_c$ condition. Similarly, we may verify that I_μ^+ satisfies $(PS)_c$ condition. By ε-Young inequality, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
(u^+)^\alpha (v^+)^\beta \le \varepsilon (u^+)^{\alpha+\beta} + C_\varepsilon (v^+)^{\alpha+\beta}. \tag{38}
$$

Since

$$
\int_\Omega |u|^{\alpha+\beta} \le C \left(\int_\Omega u^2 |\nabla u|^2 \right)^{(\alpha+\beta)/4},
$$

$$
\int_\Omega |v|^{\alpha+\beta} \le C \left(\int_\Omega v^2 |\nabla v|^2 \right)^{(\alpha+\beta)/4}. \tag{39}
$$

Then

$$
- \frac{2}{\alpha + \beta} \int_\Omega (u^+)^\alpha (v^+)^\beta
$$

$$
\ge - \frac{2}{\alpha + \beta} \varepsilon \int_\Omega (u^+)^{\alpha+\beta} - \frac{2}{\alpha + \beta} C_\varepsilon \int_\Omega (u^+)^{\alpha+\beta}
$$

$$
\ge - \frac{2C}{\alpha + \beta} \varepsilon \int_\Omega \left(\int_\Omega u^2 |\nabla u|^2 \right)^{(\alpha+\beta)/4}
$$

$$
- \frac{2C_\varepsilon}{\alpha + \beta} \left(\int_\Omega v^2 |\nabla v|^2 \right)^{(\alpha+\beta)/4} \tag{40}
$$

$$
\ge - \frac{2C}{\alpha + \beta} \varepsilon \rho^{(\alpha+\beta)/2} - \frac{2C_\varepsilon}{\alpha + \beta} \rho^{(\alpha+\beta)/2}
$$

$$
\ge - \frac{1}{\alpha + \beta} \rho^2
$$

for ε, ρ small. Thus we have

$$
I_\mu^+(u, v)
$$

$$
\ge \frac{1}{2} \int_\Omega \left(\left(1 + u^2\right) |\nabla u|^2 + u^2 \right)
$$

$$
+ \frac{1}{2} \int_\Omega \left(\left(1 + v^2\right) |\nabla v|^2 + v^2 \right) - \frac{2}{\alpha + \beta} \int_\Omega (u^+)^\alpha (v^+)^\beta
$$

$$
\ge \frac{1}{2} \rho^2 - \frac{1}{\alpha + \beta} \rho^2 = \left(\frac{1}{2} - \frac{1}{\alpha + \beta} \right) \rho^2 \tag{41}
$$

for $(u, v) \in \partial \Sigma_\rho$ and for $\rho > 0$ small enough. Choose $(\varphi, \psi) \ge (0, 0)$, $(\chi, \omega) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ and $T > 0$. Define a path $(g, h): [0, 1] \to W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)$ by $(g(t), h(t)) = (tT\varphi, tT\psi)$. When T is large enough, we have

$$
I_\mu^+(g(1), h(1)) < 0,
$$

$$
\int_\Omega \left(\left(1 + g^2(1)\right) |\nabla g(1)|^2 + g^2(1) \right)
$$

$$
+ \int_\Omega \left(\left(1 + h^2(1)\right) |\nabla h(1)|^2 + h^2(1) \right) > \rho^2, \tag{42}
$$

$$
\sup_{t \in [0,1]} I_\mu^+(g(t), h(t)) \le m
$$

for some m independent of $\mu \in (0, 1]$.

Define

$$
c_\mu = \inf_{(g,h) \in \Gamma} \sup_{t \in [0,1]} I_\mu^+(g(t), h(t)), \tag{43}
$$

where

$$\Gamma = \left\{ (g,h) \in C\left([0,1], W_0^{1,4}(\Omega) \times W_0^{1,4}(\Omega)\right) \mid (g(0), h(0)) \right.$$
$$\left. = (0,0), (g(1), h(1)) = (T\varphi, T\psi) \right\}.$$
(44)

From the mountain pass theorem we obtain that

$$c_\mu \geq \left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right) \rho^2$$
(45)

is a critical value of I_μ^+.

Let (u_μ, v_μ) be a critical point corresponding to c_μ. We have $(u_\mu, v_\mu) \geq (0,0)$. Thus (u_μ, v_μ) is a positive critical point of I_μ by the strong maximum principle. In summary, we have the following.

Proposition 6. *There exist positive constants ρ and m independent of μ such that I_μ has a positive critical point (u_μ, v_μ) satisfying*

$$\left(\frac{1}{2} - \frac{1}{\alpha + \beta}\right) \rho^2 \leq I_\mu(u_\mu, v_\mu) \leq m.$$
(46)

Finally, we give the proof of Theorem 2.

Proof of Theorem 2. For a positive solution of the system (1), the proof follows from Proposition 6 and Theorem 1. A similar argument gives a negative solution of the system (1). This completes the proof of Theorem 2. □

Acknowledgments

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