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Research Article

Multiple Nonlinear Oscillations in a $\mathbb{D}_3 \times \mathbb{D}_3$-Symmetrical Coupled System of Identical Cells with Delays

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A coupled system of nine identical cells with delays and $\mathbb{D}_3 \times \mathbb{D}_3$-symmetry is considered. The individual cells are modelled by a scalar delay differential equation which includes linear decay and nonlinear delayed feedback. By analyzing the corresponding characteristic equations, the linear stability of the equilibrium is given. We also investigate the simultaneous occurrence of multiple periodic solutions and spatiotemporal patterns of the bifurcating periodic oscillations by using the equivariant bifurcation theory of delay differential equations combined with representation theory of Lie groups. Numerical simulations are carried out to illustrate our theoretical results.

1. Introduction

Over the past decades, symmetry has become a topic of considerable attention in the research of nonlinear dynamical systems [1–11]. In general, the symmetry reflects a certain spatial invariant of the dynamical systems. The work of Golubitsky et al. [1] shows that systems with symmetry can lead to multiple patterns of oscillation, which are predictable based on the theory of equivariant bifurcations. It is well known that the introduction of time delays into some systems is more reasonable and realistic [12]. Wu and coworkers [3, 13, 14] extended the theory of equivariant Hopf bifurcation to delay differential equations.

An artificial neural network is an information processing device that is inspired by the way biological nervous systems, such as the brain, process information simultaneously. It has many applications in different areas including pattern recognition, associative memory, signal processing, and combinatorial optimization. There has been an increasing interest in the investigation of neural networks (see, e.g., [4–6, 9–11, 15]) since Hopfield [16] constructed a simplified neural network model. Ring networks have been found in a variety of neural structures such as cerebellum [17] and even in chemistry and electrical engineering. In the field of neural networks, rings are studied to gain insight into the mechanisms underlying the behavior of recurrent networks [15, 18]. The dynamical behavior of ring networks has been investigated in more detail. For example, in order to understand which patterns occur in a particular system, Huang and Wu [4] studied the following ring neural network of three identical neurons with delayed feedback:

$$\dot{x}_i(t) = -x_i(t) + f(x_i(t-\tau)) - [g(x_{i-1}(t-\tau)) + g(x_{i+1}(t-\tau))], \quad i \,(\mathrm{mod}\, 3), \tag{1}$$

where $\dot{x} = dx/dt$, $x_i(t)$ represents the state of the ith neuron at time t, f represents the nonlinear self-feedback function, g is the connection function between neurons, and $\tau \geq 0$ is the time delay. Afterward, some researchers have been studying many ring networks with $\mathbb{D}_n$-symmetry (see [5–8]). However, previous work just has considered the individual network but not investigated the interactions between multiple networks.

In fact, a wide variety of natural and artificial systems possess a hierarchic structure or functioning and can naturally be modeled by coupled subnetwork. For example, the brain may be conceived as a dynamic network of coupled neurons. In order to describe the complicated interaction between billions of neurons in large neural network, the neurons are often lumped into highly connected subnetworks [19]. Coupled networks of nonlinear dynamical systems can

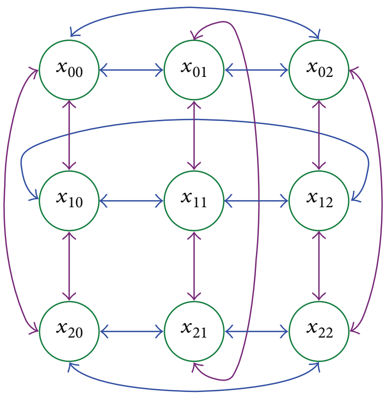

FIGURE 1: Architecture of model (2).

exhibit rich dynamics, such as synchronization, symmetric bifurcation, and chaos. The spatio-temporal dynamics of systems of several coupled nonlinear oscillators is presently receiving great attention and a significant body of research has been carried out [9–11, 20, 21]. It must be pointed out that the hierarchical network of neuronal oscillators with $\mathbb{D}_3\times\mathbb{D}_3$-symmetry investigated in [20, 21] is described by a system of ordinary differential equations (ODEs), and the effect of time delays is not considered.

Motivated by the above ideas, in this paper, we consider the two-level hierarchical system which is composed of three coupled modules of interacting nonlinear neuron oscillators with time delays, modeled by the following system of delay differential equations (DDEs):

$$\begin{aligned}\dot{x}_{i,j}(t) = &-x_{i,j}(t) + f\left(x_{i,j}(t-\tau)\right) + g\left(x_{i,j-1}(t-\tau)\right)\\ &+ g\left(x_{i,j+1}(t-\tau)\right) + h\left(x_{i-1,j}(t-\tau)\right)\\ &+ h\left(x_{i+1,j}(t-\tau)\right), \quad i,j = 0,1,2 \pmod 3,\end{aligned} \tag{2}$$

where $f, g, h \in C^1(\mathbb{R},\mathbb{R})$, $f(0) = g(0) = h(0) = 0$, and h represents the connection function between different modules. The individual elements are represented by a scalar equation, consisting of a linear decay term and a nonlinear time-delayed self-feedback. The architecture of the model is given in Figure 1.

It is easy to see that all cells are identical, all couplings within each group are identical, and all groups are identically coupled to each other in this model. Therefore, these lead to a $\mathbb{D}_3\times\mathbb{D}_3$-symmetry of the associated system. the model is a natural extension of system (1) and is a particularly simple example of a symmetric system exhibiting a hierarchical structure with two levels: a "macro" level concerning the interactions between the groups and a "micro" level concerning the interactions within the groups. On the other hand, system (2) can be regarded as a special example of the general Hopfield neural networks with delays [16].

Although model (2) is a little simple, it would be of great significance for applications to have a detailed analysis and then to understand possible mechanisms behind the observed behaviour. In this paper, our main purpose is to reveal how the time delay can affect the stability of system (2), the simultaneous occurrence of multiple periodic solutions, and spatio-temporal patterns of the bifurcating periodic oscillations depending on the $\mathbb{D}_3\times\mathbb{D}_3$-symmetry.

The rest of the paper is organized as follows. In Section 2, the associated characteristic equation is analyzed and the linear stability of the equilibrium is given. In Section 3, we discuss the existence of multiple branches of periodic oscillations and their spatio-temporal patterns with the help of symmetric bifurcation theory of delay differential equations coupled with representation theory of Lie groups. An example and numerical simulations are presented to illustrate the results in Section 4. In Section 5, a brief discussion is drawn to conclude this paper.

