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Research Article

Well-Posedness for Generalized Set Equilibrium Problems

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We study the well-posedness for generalized set equilibrium problems (GSEP) and propose two types of the well-posed concepts for these problems in topological vector space settings. These kinds of well-posedness arise from some well-posedness in the vector settings. We also study the relationship between these well-posedness concepts and present several criteria for the well-posedness of GSEP. Our results are new or include as special cases recent existing results.

1. Introduction and Preliminaries

Let X, Y, Z be three topological vector spaces, K a nonempty closed convex subset of $X, C \subset Y$ a closed convex and pointed cone with apex at the origin, and $\operatorname{int} C \neq \emptyset$; that is, C is properly closed with nonempty interior and satisfies $\lambda C \subseteq C$, for all $\lambda > 0$; $C + C \subseteq C$; and $C \cap (-C) = \{0\}$.

The set-valued mapping $f : Z \times K \times K \rightrightarrows Y$ satisfies $f(s, x, x) = \{0\}$ for all $x \in K$ and for all $s \in Z$, and a set-valued mapping $T : K \rightrightarrows Z$ is given. The generalized set equilibrium problem (GSEP) is to find an $\overline{x} \in K$ with some $\overline{s} \in T(\overline{x})$ such that

$$f(\overline{s}, \overline{x}, y) \cap (-\operatorname{int} C) = \emptyset \quad \forall y \in K. \tag{1}$$

We denote the set of all solutions for (GSEP) by Ω.

The concept of well-posedness is inspired by numerical methods producing optimizing sequences for optimization problems [1]. There are many cases so that the solutions may not be unique for a minimization problem. A naturally generalized concept of well-posedness which permits the existence but not uniqueness of minimizers and the convergence of some subsequence of every minimizing sequence toward a minimizer. Other more general notions of well-posedness have been introduced in [2] and there are many others in the literature; see, for example, [3–15]. Our main purpose is to derive some properties of well-posedness for the generalized set equilibrium problems. We also study the relations between these properties.

A minimizing mapping $\Phi : Z \times X \rightrightarrows Y$ is defined by

$$\Phi(s, x) = \operatorname{Min}_w f(s, x, K) \tag{2}$$

for all $(s, x) \in Z \times X$, where $\operatorname{Min}_w A = \{\eta \in A : A \cap (\eta - \operatorname{int} C) = \emptyset\}$ and $f(s, x, K) = \bigcup_{y \in K} \{f(s, x, y)\}$ for all $(s, x) \in Z \times X$. Assume that $\operatorname{Dom}(\Phi) \neq \emptyset$. We note that $0 \in f(s, x, K)$ for all $(s, x) \in Z \times X$ since $f(s, x, x) = \{0\}$ for all $x \in K$ and for all $s \in Z$. $\mathfrak{N}_X(x_0)$ denotes the collection of neighborhoods around x_0 in X, similar notations for $\mathfrak{N}_Y(y_0)$ and $\mathfrak{N}_Z(s_0)$. For any mapping F, $F(A)$ denotes the union $\bigcup_{x \in A} F(x)$.

We propose some properties that can be easily derived from the definition. For the sake of clarity, we give the following proof.

Lemma 1. *(i) $\Phi(s, x) \cap \operatorname{int} C = \emptyset$ for all $x \in K$ and for all $s \in T(x)$.*

(ii) $\overline{x} \in \Omega$ with $\overline{s} \in T(\overline{x})$ if and only if $0 \in \Phi(\overline{s}, \overline{x})$.

(iii) $\overline{x} \in \Omega$ with $\overline{s} \in T(\overline{x})$ if and only if $\Phi(\overline{s}, \overline{x}) \cap C \neq \emptyset$.

Proof. (i) If not, there exists $\tau \in \Phi(\overline{s}, \overline{x}) \cap \operatorname{int} C$ for some $\overline{x} \in K$ and $\overline{s} \in T(\overline{x})$. Then $f(\overline{s}, \overline{x}, K) \cap (\tau - \operatorname{int} C) = \emptyset$ and $\tau \in \operatorname{int} C$. Hence, $0 \notin f(\overline{s}, \overline{x}, K)$ which contradicts the fact that $0 \in f(s, x, K)$ for all $x \in K$ and for all $s \in T(x)$.

(ii) $0 \in \Phi(\overline{s}, \overline{x})$ if and only if $f(\overline{s}, \overline{x}, K) \cap (-\operatorname{int} C) = \emptyset$ if and only if $\overline{x} \in \Omega$.

(iii) By (i) and (ii), we have $\overline{x} \in \Omega$ with $\overline{s} \in T(\overline{x})$ if and only if $0 \in \Phi(\overline{s}, \overline{x})$ if and only if $\Phi(\overline{s}, \overline{x}) \cap C \neq \emptyset$. □

Definition 2. A sequence $\{(s_n, x_n) \in Z \times K : s_n \in T(x_n)\}$ is a minimizing sequence for Φ if for every neighborhood $U_Y \in \mathfrak{N}_Y(0)$ of 0, there is $n_0 \in \mathbb{N}$, such that $\Phi(s_n, x_n) \cap U_Y \neq \emptyset$ for all $n \geq n_0$.

Definition 3. (GSEP) is M-well-posed if it satisfies the following conditions:

(i) there exists at least one solution, that is, the set $\Omega \neq \emptyset$;

(ii) for every minimizing sequence $\{(s_n, x_n)\}$ and for every $U_X \in \mathfrak{N}_X(0)$, there is $n_0 \in \mathbb{N}$ such that $x_n \in \Omega + U_X$ for all $n \geq n_0$.

Definition 4. For $\epsilon \in C$, the ϵ-approximate solution set of (GSEP) is defined by $\Omega(\epsilon) = \{x \in K : \Phi(s, x) \cap (C - \epsilon) \neq \emptyset$ for some $s \in T(x)\}$.

