Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 417020, 7 pages
http://dx.doi.org/10.1155/2013/417020

Research Article

Homoclinic Solutions for a Second-Order Nonperiodic Asymptotically Linear Hamiltonian Systems

Qiang Zheng

College of Computer Science and Technology, Shandong University of Technology, Zibo, Shandong 255049, China

Correspondence should be addressed to Qiang Zheng; zqsci2012@163.com

Received 18 November 2012; Accepted 21 December 2012

Academic Editor: Juntao Sun

We establish a new existence result on homoclinic solutions for a second-order nonperiodic Hamiltonian systems. This homoclinic solution is obtained as a limit of solutions of a certain sequence of nil-boundary value problems which are obtained by the minimax methods. Some recent results in the literature are generalized and extended.

1. Introduction

Consider the following second-order Hamiltonian system:

$$\ddot{u}(t) + \nabla V(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{HS}$$

where $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$, $V(t,u) = -K(t,u) + W(t,u)$, $K, W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are C^1 maps. We will say that a solution $u : \mathbb{R} \to \mathbb{R}^N$ of (HS) is homoclinic (to 0), if $u(t) \to 0$, as $|t| \to \infty$. In addition, if $u \not\equiv 0$, then u is called a nontrivial homoclinic solution.

Inspired by the excellent monographs [1, 2], by now, the existence and multiplicity of homoclinic solutions for Hamiltonian systems have been extensively investigated in many papers via variational methods; see [3–7] for the first order systems and [8–19] for the second systems, and most of them treat the following system:

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{1}$$

where $L(t)$ is a symmetric matrix-valued function and $W \in C^1(\mathbb{R}, \mathbb{R}^N)$.

For the periodic case, the periodicity is used to control the lack of compactness due to the fact that (1) is set on all $\mathbb{R}$. In 1990, Rabinowitz [12] first proved that (1) has a $2kT$-periodic solution u_k, which is bounded uniformly for k, and obtained a homoclinic solution for (1) as a limit of $2kT$-periodic solution.

For the nonperiodic case, the problem is quite different from the one described in nature. Rabinowitz and Tanaka [13] introduced a type of coercive condition on the matrix L:

(L_1) $l(t) := \inf_{|x|=1} L(t)x \cdot x \to +\infty$, as $|t| \to \infty$.

They first obtained the existence of homoclinic solution for the nonperiodic system (1) under the well-known (AR) growth condition by using Ekeland's variational principle.

In 1995, Ding [8] strengthened condition (L_1) by

(L_2) there exists a constant $\alpha > 0$ such that

$$l(t)|t|^{-\alpha} \longrightarrow +\infty \quad \text{as } |t| \longrightarrow \infty. \tag{2}$$

Under the condition (L_2) and some subquadratic conditions on $W(t,u)$, Ding proved the existence and multiplicity of homoclinic solutions for the system (1). From then on, the condition (L_1) or (L_2) is extensively used in nonperiodic second-order Hamiltonian systems. However, the assumption (L_1) or (L_2) is a rather restrictive and not very natural condition as it excludes, for example, the case of constant matrices I_N.

In 2005, Izydorek and Janczewska [9] first presented the "pinching" condition (see the following (V_2)) and relaxed the conditions (L_1) and (L_2). They studied the general periodic Hamiltonian system

$$\ddot{u}(t) + \nabla V(t, u(t)) = f(t), \quad t \in \mathbb{R}, \tag{3}$$

where $V(t,u) = -K(t,u) + W(t,u)$ and obtained the following result.

Theorem A (see [9]). *Let the following conditions hold:*

(V_1) *$V(t,u) = -K(t,u) + W(t,u)$, where V is continuous and T periodic with respect to t, $T > 0$;*

(V_2) *there exist $b_1, b_2 > 0$ such that*

$$b_1|u|^2 \le K(t,u) \le b_2|u|^2, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N; \tag{4}$$

(V_3) *$K(t,u) \le (u, \nabla K(t,u)) \le 2K(t,u)$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$;*

(V_4) *$\nabla W(t,u) = o(|u|)$ as $u \to 0$ uniformly in t;*

(V_5) *there is a constant $\mu > 2$ such that*

$$0 < \mu W(t,u) \le (\nabla W(t,u), u), \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N; \tag{5}$$

(V_6) *$\bar{b}_1 := \min\{1, 2b_1\} > 2M$ and $\|f\|_{L^2} < (\bar{b}_1 - 2M)/2C^*$, where $M = \sup\{W(t,u) : t \in [0,T], |u| = 1\}$ and C^* is a positive constant depending on T.*

Then the system (3) *possesses a nontrivial homoclinic solution $u \in W^{1,2}(\mathbb{R},\mathbb{R}^N)$ such that $\dot{u}(t) \to 0$ as $t \to \pm\infty$.*

From then on, following the idea of [9], some researchers are devoted to relaxing the conditions (L_1) and (L_2) and studying the existence of homoclinic solutions of system (HS) or (3) under the periodicity assumption of the potential, such as [10, 11, 16, 19].