2. Distribution of Characteristic Roots and Linear Stability

It is clear that (2) admits the trivial solution, $\hat{x} = 0$. The linearization of (2) at this equilibrium point is given by

$$\begin{aligned}\dot{x}_{i,j}(t) = &-x_{i,j}(t) + a x_{i,j}(t-\tau) + b x_{i,j-1}(t-\tau)\\ &+ b x_{i,j+1}(t-\tau) + c x_{i-1,j}(t-\tau)\\ &+ c x_{i+1,j}(t-\tau), \quad i,j = 0,1,2 \pmod 3,\end{aligned} \tag{3}$$

where $a = f'(0)$, $b = g'(0)$, and $c = h'(0)$. Regarding τ as the parameter, let $A(\tau)$ denote the infinitesimal generator of the semigroup generated by linear system (3). We first determine when $A(\tau)$ has a pair of purely imaginary eigenvalues.

The characteristic matrix of (3) is

$$\Delta(\tau,\lambda) = (\lambda+1)\,\mathrm{Id}_9 - M e^{-\lambda\tau}, \quad \lambda\in\mathbb{C}, \tag{4}$$

where Id_n denotes the identity matrix of order n, $M = \mathrm{circ}(M_1, c\mathrm{Id}_3, c\mathrm{Id}_3)$ is a circle block matrix, and $M_1 = \mathrm{circ}(a,b,b)$ is a circulant matrix of order 3. Then we have the following lemma.

Lemma 1. *The associated characteristic equation of* (3) *is*

$$\det\Delta(\tau,\lambda) = \prod_{q=0}^{2}\prod_{p=0}^{2}\Delta_{pq} = 0, \tag{5}$$

where $\Delta_{pq} = \lambda + 1 - (a + 2b\cos(2q\pi/3) + 2c\cos(2p\pi/3))e^{-\lambda\tau}$.

Proof. Let $\chi = e^{i(2\pi/3)}$, $v_q = (1, \chi^q, \chi^{2q})^T$, and $v_{pq} = (v_q, \chi^p v_q, \chi^{2p} v_q)^T$. Then

$$M v_{pq} = \left(a + 2b\cos\frac{2q\pi}{3} + 2c\cos\frac{2p\pi}{3}\right) v_{pq}. \tag{6}$$

Hence

$$\begin{aligned}\Delta(\tau,\lambda)\, v_{pq} &= \left[\lambda + 1 - \left(a + c\chi^p + c\chi^{-p} + b\chi^q + b\chi^{-q}\right)e^{-\lambda\tau}\right] v_{pq}\\ &= \left[\lambda + 1 - \left(a + 2b\cos\frac{2q\pi}{3} + 2c\cos\frac{2p\pi}{3}\right)e^{-\lambda\tau}\right] v_{pq}.\end{aligned} \tag{7}$$

The conclusion is obtained. □

Note that

$$\begin{aligned}\Delta_{00} &= \lambda + 1 - (a+2b+2c)\,\mathrm{e}^{-\lambda\tau},\\ \Delta_{01} = \Delta_{02} &= \lambda + 1 - (a+2c-b)\,\mathrm{e}^{-\lambda\tau},\\ \Delta_{10} = \Delta_{20} &= \lambda + 1 - (a+2b-c)\,\mathrm{e}^{-\lambda\tau},\\ \Delta_{11} = \Delta_{12} = \Delta_{21} = \Delta_{22} &= \lambda + 1 - (a-b-c)\,\mathrm{e}^{-\lambda\tau},\end{aligned} \tag{8}$$

we have

$$\det\Delta(\tau,\lambda) = \Delta_{00}(\Delta_{01})^2(\Delta_{10})^2(\Delta_{11})^4. \tag{9}$$

In this paper, for the sake of simplicity, we consider only the case where characteristic equation (5) may have a pair of purely imaginary roots of multiplicity 4, that is, focus on the distribution of zeros of the factor Δ_{11}. The other cases are simpler and can be handled in a similar way, and we omit them. Therefore, throughout this paper we suppose

(**H**) $|a+2b+2c| < 1$, $|a+2c-b| < 1$, $|a+2b-c| < 1$, $a-b-c < -1$.

The following result about the distribution of the characteristic roots is obtained.

Lemma 2. *Assume that* (**H**) *holds. Define*

$$\tau_s = \frac{\arccos(1/(a-b-c)) + 2s\pi}{\sqrt{(a-b-c)^2-1}}, \quad s = 0,1,2,\dots, \tag{10}$$

$$\beta = \sqrt{(a-b-c)^2-1}. \tag{11}$$

Then

(i) *for all $\tau \geqslant 0$, all zeros of the factors Δ_{00}, Δ_{10}, and Δ_{01} have negative real parts,*

(ii) *when $\tau \in [0,\tau_0)$, all roots of the characteristic equation (5) have negative real parts; when $\tau \in (\tau_s,\tau_{s+1})$, the characteristic equation (5) has exactly $2s+2$ roots with positive real parts; the other roots have negative real parts; at (and only at) $\tau = \tau_s$, $A(\tau)$ has a pair of purely imaginary eigenvalues $\pm i\beta$ of multiplicity 4, and all other eigenvalues of $A(\tau)$ are not integer multiples of $\pm i\beta$,*

(iii) *for each fixed $s \in \mathbb{N}$, there exist $\delta > 0$ and C^1-smooth mapping $\lambda : (\tau_s-\delta,\tau_s+\delta) \to \mathbb{C}$ such that $\lambda(\tau_s) = i\beta$ and $\lambda(\tau) + 1 - (a-b-c)e^{-\tau\lambda(\tau)} = 0$ for all $\tau \in (\tau_s - \delta, \tau_s+\delta)$. Moreover, $(d/d\tau)\,\mathrm{Re}\,\lambda(\tau)|_{\tau=\tau_s} > 0$,*

(iv) *the generalized eigenspace $U_{\pm i\beta}(A(\tau_s))$ consists of eigenvector of $A(\tau_s)$ associated with $\pm i\beta$. Moreover,*

$$U_{\pm i\beta}(A(\tau_s)) = \left\{\sum_{q=1}^{2}\sum_{p=1}^{2}\left(y_{pq}\epsilon^{pq} + z_{pq}\varsigma^{pq}\right),\ y_{pq}, z_{pq} \in \mathbb{R},\ p,q = 1,2\right\}, \tag{12}$$

where

$$\begin{aligned}\epsilon^{pq}(\theta) &= \mathrm{Re}\left\{e^{i\beta\theta}v_{pq}\right\}\\ &= \cos(\beta\theta)\,\mathrm{Re}\left\{v_{pq}\right\} - \sin(\beta\theta)\,\mathrm{Im}\left\{v_{pq}\right\},\\ \varsigma^{pq}(\theta) &= \mathrm{Im}\left\{e^{i\beta\theta}v_{pq}\right\}\\ &= \sin(\beta\theta)\,\mathrm{Re}\left\{v_{pq}\right\} + \cos(\beta\theta)\,\mathrm{Im}\left\{v_{pq}\right\},\\ &\qquad \textit{for } \theta \in [-\tau, 0].\end{aligned} \tag{13}$$