We can easily see that $\Omega(0) = \Omega$ in Definition 4. Indeed, $\overline{x} \in \Omega(0)$ with some $\overline{s} \in T(\overline{x})$ if and only if $\Phi(\overline{s}, \overline{x}) \cap C \neq \emptyset$ if and only if $\overline{x} \in \Omega$ from Lemma 1(iii).

Definition 5. (GSEP) is B-well-posed if it satisfies the following conditions:

(i) there exists at least one solution, that is, the set $\Omega \neq \emptyset$;

(ii) the mapping $\Omega : C \rightrightarrows X$ is upper Hausdorff continuous at $\epsilon = 0$; that is, for every $U_X \in \mathfrak{N}_X(0)$, there exists $U_Y \in \mathfrak{N}_Y(0)$ such that $\Omega(\epsilon) \subset \Omega + U_X$ for every $\epsilon \in U_Y \cap C$.

Definition 3 arises from [8], and Definition 5 is originally proposed by [9].

Definition 6 (see [16, 17]). A set-valued mapping $T : X \rightrightarrows Z$ is

(i) upper semicontinuous if for every $x \in X$ and every open set V in Y with $T(x) \subset V$, there exists a neighborhood $W(x)$ of x such that $T(W(x)) \subset V$;

(ii) lower semicontinuous if for every $x \in X$ and every open neighborhood $V(y)$ of every $y \in T(x)$, there exists a neighborhood $W(x)$ of x such that $T(u) \cap V(y) \neq \emptyset$ for all $u \in W(x)$;

(iii) continuous if it is both upper semicontinuous and lower semicontinuous.

We note that T is upper semicontinuous at x_0 and $T(x_0)$ is compact; then for any net $\{x_\nu\} \subset X$, $x_\nu \to x_0$, and for any net $y_\nu \in T(x_\nu)$ for each ν, there exists $y_0 \in T(x_0)$ and a subnet $\{y_{\nu_\alpha}\}$ such that $y_{\nu_\alpha} \to y_0$. We can refer to [18] for more details. We also note that T is lower semicontinuous at x_0 if for any net $\{x_\nu\} \subset X$, $x_\nu \to x_0$, $y_0 \in T(x_0)$ implies that there exists net $y_\nu \in T(x_\nu)$ such that $y_\nu \to y_0$. For more details, we refer the reader to [16] or [17]. Another more weaker upper semi-continuity is said above C-upper Hausdorff semicontinuous [9]. A mapping $T : X \rightrightarrows Z$ is above C-upper Hausdorff semicontinuous if for every $x \in X$ and every open set $W_Y \in \mathfrak{N}_Y(0)$, there exists a neighborhood $V_X \in \mathfrak{N}_X(0)$ such that $T(x + V_Y) \subset T(x) + W_Y - C$. Obviously, the upper Hausdorff continuity is weaker than the upper semi-continuity, and an upper Hausdorff continuous mapping is an above C-upper Hausdorff semi-continuous mapping.

2. B-Well-Posed and M-Well-Posed

In this section, we will discuss the relationship between these two kinds of well-posedness. The first one is given as follows.

Example 7. (i) There is an example that satisfies M-well-posed, but not B-well-posed. (ii) There is an example that satisfies both B-well-posed and M-well-posed.

Solution. (i) The first one is inspired by the example of [10]. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $K = \mathbb{R}_+$, $T : K \rightrightarrows Z$ satisfy $T(x) = [x/2, x]$ for all $x \in K$. The set-valued mapping $f : Z \times K \times K \rightrightarrows Y$ is defined by

$$f(s, x, y) = \begin{cases} \{(0,0)\}, & \text{if } x = 0,\ y \geq 0; \\ \{(y - s, 0)\}, & \text{if } x > 0,\ 0 \leq y < 2s; \\ \left\{\left(s, -\dfrac{8s}{y^3}\right), \left(-s, -\dfrac{8s}{y^3}\right)\right\}, & \text{otherwise,} \end{cases} \tag{3}$$

for all $(s, x, y) \in Z \times K \times K$ with $s \in T(x)$.

Then the set of all solutions for (GSEP) is $\Omega = \{0\}$. For any $x \in K$ and any $s \in T(x)$, the minimizing mapping is

$$\Phi(s, x) = \begin{cases} \{(0,0)\} & \text{if } x = 0; \\ \left\{\left(\{-s\} \times \left[-\dfrac{1}{s^2}, 0\right]\right) \cup \left\{\left(s, -\dfrac{1}{s^2}\right)\right\}\right\}, & \text{if } x > 0. \end{cases} \tag{4}$$

If we choose $U_X = (-1/2, 1/2)$, a neighborhood of 0, and $x_n = n$, $\epsilon_n = (0, 1/n^2)$ for all $n \in \mathbb{N}$, we can easily see that $\epsilon_n \to (0,0)$ as $n \to \infty$, and $n \in \Omega(\epsilon_n) \setminus (\Omega + U_X)$ for all $n \in \mathbb{N}$. Thus, (GSEP) is not B-well-posed.

Nevertheless, for any minimizing sequence $\{(s_n, x_n)\}$ for Φ with $s_n \in T(x_n)$ for all n and for every $U_Y \in \mathfrak{N}_Y(0)$, there exists $n_0 \in \mathbb{N}$ such that $\Phi(s_n, x_n) \cap U_Y \neq \emptyset$ for $n \geq n_0$. This will force that $x_n \to 0$ as $n \to \infty$, and hence $x_n \in \Omega + U_X$ for all $n \geq n_0$. Therefore, (GSEP) is M-well-posed.