Very recently, Daouas [3] removed the periodicity condition and studied the existence of homoclinic solutions for the nonperiodic system (3), when W is superquadratic at the infinity. Motivated by [3], in this work, we will study the existence of homoclinic solutions of the nonperiodic system (HS), when W satisfies the asymptotically quadratic condition at the infinity. It is worth noticing that there are few works concerning this case for system (HS) or (3) up to now.

Our result is presented as follows.

Theorem 1. *Let $A := \sup\{K(t,u) : t \in \mathbb{R}, |u| \le 1\} < +\infty$ hold. Moreover, assume that the following conditions hold:*

(H_1) *$K(t,0) \equiv 0$, and there exists a constant $a > 0$ such that*

$$K(t,u) \ge a|u|^2, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N; \tag{6}$$

(H_2) *there exists $\beta \in (1,2]$ such that*

$$K(t,u) \le (u, \nabla K(t,u)) \le \beta K(t,u), \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N; \tag{7}$$

(H_3) *$W(t,0) \equiv 0$ and $\nabla W(t,u) = o(|u|)$ as $u \to 0$ uniformly in t, and there exist, $M_0 > 0$ such that*

$$\frac{|\nabla W(t,u)|}{|u|} \le M_0, \tag{8}$$

for any $t \in \mathbb{R}$ and $u \in \mathbb{R}^N$;

(H_4) *$W(t,u) - w(t)|u|^2 = o(|u|^2)$ as $|u| \to \infty$ uniformly in t, where $w \in L^\infty(\mathbb{R},\mathbb{R})$ with $w_\infty := \inf_{t\in\mathbb{R}} w(t) > 2A$;*

(H_5) *$\widetilde{W}(t,u) := (1/2)(\nabla W(t,u), u) - W(t,u) \to +\infty$ as $|u| \to +\infty$, and*

$$\inf\left\{\frac{\widetilde{W}(t,u)}{|u|^2} : t \in \mathbb{R} \text{ with } c \le |u| < d\right\} > 0, \tag{9}$$

for any $c, d > 0$.

Then the system (HS) *possesses a nontrivial homoclinic solution $u \in W^{1,2}(\mathbb{R},\mathbb{R}^N)$ such that $\dot{u}(t) \to 0$ as $t \to \pm\infty$.*

Remark 2. Theorem 1 treats the asymptotically quadratic case on W. Consider the functions

$$K(t,u) = \left(1 + e^{-|t|}\right)|u|^2,$$
$$W(t,u) = d(t)|u|^2\left(1 - \frac{1}{\ln(e + |u|)}\right), \tag{10}$$

where $d \in L^\infty(\mathbb{R},\mathbb{R})$ and $\inf_{t\in\mathbb{R}} d(t) > 4 + 32\pi^2$.

A straightforward computation shows that K and W satisfy the assumptions of Theorem 1, but K does not satisfy the conditions (L_1) and (L_2). Hence, Theorem 1 also extends the results in [8, 13].

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorem 1.

2. Preliminaries

Following the similar idea of [20], consider the following nil-boundary value problems:

$$\ddot{u}(t) + \nabla V(t, u(t)) = 0, \quad \forall t \in [-T,T],$$
$$u(-T) = u(T) = 0. \tag{11}$$

For each $T > 0$, let $E_T = W^{1,2}([-T,T],\mathbb{R}^N)$, where

$$W^{1,2}\left([-T,T],\mathbb{R}^N\right)$$
$$=\{u : [-T,T] \longrightarrow \mathbb{R}^N \text{ is an absolutely continuous function,}$$
$$u(-T) = u(T) = 0 \text{ and } \dot{u} \in L^2\left([-T,T],\mathbb{R}^N\right)\}, \tag{12}$$

equipped with the norm

$$\|u\| = \left(\int_{-T}^{T}\left[|\dot{u}(t)|^2 + |u(t)|^2\right]dt\right)^{1/2}. \tag{13}$$

Furthermore, for $p > 1$, let $L_T^p = L^p([-T,T],\mathbb{R}^N)$ and $L_T^\infty = L^\infty([-T,T],\mathbb{R}^N)$ under their habitual norms. We need the following result.