Proof. For $\gamma \in \mathbb{R}$, let $q_\gamma(\lambda) = \lambda + 1 - \gamma\mathrm{e}^{-\lambda\tau}$. If $\lambda = \alpha + \mathrm{i}\beta$ is a zero of $q_\gamma(\lambda)$, then $1 + \alpha + \mathrm{i}\beta = \gamma\mathrm{e}^{-(\alpha+\mathrm{i}\beta)\tau}$, from which it follows that

$$(1+\alpha)^2 + \beta^2 = \gamma^2\mathrm{e}^{-2\alpha\tau}. \tag{14}$$

Thus, we claim that all the zeros of $q_\gamma(\lambda)$ have negative real parts provided that $|\gamma| < 1$. Noticing that (**H**) holds and applying the above discussions to the factors Δ_{00}, Δ_{10}, and Δ_{01}, we obtain conclusion (i).

In what follows, we consider the distribution of zeros of the factor Δ_{11}. For $\tau = 0$, $\Delta_{11} = 0$ becomes $\lambda = a-b-c-1 < -2$. Let $\mathrm{i}\beta$ ($\beta > 0$) be a root of $\Delta_{11} = 0$, then

$$\begin{aligned}&\mathrm{i}\beta + 1 - (b+c-a)\,\mathrm{e}^{\mathrm{i}(\pi-\tau\beta)}\\ &= \sqrt{1+\beta^2}\mathrm{e}^{\mathrm{i}\arccos(1/\sqrt{1+\beta^2})} - (b+c-a)\,\mathrm{e}^{\mathrm{i}(\pi-\tau\beta)} = 0.\end{aligned} \tag{15}$$

Thus,

$$\begin{aligned}\sqrt{1+\beta^2} &= b+c-a,\\ \arccos\frac{1}{\sqrt{1+\beta^2}} &= \pi - \tau\beta + 2s\pi, \quad s \in \mathbb{Z}.\end{aligned} \tag{16}$$

Therefore, $\Delta_{11} = 0$ has a pair of purely imaginary roots $\pm\mathrm{i}\beta$ if and only if

$$\begin{aligned}\beta &= \sqrt{(a-b-c)^2-1},\\ \tau &= \frac{\arccos(1/(a-b-c)) + 2s\pi}{\sqrt{(a-b-c)^2-1}}\\ &:= \tau_s, \quad \text{for some } s \in \mathbb{N}.\end{aligned} \tag{17}$$

Clearly, for each fixed s, $\tau_s \geqslant 0$. Since

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta_{11}(\lambda)\right|_{\lambda=\mathrm{i}\beta,\tau=\tau_s} &= 1 + \left.(a-b-c)\tau\mathrm{e}^{-\lambda\tau}\right|_{\lambda=\mathrm{i}\beta,\tau=\tau_s}\\ &= 1 + \tau_s(\mathrm{i}\beta+1)\\ &\neq 0,\end{aligned} \tag{18}$$

there exist $\delta > 0$ and C^1-smooth mapping $\lambda : (\tau_s-\delta,\tau_s+\delta) \to \mathbb{C}$ such that $\lambda(\tau_s) = \mathrm{i}\beta$ and $\lambda(\tau) + 1 - (a-b-c)\mathrm{e}^{-\tau\lambda(\tau)} = 0$ for all $\tau \in (\tau_s - \delta, \tau_s + \delta)$. Differentiating $\Delta_{11} = 0$ with respect to τ, we have

$$\lambda'(\tau_s) = \frac{-(a-b-c)\lambda(\tau_s)e^{-\lambda(\tau_s)\tau_s}}{1+(a-b-c)\tau_s e^{-\lambda(\tau_s)\tau_s}} = \frac{-\lambda(\tau_s)[\lambda(\tau_s)+1]}{1+\tau_s[\lambda(\tau_s)+1]}. \tag{19}$$

Therefore,

$$\operatorname{Re}\lambda'(\tau_s) = \frac{\beta^2}{(1+\tau_s)^2+\tau_s^2\beta^2} > 0. \tag{20}$$

So far, the proof of (ii) and (iii) is complete (for more details see XI.2 in [22]). It remains to verify (iv).

Note that $v_{11} = \overline{v_{22}}$, $v_{12} = \overline{v_{21}}$; it follows from the proof of Lemma 1 and the above discussions that the eigenspace of $A(\tau_s)$ associated with $\pm i\beta$ is spanned by $e^{i\beta\theta}v_{11}$, $e^{i\beta\theta}v_{12}$, $e^{i\beta\theta}v_{21}$, $e^{i\beta\theta}v_{22}$, $e^{-i\beta\theta}v_{11}$, $e^{-i\beta\theta}v_{12}$, $e^{-i\beta\theta}v_{21}$, and $e^{-i\beta\theta}v_{22}$. Hence, the space has the real basis $\{\epsilon^{11}, \varsigma^{11}, \epsilon^{12}, \varsigma^{12}, \epsilon^{21}, \varsigma^{21}, \epsilon^{22}, \varsigma^{22}\}$. On the other hand, the eigenspace $A(\tau_s)$ associated with $i\beta_{pq}$ is of dimension 4, and the multiplicity of the characteristic root $\lambda = i\beta_{pq}$ is also 4. According to the folk theorem in functional differential equations (see [23]), $U_{\pm i\beta}(A(\tau_s))$ must coincide with the eigenspace of $A(\tau_s)$ associated with $\pm i\beta$. Therefore, conclusion (iv) is correct. This completes the proof. □

From Lemma 2 (ii), we can draw the following conclusion about the linear stability of system (2).