(ii) We modify the above example as follows. Let X, Z, Y, C, K, T be given the same as in (i). The set-valued mapping $f : Z \times K \times K \rightrightarrows Y$ is defined by

$$f(s, x, y) = \begin{cases} \{(0,0)\}, & \text{if } x = 0,\ y \geq 0; \\ \{(y - s, 0)\}, & \text{if } x > 0,\ 0 \leq y < 2s; \\ \left\{\left(s, -\dfrac{8s}{y^3}\right)\right\}, & \text{otherwise,} \end{cases} \tag{5}$$

for all $(s, x, y) \in Z \times K \times K$ with $s \in T(x)$.

Then the set of all solutions for (GSEP) is $\Omega = K$. For any $x \in K$ and any $s \in T(x)$, the minimizing mapping is

$$\Phi(s,x) = \begin{cases} \{(0,0)\} & \text{if } x = 0; \\ \left\{\left(s, -\frac{1}{s^2}\right)\right\}, & \text{if } x > 0. \end{cases} \tag{6}$$

Since for any minimizing sequence $\{(s_n, x_n)\}$, and for every $U_X \in \mathfrak{N}_X(0)$, we always have $x_n \in \Omega + U_X$. Thus, (GSEP) is M-well-posed. Furthermore, since $\Omega + U_X = K + U_X$, for all $\epsilon \in \mathbb{R}^2_+$, we always have $\Omega(\epsilon) \subset \Omega + U_X$. Hence, (GSEP) is B-well-posed.

From the above observation, the M-well-posed is weaker than B-well-posed for (GSEP). What conditions need to be added so that the converse statement can be valid? The following results will be one of the answers.

Proposition 8. *(a) If (GSEP) is B-well-posed, then it is M-well-posed. (b) If (GSEP) is M-well-posed, and for every $W_Y \in \mathfrak{N}_Y(0)$, there exists $U_Y \in \mathfrak{N}_Y(0)$ such that*

$$\Phi(h(K \setminus \mathrm{cl}(\Omega))) \cap (C + U_Y) \subset W_Y, \tag{7}$$

where $h(x) = T(x) \times \{x\}$ for all $x \in K \setminus \mathrm{cl}(\Omega)$, then (GSEP) is B-well-posed.

Proof. For the idea of the proof, we can use the similar direction of [10, Propositions 3 and 4]. For the sake of clarity, we give the proof of (b) as follows. Suppose that (GSEP) is not B-well-posed. Then there is a neighborhood $\overline{U}_X$ of 0, and sequences $\{\epsilon_n\} \subset C$ with $\epsilon_n \to 0$ and $x_n \in \Omega(\epsilon_n)$ such that

$$x_n \notin \Omega + \overline{U}_X, \quad \forall n \in \mathbb{N}. \tag{8}$$

This means

$$x_n \notin \mathrm{cl}(\Omega), \quad \forall n \in \mathbb{N}. \tag{9}$$

Since $x_n \in \Omega(\epsilon_n)$, there exists $s_n \in T(x_n)$ such that

$$\Phi(s_n, x_n) \cap (C - \epsilon_n) \neq \emptyset, \quad \forall n \in \mathbb{N}. \tag{10}$$

Now, we separate into two cases.

Case 1. If the sequence $\{(s_n, x_n)\}$ is a minimizing sequence, then by M-well-posedness, for this $\overline{U}_X$, there is a $n_0 \in \mathbb{N}$ such that $x_n \in \Omega + \overline{U}_X$, for all $n \geq n_0$, which contradicts (8).

Case 2. If the sequence $\{(s_n, x_n)\}$ is not a minimizing sequence, then there is a $W_Y \in \mathfrak{N}_Y(0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with a corresponding subsequence $\{s_{n_k}\} \in T(\{x_{n_k}\})$ such that

$$\Phi(s_{n_k}, x_{n_k}) \cap W_Y = \emptyset, \quad \forall k \in \mathbb{N}. \tag{11}$$

By relation (10), we have $\Phi(s_{n_k}, x_{n_k}) \cap (C - \epsilon_{n_k}) \neq \emptyset$, for all $k \in \mathbb{N}$. For this W_Y and condition (7), there is a symmetric neighborhood $U_Y \in \mathfrak{N}_Y(0)$ such that $\Phi(s_{n_k}, x_{n_k}) \cap (C + U_Y) \subset W_Y$, for all $k \in \mathbb{N}$. For k large enough, $-\epsilon_{n_k} \in U_Y$. Taking $\eta_{n_k} \in \Phi(s_{n_k}, x_{n_k}) \cap (C + U_Y)$ for all $k \in \mathbb{N}$. This implies that, for k large enough, $\eta_{n_k} \in \Phi(s_{n_k}, x_{n_k}) \cap W_Y$ which contradicts (11). This completes the proof. □

We need the following lemma for the next criterion for M-well-posedness of (GSEP).

Lemma 9. *Let Y be a regular topological vector spaces, and let A be a nonempty compact subset of Y. Suppose that $A \cap (-C) = \emptyset$; then there is a neighborhood W_Y of 0 such that $(A + W_Y) \cap (W_Y - C) = \emptyset$. In particular, $(A + W_Y) \cap W_Y = \emptyset$.*

Proof. Suppose that $A \cap (-C) = \emptyset$. For all $\eta \in A$, we have $\eta \notin -C$. Since Y is regular, there is a neighborhood U_Y^{η} of 0 such that

$$(\eta + U_Y^{\eta}) \bigcap (U_Y^{\eta} - C) = \emptyset. \tag{12}$$

Since A is a nonempty compact subset, the set

$$A \subset \bigcup_{\eta \in A} (\eta + U_Y^{\eta}). \tag{13}$$

There exist $\eta_1, \eta_2, \ldots, \eta_n \in A$, such that

$$A \subset \bigcup_{i=1}^{n} (\eta_i + U_Y^{\eta_i}) \subset A + \bigcup_{i=1}^{n} U_Y^{\eta_i}. \tag{14}$$

Let $W_Y = \bigcap_{i=1}^{n} U_Y^{\eta_i}$. Then

$$\left(A + \bigcup_{i=1}^{n} U_Y^{\eta_i}\right) \bigcap (W_Y - C) = \emptyset. \tag{15}$$

Since $W_Y \subset \bigcup_{i=1}^{n} U_Y^{\eta_i}$, we have

$$(A + W_Y) \cap (W_Y - C) = \emptyset. \tag{16}$$

□

We note that, although every compact regular space is a normal space, Lemma 9 is not so intuitive. Furthermore, if the set A is not compact, the conclusion may not hold. For example, we choose $A = \{(x, y) \in \mathbb{R}^2 : y \geq x^{-2}, x < 0\}$ and $C = \mathbb{R}^2_+$.