Proposition 3 (see [9]). *There is a positive constant C such that for each $T > 0$ and $u \in E_T$ the following inequality holds:*

$$\|u\|_{L_T^\infty} \le C\|u\|. \tag{14}$$

Note that the inequality (14) holds true with constant $C = \sqrt{2}$ if $T \geq 1/2$ (see [9]). Subsequently, we may assume this condition is fulfilled.

Consider a functional $I : E_T \to \mathbb{R}$ defined by

$$\begin{aligned} I(u) &= \int_{-T}^{T} \left[\frac{1}{2}|\dot{u}(t)|^2 - V(t,u(t))\right] dt \\ &= \frac{1}{2}\int_{-T}^{T} |\dot{u}(t)|^2 dt + \int_{-T}^{T} K(t,u(t))\, dt \\ &\quad - \int_{-T}^{T} W(t,u(t))\, dt. \end{aligned} \tag{15}$$

Then $I \in C^1(E_T, \mathbb{R})$, and it is easy to show that for all $u, v \in E_T$, we have

$$\begin{aligned} I'(u)v &= \int_{-T}^{T} [(\dot{u}(t), \dot{v}(t)) - (W(t,u(t)), \nabla V(t,u(t)), v(t))]\, dt \\ &= \int_{-T}^{T} [(\dot{u}(t), \dot{v}(t)) + (\nabla K(t,u(t)), v(t)) \\ &\quad - (\nabla W(t,u(t)), v(t))]\, dt. \end{aligned} \tag{16}$$

It is well known that critical points of I are classical solutions of the problem (11). We will obtain a critical point of I by using an improved version of the Mountain Pass Theorem. For completeness, we give this theorem.

Recall that a sequence $\{u_j\}$ is a (C)-sequence for the functional φ if $\varphi(u_j)$ is bounded and $(1 + \|u_j\|)\varphi'(u_j) \to 0$. A functional φ satisfies the (C)-condition if and only if any (C)-sequence for φ contains a convergent subsequence.

Theorem 4 (see [21]). *Let E be a real Banach space, and let $\varphi \in C^1(E, \mathbb{R})$ satisfy the (C)-condition and $\varphi(0) = 0$. If φ satisfies the following conditions:*

(A_1) *there exist constants $\rho, \alpha > 0$ such that $\varphi|_{\partial B_\rho(0)} \geq \alpha$;*

(A_2) *there exists $e \in E \setminus \overline{B}_\rho(0)$ such that $\varphi(e) \leq 0$, then φ possesses a critical value $c \geq \alpha$ given by*

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \varphi(f(s)), \tag{17}$$

where $B_\rho(0)$ is an open ball in E of radius ρ at about 0, and

$$\Gamma = \{f \in C([0,1], E) : f(0) = 0,\ f(1) = e\}. \tag{18}$$

Proof. As shown in Bartolo et al. [22], a deformation lemma can be proved with the (C)-condition replacing the usual (PS)-condition, and it turns out that the standard version Mountain Pass Theorem (see Rabinowitz [21]) holds true under the (C)-condition. □

Lemma 5. *Assume that (H_2) holds, then*

$$K(t,u) \leq K\left(t, \frac{u}{|u|}\right)|u|^\beta, \quad \forall t \in \mathbb{R},\ |u| \geq 1. \tag{19}$$

Proof. From (H_2) it follows that for $u \neq 0$ a map given by

$$(0, \infty) \ni \nu \mapsto W(t, \nu^{-1}u) \tag{20}$$

is nondecreasing. Similar to the proof in [12], we can get the conclusion. □

Lemma 6 (see [9]). *Let $u : \mathbb{R} \to \mathbb{R}^N$ be a continuous map such that $\dot{u}$ is locally square integrable. Then, for all $t \in \mathbb{R}$, one has*

$$|u(t)| \leq \sqrt{2}\left(\int_{t-1/2}^{t+1/2} \left(|u(s)|^2 + |\dot{u}(s)|^2\right) ds\right)^{1/2}. \tag{21}$$

3. Proof of Theorem 1

Lemma 7. *Under the assumptions of Theorem 1, the problem* (11) *possesses a nontrivial solution.*

Proof. It suffices to prove that the functional I satisfies all the assumptions of Theorem 4.