Theorem 3. *If the assumption* (**H**) *is satisfied, then the equilibrium $\hat{x} = 0$ of system* (2) *is asymptotically stable for $\tau \in [0, \tau_0)$; the equilibrium $\hat{x} = 0$ is unstable for $\tau > \tau_0$.*

3. Multiple Patterns of Oscillation

It follows from Lemma 2 that $A(\tau)$ has a pair of purely imaginary eigenvalues $\pm i\beta$ of multiplicity 4 and all other eigenvalues of $A(\tau)$ are not integer multiples of $\pm i\beta$ at $\tau = \tau_s$ $(s = 0, 1, \ldots)$. Thus, the Hopf bifurcation may provide some asynchronous periodic solutions at each τ_s.

The symmetry of a system is important in determining the patterns of oscillation. In order to explore the symmetry of system and analyze the spatio-temporal patterns of the bifurcated periodic solutions, we need the following definition.

Let $\mathbb{D}_3$ denote the dihedral group of order 6, which is generated by cyclic group $\mathbb{Z}_3$ together with the flip κ of order 2. Define the action of $\Gamma := \mathbb{D}_3 \times \mathbb{D}_3$ on $\mathbb{R}^9$ by

$$((\rho,1)\cdot x)_{i,j} = x_{i+1,j}, \qquad ((1,\rho)\cdot x)_{i,j} = x_{i,j+1},$$
$$((\kappa,1)\cdot x)_{i,j} = x_{3-i,j}, \qquad ((1,\kappa)\cdot x)_{i,j} = x_{i,3-j}, \qquad i, j \mod 3, \tag{21}$$

where ρ is the generator of $\mathbb{Z}_3$ and κ denotes the flip.

Lemma 4. *System* (2) *is Γ-equivalent.*

Proof. Let mapping $\mathcal{F} : C([-\tau, 0], \mathbb{R}^9) \to \mathbb{R}^9$ be

$$\begin{aligned}(\mathcal{F}(\phi))_{i,j} = {} & -(\phi)_{i,j}(0) + f((\phi)_{i,j}(-\tau)) \\ & + g((\phi)_{i,j-1}(-\tau)) + g((\phi)_{i,j+1}(-\tau)) \\ & + h((\phi)_{i-1,j}(-\tau)) \\ & + h((\phi)_{i+1,j}(-\tau)), \quad i, j \pmod 3,\end{aligned} \tag{22}$$

where

$$\begin{aligned}\phi &= (\phi_{00}, \phi_{01}, \phi_{02}, \phi_{10}, \phi_{11}, \phi_{12}, \phi_{20}, \phi_{21}, \phi_{22})^{T} \in C([-\tau, 0], \mathbb{R}^9), \\ \mathcal{F}(\phi) &= ((\mathcal{F}(\phi))_{00}, (\mathcal{F}(\phi))_{01}, (\mathcal{F}(\phi))_{02}, \\ &\quad (\mathcal{F}(\phi))_{10}, (\mathcal{F}(\phi))_{11}, (\mathcal{F}(\phi))_{12}, \\ &\quad (\mathcal{F}(\phi))_{20}, (\mathcal{F}(\phi))_{21}, (\mathcal{F}(\phi))_{22})^{T} \in \mathbb{R}^9.\end{aligned} \tag{23}$$

Then

$$\begin{aligned}&(\mathcal{F}((\rho,1)\phi))_{i,j} \\ &= -((\rho,1)\phi)_{i,j}(0) + f(((\rho,1)\phi)_{i,j}(-\tau)) \\ &\quad + g(((\rho,1)\phi)_{i,j-1}(-\tau)) + g(((\rho,1)\phi)_{i,j+1}(-\tau)) \\ &\quad + h(((\rho,1)\phi)_{i-1,j}(-\tau)) + h(((\rho,1)\phi)_{i+1,j}(-\tau)) \\ &= -(\phi)_{i+1,j}(0) + f((\phi)_{i+1,j}(-\tau)) \\ &\quad + g((\phi)_{i+1,j-1}(-\tau)) \\ &\quad + g((\phi)_{i+1,j+1}(-\tau)) + h((\phi)_{i,j}(-\tau)) \\ &\quad + h((\phi)_{i+2,j}(-\tau)) \\ &= ((\rho,1)\mathcal{F}(\phi))_{i,j}, \\ &(\mathcal{F}((\kappa,1)\phi))_{i,j} \\ &= -((\kappa,1)\phi)_{i,j}(0) + f(((\kappa,1)\phi)_{i,j}(-\tau)) \\ &\quad + g(((\kappa,1)\phi)_{i,j-1}(-\tau)) + g(((\kappa,1)\phi)_{i,j+1}(-\tau)) \\ &\quad + h(((\kappa,1)\phi)_{i-1,j}(-\tau)) + h(((\kappa,1)\phi)_{i+1,j}(-\tau)) \\ &= -(\phi)_{3-i,j}(0) + f((\phi)_{3-i,j}(-\tau)) \\ &\quad + g((\phi)_{3-i,j-1}(-\tau)) + g((\phi)_{3-i,j+1}(-\tau)) \\ &\quad + h((\phi)_{4-i,j}(-\tau)) + h((\phi)_{2-i,j}(-\tau)) \\ &= ((\kappa,1)\mathcal{F}(\phi))_{i,j}.\end{aligned} \tag{24}$$

Similarly, we can prove that

$$\begin{aligned}(\mathscr{F}((1,\rho)\phi))_{i,j} &= ((1,\rho)\mathscr{F}(\phi))_{i,j},\\ (\mathscr{F}((1,\kappa)\phi))_{i,j} &= ((1,\kappa)\mathscr{F}(\phi))_{i,j}.\end{aligned} \quad (25)$$

Therefore, $\mathscr{F}$ is $\mathbb{D}_3\times\mathbb{D}_3$-equivalent. This completes the proof. □

Lemma 5. *Let* Γ *act on* $\mathbb{R}^4$ *by*

$$(\rho,1)\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0\\ 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0\\ 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},$$

$$(1,\rho)\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0\\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0\\ 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},$$

$$(\kappa,1)\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}x_1\\x_2\\-x_3\\-x_4\end{pmatrix},$$

$$(1,\kappa)\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}x_1\\-x_2\\x_3\\-x_4\end{pmatrix}. \quad (26)$$

Then $\mathbb{R}^4$ *is an absolutely irreducible representation of* Γ*, and the restricted action of* Γ *on* $\operatorname{Ker}\Delta(\tau, i\beta)$ *is isomorphic to the action of* Γ *on* $\mathbb{R}^4\oplus\mathbb{R}^4$.