Now, we present first criterion of M-well-posedness for (GSEP).

Theorem 10. *Let X, Y, Z be three Hausdorff topological vector spaces where X is a finite dimensional space and Y is regular, let K be a nonempty closed convex subset of X, and let $C \subset Y$ be a closed convex and pointed cone with apex at the origin and $\mathrm{int}\, C \neq \emptyset$. The mapping $f : Z \times K \times K \rightrightarrows Y$ is upper semi-continuous with nonempty compact values and satisfies $f(s, x, x) = \{0\}$ for all $x \in K$ and for all $s \in T(x)$, and the mapping $T : K \rightrightarrows Z$ is upper semi-continuous with nonempty compact values, such that*

(i) *the solution set Ω of (GSEP) is nonempty and bounded;*

(ii) *the minimizing mapping Φ is upper Hausdorff continuous on $T(K) \times K$;*

(iii) *$f(s, x, y) \cap (-C) = \emptyset$ for all $x \in \Omega$, for all $s \in T(K)$ and for all $y \in K \setminus \Omega$;*

(iv) *the mapping* $(s, x) \rightarrow f(s, x, y)$ *is above-C-upper Hausdorff continuous on* $Z \times K$ *for every* $y \in K$, *and the mapping* $x \rightarrow f(s, x, y)$ *is above-C-concave [19] on* K *for every* $s \in T(K)$ *and* $y \in K$;

(v) *for every minimizing sequence* $\{(s_n, x_n)\} \subset T(K) \times K$, *and for each* $(s, y) \in T(K) \times K$, *there is a sequence* $\{\zeta_n\}$ *with* $\zeta_n \in f(s, x_n, y)$ *for each* $n \in \mathbb{N}$ *is a bounded sequence in* Y.

Then (GSEP) is M-well-posedness.

Proof. We prove it by contradiction. Suppose that (GSEP) is not M-well-posedness. Then there exists a minimizing sequence $\{(s_n, x_n)\} \subset TK \times K$ and $\epsilon > 0$ such that

$$x_n \notin \Omega + \epsilon B, \tag{17}$$

for infinitely many n, where B denotes the unit open ball in X. Let us choose a subsequence from $\{(s_n, x_n)\}$ so that the relation (17) holds for all elements of the subsequence. Such a subsequence is still a minimizing sequence, and we still denote it by $\{(s_n, x_n)\}$ if there is no any confusion. Now, we separate our discussion into two cases.

Case 1. If the sequence $\{x_n\}$ is bounded, then it has a convergent subsequence $\{x_{n_k}\}$ that converges to some point $x^\star \in X$ with a corresponding subsequence $\{s_{n_k}\}$ with $s_{n_k} \in T(x_{n_k})$ for every $k \in \mathbb{N}$. By the upper semi-continuity of T, there exists a convergent subsequence of $\{s_{n_k}\}$ (without any confuse, we still denote it by $\{s_{n_k}\}$) converges to some point $s^\star \in T(x^\star)$. From (17), $x_{n_k} \notin \Omega + \epsilon B$ for every $k \in \mathbb{N}$. Hence, $x^\star \notin \Omega$, and by Lemma 1, we have $0 \notin \Phi(s^\star, x^\star)$. By Lemma 9, there is a neighborhood $U_Y \in \mathfrak{N}_Y(0)$ such that

$$(\Phi(s^\star, x^\star) + W_Y) \cap W_Y = \emptyset. \tag{18}$$

Since $\{(s_{n_k}, x_{n_k})\}$ is a minimizing sequence, for each k, we can choose $\eta_{n_k} \in \Phi(s_{n_k}, x_{n_k})$ such that $\eta_{n_k} \rightarrow 0$. Since Φ is upper Hausdorff continuity of Φ at $(s^\star, x^\star)$, we have

$$\Phi\left(s_{n_k}, x_{n_k}\right) \subset \Phi(s^\star, x^\star) + W_Y. \tag{19}$$

Hence, for k large enough, $\eta_{n_k} \in (\Phi(s^\star, x^\star) + W_Y) \cap W_Y$ which contradicts (18).

Case 2. If the sequence $\{x_n\}$ is unbounded. Since Ω is bounded, so are $\Omega + \epsilon B$ and $\mathrm{cl}(\Omega + \epsilon B)$. Then the set $\mathrm{cl}(\Omega+\epsilon B)$ is compact. We denote that $\Omega \cap \partial(\Omega+\epsilon B) = \emptyset$, where $\partial(\Omega+\epsilon B)$ means the boundary of $\Omega + \epsilon B$. Since the sequence $\{x_n\}$ is unbounded, there is a subsequence $\{x_{n_k}\}$ with $\|x_{n_k}\| \rightarrow \infty$ as $k \rightarrow \infty$. Without loss of generality, we may assume that $x_n \notin \partial(\Omega + \epsilon B)$ for all $n \in \mathbb{N}$. Fix any $\overline{x} \in \Omega$ and let $v_n = \lambda_n \overline{x} + (1 - \lambda_n) x_n \in \partial(\Omega + \epsilon B)$ for any $n \in \mathbb{N}$, where $\lambda_n = \sup\{\lambda \in [0, 1] : \lambda \overline{x} + (1 - \lambda) x_n \notin \Omega + \epsilon B\}$. After a simple calculation, we can see that $\lambda_n \rightarrow 1$ as $n \rightarrow \infty$. Hence, $\{v_n\}$ has a subsequence $\{v_{n_k}\}$ that converges to some point $v^\star \in \partial(\Omega+\epsilon B)$. For the similar process in Case 1, we have a subsequence $\{t_{n_k}\}$ of the corresponding sequence $\{t_n\}$ with $t_n \in T(v_n)$ converges to some point $t^\star \in T(v^\star)$. By the above C-concavity of f in x, we have