Step 1. We show that the functional I satisfies the (C)-condition. Let

$$I(u_j) \text{ be bounded and } \left(1 + \|u_j\|\right) I'(u_j) \longrightarrow 0. \tag{22}$$

Observe that, for j large, it follows from (H_1) and (H_2) that there exists a constant C_0 such that

$$\begin{aligned} C_0 &\geq I(u_j) - \frac{1}{2} I'(u_j) u_j \\ &= \int_{-T}^{T} \left[\frac{1}{2}\left(\nabla W(t,u_j), u_j\right) - W(t,u_j)\right] dt \\ &\quad + \int_{-T}^{T} \left[K(t,u_j) - \frac{1}{2}\left(\nabla K(t,u_j), u_j\right)\right] dt \\ &\geq \int_{-T}^{T} \widetilde{W}(t,u)\, dt. \end{aligned} \tag{23}$$

Arguing indirectly, assume as a contradiction that $\|u_j\| \to \infty$. Setting $v_j = u_j/\|u_j\|$, then $\|v_j\| = 1$, and by Proposition 3, one has

$$\|v_j\|_{L_T^\infty} \leq \sqrt{2}\|v_j\| = \sqrt{2}. \tag{24}$$

Note that

$$\begin{aligned} I'(u_j) u_j &= \|\dot{u}_j\|_{L_T^2}^2 + \int_{-T}^{T} \left(\nabla K(t,u_j), u_j\right) dt \\ &\quad - \int_{-T}^{T} \left(\nabla W(t,u_j), u_j\right) dt \\ &\geq \|\dot{u}_j\|_{L_T^2}^2 + \int_{-T}^{T} K(t,u_j)\, dt \\ &\quad - \int_{-T}^{T} \left(\nabla W(t,u_j), u_j\right) dt \\ &\geq C_1\|u_j\|^2 - \int_{-T}^{T} \left|\nabla W(t,u_j)\right| \left|u_j\right| dt, \end{aligned} \tag{25}$$

where $C_1 = \min\{1, a\} > 0$. This implies that

$$\int_{-T}^{T} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|u_j\right|}{\left\|u_j\right\|^2} dt = \int_{-T}^{T} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt \longrightarrow C_1. \tag{26}$$

Set for $s \geq 0$

$$h(s) := \inf\left\{\widetilde{W}(t, u) : t \in [-T, T],\ u \in \mathbb{R}^N \text{ with } |u| \geq s\right\}. \tag{27}$$

By (H_5), $h(s) \to \infty$ as $s \to \infty$.

For $0 \leq l < m$, let

$$\Omega_j(l, m) = \left\{t \in [-T, T] : l \leq \left|u_j(t)\right| < m\right\},$$
$$C_l^m = \inf\left\{\frac{\widetilde{W}(t, u)}{|u|^2} : t \in [-T, T] \text{ with } l \leq |u(t)| < m\right\}. \tag{28}$$

Then by (H_5), $C_l^m > 0$. One has

$$\widetilde{W}\left(t, u_j\right) \geq C_l^m \left|u_j\right|^2, \quad \forall t \in \Omega_j(l, m). \tag{29}$$

It follows from (23) that

$$\begin{aligned} C_0 \geq & \int_{\Omega_j(0,l)} \widetilde{W}\left(t, u_j\right) dt + \int_{\Omega_j(l,m)} \widetilde{W}\left(t, u_j\right) dt \\ & + \int_{\Omega_j(m,\infty)} \widetilde{W}\left(t, u_j\right) dt \\ \geq & \int_{\Omega_j(0,l)} \widetilde{W}\left(t, u_j\right) dt + C_l^m \int_{\Omega_j(l,m)} \left|u_j\right|^2 dt \\ & + h(m)\left|\Omega_j(m, \infty)\right| \end{aligned} \tag{30}$$

which implies that

$$\left|\Omega_j(m, \infty)\right| \leq \frac{C_0}{h(m)} \longrightarrow 0 \tag{31}$$

as $m \to \infty$ uniformly in j, and for any fixed $0 < l < m$

$$\int_{\Omega_j(l,m)} \left|v_j\right|^2 dt = \frac{1}{\left\|u_j\right\|^2} \int_{\Omega_j(l,m)} \left|u_j\right|^2 dt \leq \frac{C_0}{C_l^m \left\|u_j\right\|^2} \longrightarrow 0 \tag{32}$$

as $j \to \infty$. Using (14) and (31), we have

$$\int_{\Omega_j(m,\infty)} \left|v_j\right|^2 dt \leq \left\|v_j\right\|_{L_T^\infty}^2 \cdot \left|\Omega_j(m, \infty)\right| \leq 2\left|\Omega_j(m, \infty)\right| \longrightarrow 0 \tag{33}$$

as $m \to \infty$ uniformly in j.