Proof. It is straightforward to verify the absolute irreducibility of the representation of Γ on $\mathbb{R}^4$ by the definition (see [1]). Note that

$$\operatorname{Ker}\Delta(\tau,\mathrm{i}\beta) = \left\{\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}, a_{pq}, b_{pq}\in\mathbb{R},\ p,q=1,2\right\}. \quad (27)$$

Define $J:\operatorname{Ker}\Delta(\tau,\mathrm{i}\beta)\to\mathbb{R}^4\oplus\mathbb{R}^4$ by

$$\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq} \longmapsto B(a_{11},b_{11},a_{12},b_{12},a_{21},b_{21},a_{22},b_{22})^{\mathrm{T}}, \quad (28)$$

where the matrix

$$B=\begin{pmatrix}1&0&1&0&1&0&1&0\\0&1&0&-1&0&1&0&-1\\0&1&0&1&0&-1&0&-1\\-1&0&1&0&1&0&-1&0\\0&1&0&1&0&1&0&1\\-1&0&1&0&-1&0&1&0\\-1&0&-1&0&1&0&1&0\\0&-1&0&1&0&1&0&-1\end{pmatrix}. \quad (29)$$

The nonsingularity of the matrix B implies that $J:\operatorname{Ker}\Delta(\tau,\mathrm{i}\beta)\cong\mathbb{R}^4$ is a linear isomorphism. It is easy to see that

$$\begin{aligned}
&(\rho,1)\cdot\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right)\\
&\quad=\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)\mathrm{e}^{\mathrm{i}((-1)^{p+1}2\pi/3)}v_{pq},\\
&(1,\rho)\cdot\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right)\\
&\quad=\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)\mathrm{e}^{\mathrm{i}((-1)^{q+1}2\pi/3)}v_{pq},\\
&(\kappa,1)\cdot\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right)\\
&\quad=\left(a_{21}+\mathrm{i}b_{21}\right)v_{11}+\left(a_{22}+\mathrm{i}b_{22}\right)v_{12}\\
&\qquad+\left(a_{11}+\mathrm{i}b_{11}\right)v_{21}+\left(a_{12}+\mathrm{i}b_{12}\right)v_{22},\\
&(1,\kappa)\cdot\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right)\\
&\quad=\left(a_{12}+\mathrm{i}b_{12}\right)v_{11}+\left(a_{11}+\mathrm{i}b_{11}\right)v_{12}\\
&\qquad+\left(a_{22}+\mathrm{i}b_{22}\right)v_{21}+\left(a_{21}+\mathrm{i}b_{21}\right)v_{22}.
\end{aligned} \quad (30)$$

Therefore, a straightforward calculation shows that

$$\begin{aligned}
&J\left((\rho,1)\cdot\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right)\right)\\
&\quad=(\rho,1)\cdot J\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right),\\
&J\left((1,\rho)\cdot\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right)\right)\\
&\quad=(1,\rho)\cdot J\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right),
\end{aligned}$$

$$\begin{aligned} &J\left((\kappa,1)\cdot\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right)\right) \\ &\quad=(\kappa,1)\cdot J\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right), \\ &J\left((1,\kappa)\cdot\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right)\right) \\ &\quad=(1,\kappa)\cdot J\left(\sum_{p=1}^{2}\sum_{q=1}^{2}\left(a_{pq}+\mathrm{i}b_{pq}\right)v_{pq}\right). \end{aligned} \tag{31}$$

This concludes the proof. □

Let $\omega = 2\pi/\beta$ and let P_ω be the Banach space of all continuous ω-periodic functions $x:\mathbb{R}\to\mathbb{R}^9$. Then for the circle group S^1, $\Gamma\times S^1$ acts on P_ω by

$$\begin{gathered} \left(\gamma,\mathrm{e}^{\mathrm{i}\theta}\right)\cdot x(t)=\gamma\cdot x\left(t+\frac{\omega}{2\pi}\theta\right), \\ \left(\gamma,\mathrm{e}^{\mathrm{i}\theta}\right)\in\Gamma\times S^1,\ x\in P_\omega. \end{gathered} \tag{32}$$

Denote by SP_ω the subspace of P_ω consisting of all ω-periodic solutions of system (3) with $\tau=\tau_s$. Then

$$\begin{aligned} &SP_\omega \\ &=\left\{\sum_{q=1}^{2}\sum_{p=1}^{2}\left(y_{pq}\epsilon^{pq}+z_{pq}\varsigma^{pq}\right),\ y_{pq},z_{pq}\in\mathbb{R},\ p,q=1,2\right\}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} \epsilon^{pq}(t)&=\operatorname{Re}\left\{\mathrm{e}^{\mathrm{i}\beta t}v_{pq}\right\} \\ &=\cos(\beta t)\operatorname{Re}\left\{v_{pq}\right\}-\sin(\beta t)\operatorname{Im}\left\{v_{pq}\right\}, \\ \varsigma^{pq}(t)&=\operatorname{Im}\left\{\mathrm{e}^{\mathrm{i}\beta t}v_{pq}\right\} \\ &=\sin(\beta t)\operatorname{Re}\left\{v_{pq}\right\}+\cos(\beta t)\operatorname{Im}\left\{v_{pq}\right\}. \end{aligned} \tag{34}$$

Therefore, for $\epsilon^{pq}(t)$, $\varsigma^{pq}(t)$ $(p,q=1,2)$, we have the following properties.

Lemma 6.

$$\begin{gathered} (\rho,1)\cdot\epsilon^{pq}=\epsilon^{pq}\cos\frac{2p\pi}{3}-\varsigma^{pq}\sin\frac{2p\pi}{3}, \\ (\rho,1)\cdot\varsigma^{pq}=\varsigma^{pq}\cos\frac{2p\pi}{3}+\epsilon^{pq}\sin\frac{2p\pi}{3}, \\ (1,\rho)\cdot\epsilon^{pq}=\epsilon^{pq}\cos\frac{2q\pi}{3}-\varsigma^{pq}\sin\frac{2q\pi}{3}, \\ (1,\rho)\cdot\varsigma^{pq}=\varsigma^{pq}\cos\frac{2q\pi}{3}+\epsilon^{pq}\sin\frac{2q\pi}{3}, \\ (\kappa,1)\cdot\epsilon^{pq}=\epsilon^{3-p,q},\qquad(\kappa,1)\cdot\varsigma^{pq}=\varsigma^{3-p,q}, \\ (1,\kappa)\cdot\epsilon^{pq}=\epsilon^{p,3-q},\qquad(1,\kappa)\cdot\varsigma^{pq}=\varsigma^{p,3-q}. \end{gathered} \tag{35}$$