$$\begin{aligned} &\lambda_{n_k} f\left(t_{n_k}, \overline{x}, v^\star\right) + \left(1 - \lambda_{n_k}\right) f\left(t_{n_k}, x_{n_k}, v^\star\right) \\ &\quad \subset f\left(t_{n_k}, v_{n_k}, v^\star\right) - C. \end{aligned} \tag{20}$$

By condition (v), there is a bounded sequence $\{\zeta_{n_k}\}$ with $\zeta_{n_k} \in f(t_{n_k}, x_{n_k}, v^\star)$ for each $k \in \mathbb{N}$ in Y. Hence, by (20), we have

$$\lambda_{n_k} f\left(t_{n_k}, \overline{x}, v^\star\right) + \left(1 - \lambda_{n_k}\right) \zeta_{n_k} \subset f\left(t_{n_k}, v_{n_k}, v^\star\right) - C. \tag{21}$$

Next, we claim that

$$f(t^\star, \overline{x}, v^\star) \cap (-C) \neq \emptyset. \tag{22}$$

Indeed, if $f(t^\star, \overline{x}, v^\star) \cap (-C) = \emptyset$. By Lemma 9, there is a neighborhood $U_Y \in \mathfrak{N}_Y(0)$ such that

$$(f(t^\star, \overline{x}, v^\star) + U_Y) \cap (U_Y - C) = \emptyset. \tag{23}$$

For this U_Y, by the above C-upper Hausdorff continuity of f and the fact that $f(t^\star, v^\star, v^\star) = \{0\}$, we have

$$\begin{aligned} &f\left(t_{n_k}, v_{n_k}, v^\star\right) \subset U_Y - C, \\ &f\left(t_{n_k}, \overline{x}, v^\star\right) \subset f(t^\star, \overline{x}, v^\star) + U_Y \end{aligned} \tag{24}$$

for k large enough. Since the sequence $\{\zeta_{n_k}\}$ is bounded, the left-hand side of (21) will fell into $f(t^\star, \overline{x}, v^\star) + U_Y$ for k large enough. But in this situation, we can see that $(f(t^\star, \overline{x}, v^\star) + U_Y) \cap (U_Y - C) \neq \emptyset$ which contradicts (23). Thus, the relation (22) holds. Since $\overline{x} \in \Omega$ and $t^\star \in T(K)$, by condition (iii) we have $v^\star \in \Omega$ which contradicts the fact that $v^\star \in \partial(\Omega + \epsilon B)$.

From the discussions of above two cases, (GSEP) is M-well-posedness. □

Remark 11. Theorem 10 generalize the Theorem 1 of [10] to (GSEP).

Lemma 12. *Suppose that* $F : K \rightrightarrows Y$ *and* $f : K \times K \rightrightarrows Y$ *satisfy* $f(x, y) = F(y) - F(x)$ *for all* $x, y \in K$, *then*

$$\mathrm{Min}_w f(x, K) \subset \mathrm{Min}_w F(K) - F(x) \tag{25}$$

for all $x \in K$.

Proof. For any fixed $x \in K$. Choose any $\xi \in \mathrm{Min}_w f(x, K)$; we have $\xi \in F(K) - F(x)$ and $(F(K) - F(x)) \cap (\xi - \mathrm{int}\, C) = \emptyset$. There exist $\xi_1 \in F(K)$ and $\xi_2 \in F(x)$ such that $\xi = \xi_1 - \xi_2$ and $(F(K) - F(x)) \cap (\xi_1 - \xi_2 - \mathrm{int}\, C) = \emptyset$. That is, $(F(K) + \xi_2 - F(x)) \cap (\xi_1 - \mathrm{int}\, C) = \emptyset$. Since $0 \in \xi_2 - F(x)$ and $F(K) \subset F(K) + \xi_2 - F(x)$, we know that $F(K) \cap (\xi_1 - \mathrm{int}\, C) = \emptyset$. Thus, $\xi_1 \in \mathrm{Min}_w F(K)$, and hence, $\xi \in \mathrm{Min}_w F(K) - F(x)$. Therefore,

$$\mathrm{Min}_w f(x, K) \subset \mathrm{Min}_w F(K) - F(x). \tag{26}$$

□

Lemma 13. *Let X, Y, C, K be given the same as in Theorem 10, let f be as given in Lemma 12, and let $F : K \rightrightarrows Y$ be upper semi-continuous with nonempty compact values such that the set $\Omega = w-\operatorname{Eff}(F,K)$ is bounded, where $w-\operatorname{Eff}(F,K) = \{x \in K : y \in F(x), F(K) \cap (y - \operatorname{int} C) = \emptyset\}$. Any sequence $\{x_n\}$ satisfies for every neighborhood $U_Y \in \mathfrak{N}_Y(0)$ of 0; there is $n_0 \in \mathbb{N}$, such that $\operatorname{Min}_w f(x_n, K) \cap U_Y \neq \emptyset$ for all $n \geq n_0$. Then there exists a bounded sequence $\{\eta_n\}$ in Y with $\eta_n \in F(x_n)$, for all $n \in \mathbb{N}$.*