Let $0 < \epsilon < C_1/3$. By (H_3) there is $l_\epsilon > 0$ such that

$$|\nabla W(t, u)| < \frac{\epsilon}{4T} |u| \tag{34}$$

for all $|t| \leq l_\epsilon$. Consequently,

$$\begin{aligned} \int_{\Omega_j(0,l_\epsilon)} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt & \leq \frac{\epsilon}{4T} \int_{\Omega_j(0,l_\epsilon)} \left|v_j\right|^2 dt \\ & \leq \frac{\epsilon}{4T} \left\|v_j\right\|_{L_T^\infty}^2 2T < \epsilon \end{aligned} \tag{35}$$

for all j.

By (31), we can take m_ϵ large such that

$$\int_{\Omega_j(m_\epsilon,\infty)} \left|v_j\right|^2 dt < \frac{\epsilon}{M_0}. \tag{36}$$

Hence, by (H_3) one has

$$\int_{\Omega_j(m_\epsilon,\infty)} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt \leq M_0 \int_{\Omega_j(m_\epsilon,\infty)} \left|v_j\right|^2 dt < \epsilon \tag{37}$$

for all j. By (32) there exists j_0 such that

$$\int_{\Omega_j(l_\epsilon,m_\epsilon)} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt \leq M_0 \int_{\Omega_j(l_\epsilon,m_\epsilon)} \left|v_j\right|^2 dt < \epsilon \tag{38}$$

for all $j \geq j_0$. By (35)–(38), one has

$$\limsup_{j \to \infty} \int_{-T}^{T} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt \leq 3\epsilon < C_1 \tag{39}$$

which contradicts with (26). So $\{u_j\}$ is bounded in E_T. In a similar way to Proposition B. 35 in [21], we can prove that $\{u_j\}$ has a convergent subsequence. Hence I satisfies the (C)-condition.

Step 2. We show that the functional I satisfies the condition (A_1) of Theorem 4.

Observe that, by (H_3) and (H_4), given $0 < \epsilon < a$, there exists some $C_\epsilon > 0$ such that

$$|W(t, u)| \leq \varepsilon|u|^2 + C_\varepsilon|u|^p \tag{40}$$

for all $u \in \mathbb{R}^N$ and $t \in [-T, T]$, where $p > 2$. It follows from (H_1), (40), and Proposition 3 that

$$\begin{aligned} I(u) = & \frac{1}{2} \int_{-T}^{T} |\dot{u}(t)|^2 dt + \int_{-T}^{T} K(t, u(t)) dt \\ & - \int_{-T}^{T} W(t, u(t)) dt \\ \geq & \frac{1}{2} \|\dot{u}\|_{L_T^2}^2 + a\|u\|_{L_T^2}^2 - \epsilon\|u\|_{L_T^2}^2 - C_\epsilon \int_{-T}^{T} |u(t)|^p dt \\ \geq & \frac{1}{2} \|\dot{u}\|_{L_T^2}^2 + a\|u\|_{L_T^2}^2 - \epsilon\|u\|_{L_T^2}^2 - 2TC_\epsilon\|u\|_{L_T^\infty}^p \\ \geq & \min\left\{\frac{1}{2}, a - \epsilon\right\} \|u\|^2 - 2^{p/2+1} TC_\epsilon\|u\|^p. \end{aligned} \tag{41}$$

Hence there exist $\alpha > 0$ and $\rho > 0$ such that $I(u) \geq \alpha$ for all $u \in E_T$ with $\|u\| = \rho$.

Step 3. We show that the functional I satisfies the condition (A_2) of Theorem 4.

By (H_4), there exists $B > 0$ such that

$$W(t,u) \geq w_\infty |u|^2 - B, \quad \forall t \in [-T,T], \, u \in \mathbb{R}^N. \tag{42}$$

Let

$$e(t) = \zeta |\sin(\omega t)| e_0, \quad t \in [-T,T], \tag{43}$$

where $\omega = 2\pi/T$ and $e_0 = (1,0,\ldots,0)$. Clearly, $e \in E_T$. By (15), (42), and Lemma 5, one has

$$\begin{aligned} I(e) &= \frac{\zeta^2\omega^2}{2}\int_{-T}^{T} |\cos(\omega t)|^2 dt + \int_{\{t\in[-T,T];|e(t)|\leq 1\}} K(t,e(t))\,dt \\ &\quad + \int_{\{t\in[-T,T];|e(t)|\geq 1\}} K(t,e(t))\,dt - \int_{-T}^{T} W(t,e(t))\,dt \\ &\leq \frac{T\zeta^2\omega^2}{2} + 2TA + A\int_{\{t\in[-T,T];|e(t)|\geq 1\}} |e(t)|^\beta dt \\ &\quad - w_\infty \zeta^2 \int_{-T}^{T} |\sin(\omega t)|^2 dt + 2TB \\ &= \frac{T\zeta^2\omega^2}{2} + 2TA + A\zeta^2 \int_{-T}^{T} |\sin(\omega t)|^2 dt - T w_\infty \zeta^2 \\ &\quad + 2TB \\ &= T\left(\frac{\omega^2}{2} + A - w_\infty\right)\zeta^2 + 2T(A+B). \end{aligned} \tag{44}$$