Proof. For $i,j\ (\mathrm{mod}\ 3)$ and $t\in\mathbb{R}$, note that

$$\begin{aligned} \epsilon_{i,j}^{pq}(t)&=\cos\left(\beta t+\frac{2ip\pi}{3}+\frac{2jq\pi}{3}\right), \\ \varsigma_{i,j}^{pq}(t)&=\sin\left(\beta t+\frac{2ip\pi}{3}+\frac{2jq\pi}{3}\right), \end{aligned} \tag{36}$$

we have

$$\begin{aligned} &\left((\rho,1)\cdot\epsilon^{pq}(t)\right)_{i,j} \\ &\quad=\epsilon_{i+1,j}^{pq}(t)=\cos\left(\beta t+\frac{2(i+1)p\pi}{3}+\frac{2jq\pi}{3}\right) \\ &\quad=\cos\left(\beta t+\frac{2ip\pi}{3}+\frac{2jq\pi}{3}+\frac{2p\pi}{3}\right) \\ &\quad=\epsilon_{i,j}^{pq}\cos\frac{2p\pi}{3}-\varsigma_{i,j}^{pq}\sin\frac{2p\pi}{3}, \\ &\left((\rho,1)\cdot\varsigma^{pq}(t)\right)_{i,j} \\ &\quad=\varsigma_{i+1,j}^{pq}(t)=\sin\left(\beta t+\frac{2(i+1)p\pi}{3}+\frac{2jq\pi}{3}\right) \\ &\quad=\sin\left(\beta t+\frac{2ip\pi}{3}+\frac{2jq\pi}{3}+\frac{2p\pi}{3}\right) \\ &\quad=\varsigma_{i,j}^{pq}\cos\frac{2p\pi}{3}+\epsilon_{i,j}^{pq}\sin\frac{2p\pi}{3}, \\ &\left((\kappa,1)\cdot\epsilon^{pq}(t)\right)_{i,j} \\ &\quad=\epsilon_{3-i,j}^{pq}(t)=\cos\left(\beta t+\frac{2(3-i)p\pi}{3}+\frac{2jq\pi}{3}\right) \\ &\quad=\cos\left(\beta t+\frac{2i(3-p)\pi}{3}+\frac{2jq\pi}{3}\right)=\epsilon_{i,j}^{3-p,q}, \\ &\left((\kappa,1)\cdot\varsigma^{pq}(t)\right)_{i,j} \\ &\quad=\epsilon_{3-i,j}^{pq}(t)=\sin\left(\beta t+\frac{2(3-i)p\pi}{3}+\frac{2jq\pi}{3}\right) \\ &\quad=\sin\left(\beta t+\frac{2i(3-p)\pi}{3}+\frac{2jq\pi}{3}\right)=\varsigma_{i,j}^{3-p,q}. \end{aligned} \tag{37}$$

Table 1: The maximal isotropy subgroups of $\Gamma \times S^1$ and associated fixed point subspaces.

No.	Σ	$\text{Fix}(\Sigma, SP_\omega)$	dim $\text{Fix}(\Sigma, SP_\omega)$
1	$\langle(\kappa,1,1),(1,\kappa,1)\rangle$	(y,z,y,z,y,z,y,z)	2
2	$\langle(\kappa,1,1),(1,\kappa,-1)\rangle$	$(y,z,-y,-z,y,z,-y,-z)$	2
3	$\langle(\kappa,1,-1),(1,\kappa,1)\rangle$	$(y,z,y,z,-y,-z,-y,-z)$	2
4	$\langle(\kappa,1,-1),(1,\kappa,-1)\rangle$	$(y,z,-y,-z,-y,-z,y,z)$	2
5	$\langle(\rho,1,e^{-2\pi i/3}),(1,\rho,e^{-2\pi i/3})\rangle$	$(y,z,0,0,0,0,0,0)$	2
6	$\langle(\rho,1,e^{-2\pi i/3}),(1,\rho,e^{2\pi i/3})\rangle$	$(0,0,y,z,0,0,0,0)$	2
7	$\langle(\rho,1,e^{2\pi i/3}),(1,\rho,e^{-2\pi i/3})\rangle$	$(0,0,0,0,y,z,0,0)$	2
8	$\langle(\rho,1,e^{2\pi i/3}),(1,\rho,e^{2\pi i/3})\rangle$	$(0,0,0,0,0,0,y,z)$	2
9	$\langle(\rho,1,e^{-2\pi i/3}),(1,\kappa,1)\rangle$	$(y,z,y,z,0,0,0,0)$	2
10	$\langle(\rho,1,e^{-2\pi i/3}),(1,\kappa,-1)\rangle$	$(y,z,-y,-z,0,0,0,0)$	2
11	$\langle(\rho,1,e^{2\pi i/3}),(1,\kappa,1)\rangle$	$(0,0,0,0,y,z,y,z)$	2
12	$\langle(\rho,1,e^{2\pi i/3}),(1,\kappa,-1)\rangle$	$(0,0,0,0,y,z,-y,-z)$	2
13	$\langle(1,\rho,e^{-2\pi i/3}),(\kappa,1,1)\rangle$	$(y,z,0,0,y,z,0,0)$	2
14	$\langle(1,\rho,e^{-2\pi i/3}),(\kappa,1,-1)\rangle$	$(y,z,0,0,-y,-z,0,0)$	2
15	$\langle(1,\rho,e^{2\pi i/3}),(\kappa,1,1)\rangle$	$(0,0,y,z,0,0,y,z)$	2
16	$\langle(1,\rho,e^{2\pi i/3}),(\kappa,1,-1)\rangle$	$(0,0,y,z,0,0,-y,-z)$	2
17	$\langle(\rho,\rho,1),(\kappa,\kappa,1)\rangle$	$(0,0,y,z,y,z,0,0)$	2
18	$\langle(\rho,\rho,1),(\kappa,\kappa,-1)\rangle$	$(0,0,y,z,-y,-z,0,0)$	2
19	$\langle(\rho^2,\rho,1),(\kappa,\kappa,1)\rangle$	$(y,z,0,0,0,0,y,z)$	2
20	$\langle(\rho^2,\rho,1),(\kappa,\kappa,-1)\rangle$	$(y,z,0,0,0,0,-y,-z)$	2

Therefore

$$\begin{aligned}(\rho,1)\cdot\epsilon^{pq} &= \epsilon^{pq}\cos\frac{2p\pi}{3} - \varsigma^{pq}\sin\frac{2p\pi}{3},\\ (\rho,1)\cdot\varsigma^{pq} &= \varsigma^{pq}\cos\frac{2p\pi}{3} + \epsilon^{pq}\sin\frac{2p\pi}{3},\\ (\kappa,1)\cdot\epsilon^{pq} = \epsilon^{3-p,q}, &\qquad (\kappa,1)\cdot\varsigma^{pq} = \varsigma^{3-p,q}.\end{aligned} \tag{38}$$