Proof. Since Ω is bounded, its closure $\operatorname{cl}(\Omega)$ is compact. By the upper semi-continuity of F, $F(\operatorname{cl}(\Omega))$ is compact. Hence it is bounded, so is $F(\Omega)$. Fixed $W_Y \in \mathfrak{N}_Y(0)$, a symmetric neighborhood of 0. Since the sequence $\{x_n\}$ satisfies for every neighborhood $U_Y \in \mathfrak{N}_Y(0)$ of 0, there is $n_0 \in \mathbb{N}$, such that $\operatorname{Min}_w f(x_n, K) \cap U_Y \neq \emptyset$ for all $n \geq n_0$. From Lemma 12, we have

$$\left(\operatorname{Min}_w F(K) - F(x_n)\right) \cap W_Y \neq \emptyset \tag{27}$$

for all $n \geq n_0$. We can pick some points $\xi_n \in \operatorname{Min}_w F(K)$, and $\eta_n \in F(x_n)$ such that

$$\xi_n - \eta_n \in W_Y \tag{28}$$

for all $n \geq n_0$. Since W_Y is symmetric, we have

$$\begin{aligned} \eta_n &\in \xi_n + W_Y \subset \operatorname{Min}_w F(K) + W_Y \\ &= F(w - \operatorname{Eff}(F,K)) + W_Y \\ &= F(\Omega) + W_Y \end{aligned} \tag{29}$$

for all $n \geq n_0$. Since $F(\Omega)$ is bounded, so is $F(\Omega) + W_Y$. Therefore, the sequence $\{\eta_n\}$ is bounded. □

Now, let us present another criterion for M-well-posedness of (GSEP).

Theorem 14. *Let X, Y, Z, C, K, f, T be given the same as in Theorem 10. Suppose that*

(i) *the solution set Ω of (GSEP) is nonempty and compact;*

(ii) *for every $x, z \in K$ and $s \in T(K)$, if $f(s,x,z) \cap C \neq \emptyset$, then $f(s,z,x) \cap (-C) \neq \emptyset$;*

(iii) *$f(s,x,z) \cap (-C) = \emptyset$ for all $x \in \Omega$, for all $s \in T(K)$ and for all $z \in K \setminus \Omega$;*

(iv) *the mapping $(s,x) \to f(s,x,y)$ is above C-upper Hausdorff continuous on $Z \times K$ for every $y \in K$, and the mapping $x \to f(s,x,y)$ is above C-concave on K for every $s \in T(K)$ and $y \in K$;*

(v) *for each minimizing sequence $\{(s_n, x_n)\} \subset T(K) \times K$, and for each $(y,s) \in K \times Z$, there is a sequence $\{\zeta_n\}$ with $\zeta_n \in f(s, x_n, y)$ for each $n \in \mathbb{N}$ is a bounded sequence in Y;*

(vi) *for every $x \in K \setminus \Omega$ and for every $s \in T(x)$, $\operatorname{Min}_w f(s,x,K) \subset f(s,x,\Omega)$.*

Then (GSEP) is M-well-posedness.

Proof. We prove it by contradiction. Suppose that (GSEP) is not M-well-posedness. Hence there exists a minimizing sequence $\{(s_n, x_n)\} \subset T(K) \times K$ and $\epsilon > 0$ such that the relation (17) holds. If the sequence $\{x_n\}$ is unbounded, by a similar process in Case 2 of Theorem 10 we know that (GSEP) is M-well-posedness. If the sequence $\{x_n\}$ is bounded, then it has a convergent subsequence that converges to some point $x^\star \in X$ with a corresponding subsequence $\{s_n\}$ with $s_n \in T(x_n)$ for every $n \in \mathbb{N}$. We still denote it by $\{x_n\}$ if there is no confusion. The relation (17) tells us that

$$x^\star \notin \Omega. \tag{30}$$

Since $\{(s_n, x_n)\}$ is a minimizing sequence, we can choose a sequence $\{\tau_n\}$ with $\tau_n \to 0$, where $\tau_n \in \Phi(x_n) = \operatorname{Min}_w f(s_n, x_n, K)$ for all $n \in \mathbb{N}$. For the same process as in Case 1 of Theorem 10, by the upper semi-continuity of T, there exists a convergent subsequence of $\{s_n\}$ (without any confusion, we still denote it by $\{s_n\}$) that converges to some point $s^\star \in T(x^\star)$. Since $x_n \in K \setminus \Omega$, by condition (vi), $\tau_n \in \operatorname{Min}_w f(s_n, x_n, K) \subset f(s_n, x_n, \Omega)$. Then, for each $n \in \mathbb{N}$, there is a $z_n \in \Omega$ such that $\tau_n \in f(s_n, x_n, z_n)$. Since Ω is compact, there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ that converges to some point $z^\star \in \Omega$. Now we claim that $f(s^\star, x^\star, z^\star) \cap C \neq \emptyset$. Indeed, suppose that $f(s^\star, x^\star, z^\star) \cap C = \emptyset$. By Lemma 9, there is $W_Y \in \mathbb{N}_Y(0)$ such that

$$\left(f(s^\star, x^\star, z^\star) + W_Y\right) \cap (C + W_Y) = \emptyset, \tag{31}$$

or

$$\left(f(s^\star, x^\star, z^\star) + W_Y - C\right) \cap W_Y = \emptyset. \tag{32}$$

By the above C-upper Hausdorff continuity of f, for this W_Y, there is $k_0 \in \mathbb{N}$ such that

$$\tau_{n_k} \in f\left(s_{n_k}, x_{n_k}, z_{n_k}\right) \subset f(s^\star, x^\star, z^\star) + W_Y - C, \tag{33}$$

for all $k \geq k_0$. Thus, from (32), $\tau_{n_k} \notin W_Y$ which contradicts the fact that $\tau_{n_k} \to 0$. Hence, $f(s^\star, x^\star, z^\star) \cap C \neq \emptyset$. By condition (ii), $f(s^\star, z^\star, x^\star) \cap (-C) \neq \emptyset$. Since $z^\star \in \Omega$, by condition (iii), $x^\star \in \Omega$ which contradicts (30). Hence (GSEP) is M-well-posedness. □

Let us present the third criterion for M-well-posedness of (GSEP) as follows.