Since $w_\infty > 2A + 32\pi^2$ and $T > \sqrt{2/A}\pi$, then $\omega^2/2 + A - w_\infty < 0$. So $I(e) \to -\infty$ as $|\zeta| \to \infty$. So, we can choose large enough $\zeta \in \mathbb{R}$ such that $\|e\| > \rho$ and $I(e) < 0$.

Clearly $I(0) = 0$; then, by application of Theorem 4, there exists a critical point $u_T \in E_T$ of I such that $I(u_T) \geq \alpha$ for all $T > \sqrt{2/A}\pi$. □

Lemma 8. *u_T is bounded uniformly in $T > \sqrt{2/A}\pi$.*

Proof. Define the set of paths

$$\Gamma_T = \{ f \in C([0,1], E_T) \mid f(0) = 0, f(1) = e \}. \tag{45}$$

It follows from Lemma 7 that there exists a solution u_T of problem (11) at which

$$\inf_{f\in\Gamma_T} \max_{s\in[0,1]} I(f(s)) \equiv D_T \tag{46}$$

is achieved. Let $\overline{T} > T$. Since any function in E_T can be regarded as belonging to $E_{\overline{T}}$ if one extends it by zero in $[-\overline{T},\overline{T}] \setminus [-T,T]$, then $\Gamma_T \subset \Gamma_{\overline{T}}$. Therefore, for any solution u_T of problem (11), we obtain

$$I(u_T) = D_T \leq D_{1/2} \quad \text{uniformly in} \quad T > \sqrt{\frac{2}{A}}\pi. \tag{47}$$

Notice that $I'(u_T) = 0$, and together with (47), one has

$$I(u_T) \leq D_{1/2}, \qquad (1 + u_T)\left\| I'(u_T) \right\| = 0. \tag{48}$$

The rest of the proof is similar to that of Step 1 in Lemma 7. Hence there exists a constant $M_1 > 0$, independent of T such that

$$\|u_T\| \leq M_1, \quad \forall T > \sqrt{\frac{2}{A}}\pi. \tag{49}$$

The proof is complete. □

Take a sequence $T_n \to \infty$, and consider the problem (11) on the interval $[-T_n, T_n]$. By Lemma 7, there exists a nontrivial solution $u_n := u_{T_n}$ of problem (11).

Lemma 9. *Let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence given above. Then there exists a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ convergent to u_0 in $C^1_{\text{loc}}(\mathbb{R},\mathbb{R}^N)$.*

Proof. First we prove that the sequences $\|u_n\|_{L^\infty_{T_n}}$, $\|\dot{u}_n\|_{L^\infty_{T_n}}$, and $\|\ddot{u}_n\|_{L^\infty_{T_n}}$ are bounded. From (14) and (49), for n large enough, one has

$$\|u_n\|_{L^\infty_{T_n}} \leq CM_1 := M_2. \tag{50}$$

By (11) and (50), for all $t \in [-T_n, T_n]$, there exists $M_3 > 0$ independent of n such that

$$\|\ddot{u}_n\|_{L^\infty_{T_n}} \leq M_3. \tag{51}$$

It follows from the Mean Value Theorem that for every $n \in N$ and $t \in \mathbb{R}$, there exists $\tau_n \in [t-1, t]$ such that

$$\dot{u}_n(\tau_n) = \int_{t-1}^{t} \dot{u}_n(s)\,ds = u_n(t) - u_n(t-1). \tag{52}$$

Combining the above with (50), and (51) we get

$$\begin{aligned} |\dot{u}_n(t)| &= \left| \int_{\tau_n}^{t} \ddot{u}_n(s)\,ds + \dot{u}_n(\tau_n) \right| \\ &\leq \int_{t-1}^{t} |\ddot{u}_n(s)|\,ds + |u_n(t) - u_n(t-1)| \\ &\leq M_3 + 2M_2 := M_4 \end{aligned} \tag{53}$$

and hence for n large enough

$$\|\dot{u}_n\|_{L^\infty_{T_n}} \leq M_4. \tag{54}$$

Second we show that the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{\dot{u}_n\}_{n\in\mathbb{N}}$ are equicontinuous. Indeed, for any $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$, by (54), we have

$$\begin{aligned} |u_n(t_1) - u_n(t_2)| &= \left| \int_{t_2}^{t_1} \dot{u}_n(s)\,ds \right| \leq \int_{t_2}^{t_1} |\dot{u}_n(s)|\,ds \\ &\leq M_3 |t_1 - t_2|. \end{aligned} \tag{55}$$