Similarly, we can prove that

$$\begin{aligned}(1,\rho)\cdot\epsilon^{pq} &= \epsilon^{pq}\cos\frac{2q\pi}{3} - \varsigma^{pq}\sin\frac{2q\pi}{3},\\ (1,\rho)\cdot\varsigma^{pq} &= \varsigma^{pq}\cos\frac{2q\pi}{3} + \epsilon^{pq}\sin\frac{2q\pi}{3},\\ (1,\kappa)\cdot\epsilon^{pq} = \epsilon^{p,3-q}, &\qquad (1,\kappa)\cdot\varsigma^{pq} = \varsigma^{p,3-q}.\end{aligned} \tag{39}$$

□

It is clear that if x is a periodic solution of system (2), then so is $(\gamma, e^{i\theta})x$ for every $(\gamma, e^{i\theta}) \in \Gamma \times S^1$. The spatial-temporal symmetry of a bifurcation of periodic solutions $x(t)$ can be completely characterized by the isotropy group $\Sigma_x = \{(\gamma, e^{i\theta}) \in \Gamma \times S^1 \mid (\gamma, e^{i\theta})x = x\} \leq \Gamma \times S^1$, and it is easy to verify that the isotropy group of $(\gamma, e^{i\theta})x$ is $(\gamma, e^{i\theta})\Sigma_x(\gamma, e^{i\theta})^{-1}$, which is conjugate to Σ_x. The maximal isotropy subgroups Σ^m $(m = 1, 2, \ldots, 20)$ of $\Gamma \times S^1$ are listed in Table 1. For each subgroup Σ^m, the Σ^m-fixed-point set

$$\begin{aligned}&\text{Fix}(\Sigma^m, SP_\omega)\\ &\quad = \left\{x \in SP_\omega \mid (\gamma, e^{i\theta})x = x,\ \forall (\gamma, e^{i\theta}) \in \Sigma^m\right\}\end{aligned} \tag{40}$$

is a subspace of SP_ω. According to Lemma 6, it is easy to verify that dim $\text{Fix}(\Sigma^m, SP_\omega) = 2$, $m = 1, 2, \ldots, 20$.

Together with Lemma 2, Lemmas 4–6 allow us to apply the equivariant Hopf bifurcation theorem for delay differential equations due to Wu [3] to obtain the following result on the spatio-temporal patterns of the bifurcated periodic solutions.

Theorem 7. *Assume that* (**H**) *is satisfied. Then for system* (2), *near each* τ_s $(s = 0, 1, \ldots)$ *there exist* 100 *distinct branches of periodic solutions bifurcated from the equilibrium* $\hat{x} = 0$. *More precisely, for each isotropy group* Σ^m $(m = 1, 2, \ldots, 20)$ *and a chosen basis* $\{\delta_1, \delta_2\}$ *of* $\text{Fix}(\Sigma^m, SP_\omega)$, *there exist* $\alpha_0 > 0$, $\tau_0^* > 0$, $\sigma_0 > 0$, *and a* C^1*-smooth mapping* $(\tau^*, \omega^*, x^*) : \mathbb{R}^2_{\alpha_0} \to \mathbb{R} \times \mathbb{R}^+ \times C(\mathbb{R}, \mathbb{R}^9)$, *where* $\mathbb{R}^2_{\alpha_0} = \{\alpha \in \mathbb{R}^2 \mid |\alpha| < \alpha_0\}$, *such that for each* $\alpha \in \mathbb{R}^2_{\alpha_0}$, $x^* = x^*(t;\alpha)$ *is an* $\omega^*(\alpha)$*-periodic solution of system* (2) *with* $\tau = \tau_s + \tau^*(\alpha)$, *and*

$$\begin{aligned}\gamma \cdot x^*(t) &= x^*\left(t - \frac{\omega^*(\alpha)}{2\pi}\theta\right), \quad (\gamma, e^{i\theta}) \in \Sigma^m,\\ \omega^*(0) &= \frac{2\pi}{\beta}, \quad \tau^*(0) = 0,\\ x^*(t;\alpha) &= (\delta_1, \delta_2)\alpha + o(|\alpha|), \quad \text{as } |\alpha| \longrightarrow 0.\end{aligned} \tag{41}$$

Furthermore, for $|\tau - \tau_s| < \tau_0^*$, $|\widetilde{\omega} - 2\pi/\beta| < \sigma_0$, *every* $\widetilde{\omega}$*-periodic solution* $x(t)$ *of system* (2) *with* $\|x_t\| < \sigma_0$, $\gamma \cdot x(t) = x(t - (\widetilde{\omega}/2\pi)\theta)$ *for* $(\gamma, e^{i\theta}) \in \Sigma^m$ *and* $t \in \mathbb{R}$ *must be of the above type.*