Theorem 15. *Let X, Y, Z, C, K, T, f be given the same as in Theorem 10. Suppose that*

(i) *the solution set Ω of (GSEP) is nonempty and bounded;*

(ii) *$f(s,x,z) \cap (-C) = \emptyset$ and $f(s,z,x) \cap (-C) = \emptyset$ for all $z \in \operatorname{cl}(\Omega)$, for all $s \in T(K)$ and for all $x \in K \setminus \operatorname{cl}(\Omega)$;*

(iii) *the mapping $x \to f(s,x,y)$ is above C-concave on K for every $(s,y) \in T(K) \times K$, and the mapping $s \to f(s,x,y)$ is lower semi-continuous on $T(K)$, for every $(x,y) \in K \times K$;*

(iv) *for every $x \in K \setminus \operatorname{cl}(\Omega)$ and for every $s \in T(x)$, $\operatorname{Min}_w f(s,x,K) \subset f(s,x,\operatorname{cl}(\Omega))$.*

Then (GSEP) is M-well-posedness.

Proof. Suppose that (GSEP) is not M-well-posedness. Then we have a minimizing sequence $\{(s_n, x_n)\} \subset T(K) \times K$ and $\epsilon > 0$ such that the relation (17) holds for all $n \in \mathbb{N}$. We can use the same process as in the proof of Theorem 10 under the situation when we replace Ω by $\mathrm{cl}(\Omega)$. If the sequence $\{x_n\}$ is unbounded, combining Lemmas 12 and 13, we have the sequences $\{x_{n_k}\}$, $\{v_n\}$, $\{v_{n_k}\}$, $\{\lambda_n\}$, $\{t_n\}$, $\{t_{n_k}\}$, $\{\zeta_{n_k}\}$, and points $\overline{x}$, $v^\star$, $t^\star$ with the same properties as in the proof (Case 2) of Theorem 10. By condition (iii), we have the mapping $x \rightarrow f(s, x, y)$ which is above C-concave on K for every $(s, y) \in T(K) \times K$ and the relations (20) and (21) hold. Since the mapping $s \rightarrow f(s, x, y)$ is lower semi-continuous on $T(K)$ for every $(x, y) \in K \times K$ and fix any $\xi \in f(t^\star, \overline{x}, v^\star)$, there exists $\xi_{n_k} \in f(t_{n_k}, \overline{x}, v^\star)$ such that $\xi_{n_k} \rightarrow \xi$. For this ξ_{n_k}, by (21), there exists $\nu_{n_k} \in f(t_{n_k}, v_{n_k}, v^\star)$ such that

$$\lambda_{n_k}\xi_{n_k} + \left(1 - \lambda_{n_k}\right)\zeta_{n_k} \in \nu_{n_k} - C. \tag{34}$$

Since the mapping $(s, x) \rightarrow f(s, x, y)$ is upper semi-continuous, hence it is upper Hausdorff continuous on $Z \times K$ for every $y \in K$, and $(t_{n_k}, v_{n_k}) \rightarrow (t^\star, \overline{x})$, for any given $W_Y \in \mathfrak{N}_Y(0)$,

$$f\left(t_{n_k}, v_{n_k}, v^\star\right) \subset f\left(t^\star, v^\star, v^\star\right) + W_Y = W_Y \tag{35}$$

for k large enough. From (34) and (35) and the fact that $\{\lambda_{n_k}\} \rightarrow 1$, we have

$$\xi \in W_Y - C. \tag{36}$$

Next, we claim that

$$f\left(t^\star, \overline{x}, v^\star\right) \cap (-C) \neq \emptyset. \tag{37}$$

Indeed, if not, $f(t^\star, \overline{x}, v^\star) \cap (-C) = \emptyset$. By Lemma 9, there exists $N_Y \in \mathfrak{N}_Y(0)$ such that $(f(t^\star, \overline{x}, v^\star) + N_Y) \cap (N_Y - C) = \emptyset$. But this contradicts (36), and hence (37) holds. Since $\overline{x} \in \mathrm{cl}(\Omega)$, by condition (ii), $v^\star \in \mathrm{cl}(\Omega)$ which contradicts the fact $v^\star \in \partial(\mathrm{cl}(\Omega) + \epsilon B)$. Hence, (GSEP) is M-well-posed. On the other hand, if the sequence $\{x_n\}$ is bounded, the sequences $\{x_{n_k}\}$, $\{s_n\}$, $\{s_{n_k}\}$, the points $x^\star$, $s^\star$, and the number ϵ are the same as in the Case 1 of Theorem 10, so that $x_{n_k} \notin \mathrm{cl}(\Omega) + \epsilon B$ for all $k \in \mathbb{N}$. Hence $x^\star \notin \mathrm{cl}(\Omega)$. Since $\{(s_n, x_n)\}$ is minimizing, we can choose a sequence $\eta_n \in \Phi(s_n, x_n) = \mathrm{Min}_w f(s_n, x_n, K)$ with $\eta_n \rightarrow 0$. By condition (v), $\eta_n \in \mathrm{Min}_w f(s_n, x_n, K) \subset f(s_n, x_n, \mathrm{cl}(\Omega))$. For each $n \in \mathbb{N}$, there is $z_n \in \mathrm{cl}(\Omega)$ such that $\eta_n \in f(s_n, x_n, z_n)$. Since $\mathrm{cl}(\Omega)$ is compact, there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ that converges to some point $z^\star \in \mathrm{cl}(\Omega)$. Since $f : Z \times K \times K \rightrightarrows Y$ is upper semi-continuous with nonempty compact values, we have

$$0 \in f\left(s^\star, x^\star, z^\star\right). \tag{38}$$

From condition (ii) and the fact that $z^\star \in \mathrm{cl}(\Omega)$, we have $x^\star \in \Omega$ which contradicts the fact $x^\star \notin \Omega$. Hence, (GSEP) is M-well-posed. □

Example 7 tells us that if (GSEP) is M-well-posed, then it is B-well-posed. But the converse is not true. Proposition 8 proposes a possible condition so that the converse holds. To the end, we state this result as follows.