Similarly, by (51), one gets

$$\left|\dot{u}_n(t_1) - \dot{u}_n(t_2)\right| \le M_2 \left|t_1 - t_2\right|. \quad (56)$$

By using the Arzelà-Ascoli Theorem, we obtain the existence of a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ and a function u_0 such that

$$u_{n_j} \longrightarrow u_0, \quad \text{as } j \longrightarrow \infty \text{ in } C^1_{\mathrm{loc}}\left(\mathbb{R}, \mathbb{R}^N\right). \quad (57)$$

The proof is complete. □

Lemma 10. *Let $u_0 : \mathbb{R} \to \mathbb{R}^N$ be the function given by* (57). *Then u_0 is the homoclinic solution of* (*HS*).

Proof. First we show that u_0 is a solution of (HS). Let $\{u_{n_j}\}_{j\in\mathbb{N}}$ be the sequence given by Lemma 9, then

$$\ddot{u}_{n_j}(t) + \nabla V\left(t, u_{n_j}(t)\right) = 0 \quad (58)$$

for every $j \in \mathbb{N}$ and $t \in [-T_{n_j}, T_{n_j}]$. Take $b, c \in \mathbb{R}$ with $b < c$. There exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$; we get $[b, c] \subset [-T_{n_j}, T_{n_j}]$ and

$$\ddot{u}_{n_j}(t) = -\nabla V\left(t, u_{n_j}(t)\right), \quad \forall t \in [b, c]. \quad (59)$$

Integrating (59) from b to $t \in [b, c]$, we have

$$\dot{u}_{n_j}(t) - \dot{u}_{n_j}(b) = -\int_b^t \nabla V\left(s, u_{n_j}(s)\right) ds, \quad \forall t \in [b, c]. \quad (60)$$

Since $u_{n_j} \to u_0$ uniformly on $[b, c]$ and $\dot{u}_{n_j} \to \dot{u}_0$ uniformly on $[b, c]$ as $j \to \infty$, then, from (60), we obtain

$$\dot{u}_0(t) - \dot{u}_0(b) = -\int_b^t \nabla V\left(s, u_0(s)\right) ds, \quad \forall t \in [b, c]. \quad (61)$$

Because of the arbitrariness of b and c, we conclude that u_0 satisfies (HS).

Second we prove that $u_0(t) \to 0$, as $|t| \to \infty$. Note that, by (49), for $k \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ such that, for all $j > j_0$, one has

$$\int_{-T_{n_j}}^{T_{n_j}} \left[\left|u_{n_j}(t)\right|^2 + \left|\dot{u}_{n_j}(t)\right|^2\right] dt \le \left\|u_{n_j}\right\|^2 \le M_1^2. \quad (62)$$

Letting $j \to \infty$, one gets

$$\int_{-T_{n_j}}^{T_{n_j}} \left[\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right] dt \le M_1^2 \quad (63)$$

and letting $j \to \infty$, we have

$$\int_{-\infty}^{+\infty} \left[\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right] dt \le M_1^2 \quad (64)$$

and so

$$\int_{|t|\ge r} \left[\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right] dt \longrightarrow 0. \quad (65)$$

From Lemma 6 and (65), we obtain $u_0(t) \to 0$ as $|t| \to \infty$.

Next we show that $\dot{u}_0(t) \to 0$ as $|t| \to \infty$. Indeed, applying again Lemma 6 to $\dot{u}_0$, we obtain

$$\left|\dot{u}_0(t)\right| \le \sqrt{2}\left(\int_{t-1/2}^{t+1/2} \left(\left|\dot{u}_0(s)\right|^2 + \left|\ddot{u}_0(s)\right|^2\right) ds\right)^{1/2}. \quad (66)$$

Also, from (65), we get

$$\int_{t-1/2}^{t+1/2} \left|\dot{u}_0(s)\right|^2 ds \longrightarrow 0, \quad \text{as } |t| \longrightarrow \infty. \quad (67)$$

Hence, it is enough to prove that

$$\int_{t-1/2}^{t+1/2} \left|\ddot{u}_0(s)\right|^2 ds \longrightarrow 0, \quad \text{as } |t| \longrightarrow \infty. \quad (68)$$

Since u_0 is a solution of (HS), one has

$$\int_{t-1/2}^{t+1/2} \left|\ddot{u}_0(s)\right|^2 ds = \int_{t-1/2}^{t+1/2} \left|\nabla V\left(s, u_0(s)\right)\right|^2 ds. \quad (69)$$

Since $\nabla V(t, 0) = 0$ for all $t \in \mathbb{R}$ and $u_0(t) \to 0$, as $|t| \to \infty$, (68) follows from (69).