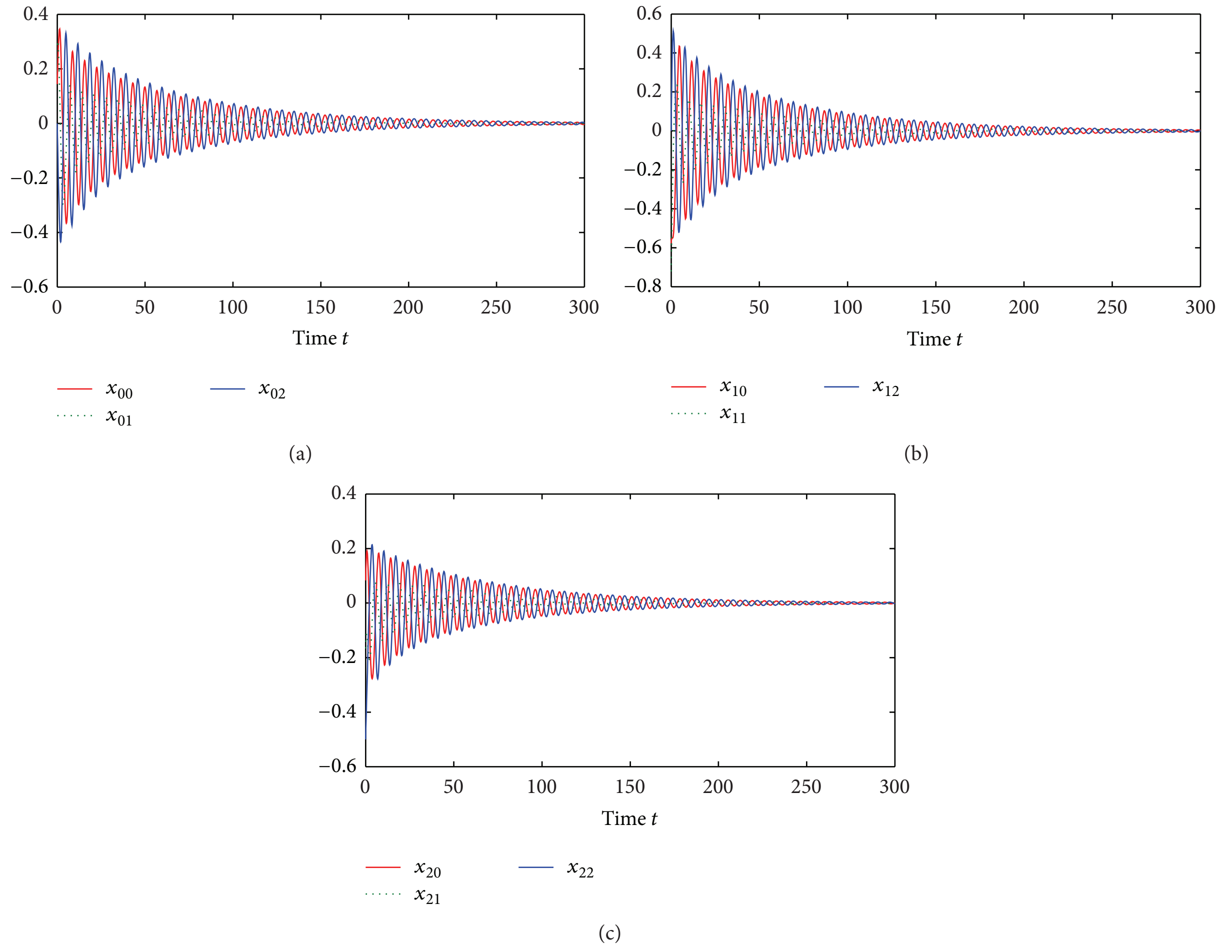

FIGURE 2: Trajectories $x_{i,j}$ $(i, j = 0, 1, 2)$ of system (42) when $\tau = 2.6$.

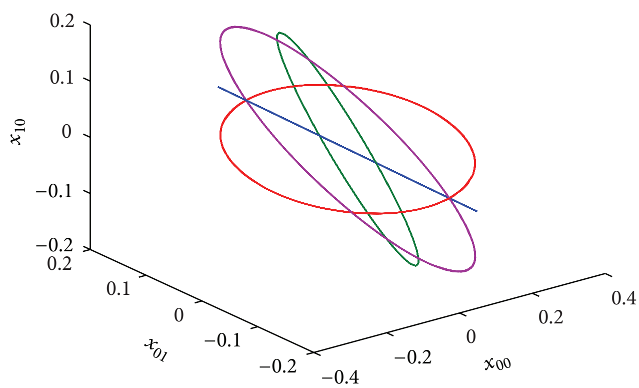

FIGURE 3: System (42) has multiple periodic solutions when $\tau = 3$.

4. An Example and Numerical Simulations

As a simple example, we consider the following specific Hopfield model with $\mathbb{D}_3 \times \mathbb{D}_3$-symmetry:

$$\begin{aligned}\dot{x}_{i,j}(t) = & -x_{i,j}(t) - 0.6\tanh\left(x_{i,j}(t-\tau)\right) \\ & + 0.3\tanh\left(x_{i,j-1}(t-\tau)\right) \\ & + 0.3\tanh\left(x_{i,j+1}(t-\tau)\right) \\ & + 0.4\tanh\left(x_{i-1,j}(t-\tau)\right) \\ & + 0.4\tanh\left(x_{i+1,j}(t-\tau)\right),\end{aligned} \tag{42}$$

where $i, j = 0, 1, 2 \pmod 3$.

Clearly, the origin is an equilibrium of system (42). It is easy to compute $a = -0.6$, $b = 0.3$, and $c = 0.4$ and verify that the hypothesis (**H**) holds. According to (10), we obtain

$$\tau_s = 2.9476 + 7.5641s, \quad s = 0, 1, \ldots. \tag{43}$$

From Theorem 3, the origin is asymptotically stable if $\tau < \tau_0 = 2.9476$. It follows from Theorem 7 that the equivariant Hopf bifurcation occurs at $\tau_0 = 2.9476$ and there exist 100 distinct branches of periodic solutions bifurcated from the origin. To illustrate the analytical results found, we give some numerical simulations. Figure 2 shows that the origin of system (42) is stable when $\tau = 2.6$. Figure 3 shows that four periodic orbits occur simultaneously when $\tau = 3$. Unfortunately, we cannot verify the existence of all other bifurcated periodic orbits since they may be unstable.

5. Discussion

In this paper, we have studied a coupled system of nine identical cells with delays and $\mathbb{D}_3 \times \mathbb{D}_3$-symmetry. By choosing the time delay τ as a bifurcation parameter and analyzing the corresponding characteristic equation, we have shown that under some assumption, the equilibrium of the model loses its stability and periodic solutions via Hopf bifurcation occur when τ passes through a critical value. This implies that the time delay can be regarded as a source of instability and oscillatory response of the networks and is able to alter the dynamics of system (2) significantly. Moreover, the spatio-temporal patterns of bifurcating periodic solutions are explored clearly by employing the symmetric bifurcation theory of delay differential equations combined with representation theory of Lie groups. From Theorem 7, we have obtained the conclusion that the small-scale network with a special structure may have a large number of periodic oscillations. Therefore, it is natural that the large-scale network possesses complicated dynamics generally.

Further investigations such as the stability, direction and global existence of the periodic solutions bifurcating from the local Hopf bifurcations are essential in order to fully understand the periodic phenomenon of the system. We can compute the normal forms directly by using the method due to Faria and Magalháes [24, 25]. However, this is a complex and prolix task. In addition, we would like to point out that codimension two mode interactions may take place if the assumption (**H**) is not satisfied. For example, if $a+2b+2c = 1$, $|a+2c-b| < 1$, $|a+2b-c| < 1$, and $a-b-c < -1$, then system (2) undergoes a fold-Hopf bifurcation when τ passes through a critical value. Therefore, it is possible to study secondary bifurcations and more complex behaviours in this coupled network. We leave them for our future work.

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