Corollary 16. *Under the framework of Theorem 14 (resp., Theorem 15) the following condition (A) holds:*

(A) for every $W_Y \in \mathfrak{N}_Y(0)$, there is a $U_Y \in \mathfrak{N}(0)$ such that

$$f\left(g\left(K \setminus \mathrm{cl}\left(\Omega\right)\right)\right) \cap \left(C + U_Y\right) \subset W_Y, \tag{39}$$

where $g(x) = h(x) \times \mathrm{cl}(\Omega)$ and h is given in (7).

Then (GSEP) is B-well-posed.

Proof. From Theorem 14 (resp., Theorem 15), (GSEP) is M-well-posed. By condition (vi) of Theorem 14 (resp., condition (iv) of Theorem 15), we have

$$\Phi\left(s, x\right) \subset f\left(s, x, \mathrm{cl}\left(\Omega\right)\right) \tag{40}$$

for every $s \in T(x)$ and $x \in K \setminus \mathrm{cl}(\Omega)$. Hence,

$$\Phi\left(h\left(K \setminus \mathrm{cl}\left(\Omega\right)\right)\right) \subset f\left(g\left(K \setminus \mathrm{cl}\left(\Omega\right)\right)\right). \tag{41}$$

Combining this with condition (A), the condition (7) holds. Hence, by Proposition 8, (GSEP) is B-well-posed. □

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References

[1] A. N. Tihonov, "Stability of a problem of optimization of functionals," *Akademija Nauk SSSR*, vol. 6, pp. 631–634, 1966.

[2] L. C. Ceng and Y. C. Lin, "Metric characterizations of α-well-posedness for a system of mixed quasivariational-like inequalities in Banach spaces," *Journal of Applied Mathematics*, vol. 2012, Article ID 264721, 22 pages, 2012.

[3] A. L. Dontchev and T. Zolezzi, *Well-Posed Optimization Problems*, Springer, Berlin, Germany, 1993.

[4] E. Bednarczuk and J.-P. Penot, "Metrically well-set minimization problems," *Applied Mathematics and Optimization*, vol. 26, no. 3, pp. 273–285, 1992.

[5] G. P. Crespi, A. Guerraggio, and M. Rocca, "Well posedness in vector optimization problems and vector variational inequalities," *Journal of Optimization Theory and Applications*, vol. 132, no. 1, pp. 213–226, 2007.

[6] M. Furi and A. Vignoli, "About well-posed minimization problems for functionals in metric spaces," *Journal of Optimization Theory and Applications* , vol. 5, no. 3, pp. 225–229, 1970.

[7] T. Zolezzi, "Well-posedness criteria in optimization with application to the calculus of variations," *Nonlinear Analysis:Theory, Methods & Applications*, vol. 25, no. 5, pp. 437–453, 1995.

[8] E. Miglierina and E. Molho, "Well-posedness and convexity in vector optimization," *Mathematical Methods of Operations Research*, vol. 58, no. 3, pp. 375–385, 2003.

[9] E. Bednarczuk, "An approach to well-posedness in vector optimization: consequences to stability," *Control and Cybernetics*, vol. 23, no. 1-2, pp. 107–122, 1994.

[10] M. Bianchi, G. Kassay, and R. Pini, "Well-posedness for vector equilibrium problems," *Mathematical Methods of Operations Research*, vol. 70, no. 1, pp. 171–182, 2009.

[11] L. C. Ceng, N. Hadjisavvas, S. Schaible, and J. C. Yao, "Well-posedness for mixed quasivariational-like inequalities," *Journal of Optimization Theory and Applications*, vol. 139, no. 1, pp. 109–125, 2008.

[12] L. C. Ceng and J. C. Yao, "Well-posedness of generalized mixed variational inequalities, inclusion problems and fixed-point problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 12, pp. 4585–4603, 2008.

[13] Y.-P. Fang, N.-J. Huang, and J.-C. Yao, "Well-posedness of mixed variational inequalities, inclusion problems and fixed point problems," *Journal of Global Optimization*, vol. 41, no. 1, pp. 117–133, 2008.

[14] K. Kimura, Y.-C. Liou, S.-Y. Wu, and J.-C. Yao, "Well-posedness for parametric vector equilibrium problems with applications," *Journal of Industrial and Management Optimization*, vol. 4, no. 2, pp. 313–327, 2008.

[15] L. Q. Anh, P. Q. Khanh, D. T. M. van, and J.-C. Yao, "Well-posedness for vector quasiequilibria," *Taiwanese Journal of Mathematics*, vol. 13, no. 2B, pp. 713–737, 2009.

[16] C. Berge, *Topological Spaces*, Macmillan, New York, NY, USA, 1963.

[17] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, Germany, 1984.

[18] F. Ferro, "Optimization and stability results through cone lower semicontinuity," *Set-Valued Analysis*, vol. 5, no. 4, pp. 365–375, 1997.

[19] Y.-C. Lin, Q. H. Ansari, and H.-C. Lai, "Minimax theorems for set-valued mappings under cone-convexities," *Abstract and Applied Analysis*, vol. 2012, Article ID 310818, 26 pages, 2012.