Finally, similar to the proof in [12], we can prove that u_0 is nontrivial, and we omit it here. The proof of Theorem 1 is complete. □

References

[1] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74, Springer, New York, NY, USA, 1989.

[2] P. H. Rabinowitz, "Variational methods for Hamiltonian systems," in *Handbook of Dynamical Systems*, vol. 1, part 1, chapter 14, pp. 1091–1127, 2002.

[3] A. Daouas, "Homoclinic orbits for superquadratic Hamiltonian systems without a periodicity assumption," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 11, pp. 3407–3418, 2011.

[4] Y. Ding, "Multiple homoclinics in a Hamiltonian system with asymptotically or super linear terms," *Communications in Contemporary Mathematics*, vol. 8, no. 4, pp. 453–480, 2006.

[5] Y. Ding and C. Lee, "Existence and exponential decay of homoclinics in a nonperiodic superquadratic Hamiltonian system," *Journal of Differential Equations*, vol. 246, no. 7, pp. 2829–2848, 2009.

[6] Y. Ding and L. Jeanjean, "Homoclinic orbits for a nonperiodic Hamiltonian system," *Journal of Differential Equations*, vol. 237, no. 2, pp. 473–490, 2007.

[7] J. Sun, H. Chen, and J. J. Nieto, "Homoclinic orbits for a class of first-order nonperiodic asymptotically quadratic Hamiltonian systems with spectrum point zero," *Journal of Mathematical Analysis and Applications*, vol. 378, no. 1, pp. 117–127, 2011.

[8] Y. H. Ding, "Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 25, no. 11, pp. 1095–1113, 1995.

[9] M. Izydorek and J. Janczewska, "Homoclinic solutions for a class of the second order Hamiltonian systems," *Journal of Differential Equations*, vol. 219, no. 2, pp. 375–389, 2005.

[10] M. Izydorek and J. Janczewska, "Homoclinic solutions for nonautonomous second order Hamiltonian systems with a coercive potential," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 2, pp. 1119–1127, 2007.

[11] X. Lv, S. Lu, and P. Yan, "Homoclinic solutions for nonautonomous second-order Hamiltonian systems with a coercive potential," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 7-8, pp. 3484–3490, 2010.

[12] P. H. Rabinowitz, "Homoclinic orbits for a class of Hamiltonian systems," *Proceedings of the Royal Society of Edinburgh A*, vol. 114, no. 1-2, pp. 33–38, 1990.

[13] P. H. Rabinowitz and K. Tanaka, "Some results on connecting orbits for a class of Hamiltonian systems," *Mathematische Zeitschrift*, vol. 206, no. 3, pp. 473–499, 1991.

[14] J. Sun, H. Chen, and J. J. Nieto, "Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 1, pp. 20–29, 2011.

[15] X. H. Tang and X. Lin, "Homoclinic solutions for a class of second-order Hamiltonian systems," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 2, pp. 539–549, 2009.

[16] X. H. Tang and L. Xiao, "Homoclinic solutions for a class of second-order Hamiltonian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 1140–1152, 2009.

[17] L.-L. Wan and C.-L. Tang, "Existence and multiplicity of homoclinic orbits for second order Hamiltonian systems without (AR) condition," *Discrete and Continuous Dynamical Systems B*, vol. 15, no. 1, pp. 255–271, 2011.

[18] Z. Zhang and R. Yuan, "Homoclinic solutions for a class of nonautonomous subquadratic second-order Hamiltonian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 4125–4130, 2009.

[19] Z. Zhang and R. Yuan, "Homoclinic solutions for some second order non-autonomous Hamiltonian systems with the globally superquadratic condition," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1809–1819, 2010.

[20] P. Korman and A. C. Lazer, "Homoclinic orbits for a class of symmetric Hamiltonian systems," *Electronic Journal of Differential Equations*, vol. 1994, pp. 1–10, 1994.

[21] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS*, 1986.

[22] P. Bartolo, V. Benci, and D. Fortunato, "Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 7, no. 9, pp. 981–1012, 1983.