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Research Article

Solution and Stability of the Multiquadratic Functional Equation

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We consider the multiquadratic functional equation. We establish its general solution and provide a characterization for this functional equation. Finally, we prove the Hyers-Ulam-Rassias stability of this functional equation.

1. Introduction

In 1940, Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin, in which he discussed a number of unsolved problems. The stability of a functional equation originated from a question raised by Ulam: "when is it true that the solution of an equation differing slightly from a given one must of necessity be close to the solution of the given equation?" This question was solved by Hyers [2] in the case of the approximately additive functions between Banach spaces. In 1978, Rassias [3] provided a generalized version of Hyers' result by allowing the Cauchy difference to be unbounded. The paper of Rassias [3] has provided a lot of influence in the development of the stability of functional equations, and this new concept is known as generalized Hyers-Ulam-Rassias stability or Hyers-Ulam-Rassias stability. Since then, the stability problems have been widely studied and extensively developed by many authors for a number of functional equations; see, for example, [4–10] and the books [11–14].

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

is called the quadratic functional equation, and every solution of the quadratic functional equation is said to be a quadratic function. It is well known that a quadratic function $f: E_1 \to E_2$ between vector spaces can be expressed by a symmetric biadditive (i.e., additive for each fixed one variable) function $B: E_1 \times E_1 \to E_2$. On the other hand, the stability problem for the quadratic functional equation has been studied by many mathematicians under various degrees of generality imposed on the equation or on the underlying space; see, for example, [15–20] and the references therein.

In [21], Park and Bae obtained the general solution and the generalized Hyers-Ulam-Rassias stability of the biquadratic functional equation. Let X and Y be vector spaces. Recall from [21] that a mapping $f: X \times X \to Y$ is called biquadratic if f satisfies the system of equations

$$\begin{aligned} f(x+y,z)+f(x-y,z)&=2f(x,z)+2f(y,z)\\ f(x,y+z)+f(x,y-z)&=2f(x,y)+2f(x,z) \end{aligned} \tag{2}$$

for all $x, y, z \in X$; that is, f is quadratic for each fixed one variable.

A general version of the biquadratic functional equation is the multiquadratic functional equation. Recall from [22] that a mapping $f: V^n \to W$, where V is a commutative group, W is a linear space, and $n \geq 2$ is an integer, is called multiquadratic if it is quadratic in each variable. On the other hand, for more details about the multiadditive (resp., the multi-Jensen mappings) (i.e., mappings satisfying Cauchy's (resp., Jensen's) functional equation in each variable) and the stability for them, one can see [23–28] and the references given there.

The stability of the multiquadratic functional equation was also studied by some authors. For example, Park [29] proved the stability of the multiquadratic functional equation in Banach spaces. Ciepliński [22] proved the stability of this functional equation in complete non-Archimedean spaces as well as in Banach spaces but using the fixed point method. However, to our knowledge, not many results are known about the solution of this functional equation.

In the present paper, we establish the general solution of the multiquadratic functional equation and provide a sufficient and necessary condition for a mapping to be multiquadratic. Finally, we prove its Hyers-Ulam-Rassias stability.

2. General Solution

Throughout this section, let V and W be vector spaces, and let n be a positive integer. We begin with the following useful proposition.

Proposition 1 (see [11]). *A function $f : V \to W$ is quadratic if and only if there exists a unique symmetric biadditive function $B : V \times V \to W$ such that $f(x) = B(x, x)$ for any $x \in V$. The biadditive function B is given by*

$$B(x, y) = \frac{1}{4}[f(x + y) - f(x - y)] \quad \forall x, y \in V. \tag{3}$$

In the following, we give the general solution of the multiquadratic functional equation.

Theorem 2. *A mapping $f : V^n \to W$ is multiquadratic if and only if there exists a multiadditive mapping $M : V^{2n} \to W$ such that*

$$f(x_1, x_2, \ldots, x_n) = M(x_1, x_1, x_2, x_2, \ldots, x_n, x_n) \tag{4}$$

for all $x_1, \ldots, x_n \in V$, and M satisfies the following symmetric condition

$$\begin{aligned} &M(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}) \\ &\quad = M(x_{11}, x_{12}, \ldots, x_{i2}, x_{i1}, \ldots, x_{n1}, x_{n2}) \end{aligned} \tag{5}$$

for all $x_{ij} \in V$, where $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2\}$. Moreover, the mapping M is given by

$$\begin{aligned} &M(x_{11}, x_{12}, x_{21}, x_{22}, \ldots, x_{n1}, x_{n2}) \\ &\quad = \frac{1}{4^n} \sum_{i_1, \ldots, i_n \in \{1, -1\}} (i_1 \cdots i_n) f(x_{11} + i_1 x_{12}, \ldots, x_{n1} + i_n x_{n2}), \end{aligned} \tag{6}$$

where $x_{ij} \in V$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2\}$.

Proof. We prove this theorem by using induction on n. Clearly, Theorem 2 is true for $n = 1$ thanks to Proposition 1. Now, we assume that the present theorem is true for some $n \geq 2$, and we consider the case for $n + 1$.

We first assume that there exists a multiadditive mapping $M : V^{2(n+1)} \to W$ such that

$$\begin{aligned} &f(x_1, x_2, \ldots, x_n, x_{n+1}) \\ &\quad = M(x_1, x_1, x_2, x_2, \ldots, x_n, x_n, x_{n+1}, x_{n+1}) \end{aligned} \tag{7}$$

for all $x_1, \ldots, x_n, x_{n+1} \in V$, and M satisfies the following symmetric condition:

$$\begin{aligned} &M(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \\ &\quad = M(x_{11}, x_{12}, \ldots, x_{i2}, x_{i1}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \end{aligned} \tag{8}$$

for all $x_{ij} \in V$, where $i \in \{1, 2, \ldots, n, n+1\}$ and $j \in \{1, 2\}$. Then, for each $i \in \{1, 2, \ldots, n, n+1\}$, we have that

$$\begin{aligned} &f(x_1, \ldots, x_{i-1}, x_i + x_i', x_{i+1}, \ldots, x_n, x_{n+1}) \\ &\quad + f(x_1, \ldots, x_{i-1}, x_i - x_i', x_{i+1}, \ldots, x_n, x_{n+1}) \\ &= M(x_1, x_1, \ldots, x_{i-1}, x_{i-1}, x_i + x_i', x_i + x_i', x_{i+1}, \\ &\qquad x_{i+1}, \ldots, x_n, x_n, x_{n+1}, x_{n+1}) \\ &\quad + M(x_1, x_1, \ldots, x_{i-1}, x_{i-1}, x_i - x_i', x_i - x_i', \\ &\qquad x_{i+1}, x_{i+1}, \ldots, x_n, x_n, x_{n+1}, x_{n+1}) \\ &= 2M(x_1, x_1, \ldots, x_{i-1}, x_{i-1}, x_i, x_i, x_{i+1}, \\ &\qquad x_{i+1}, \ldots, x_n, x_n, x_{n+1}, x_{n+1}) \\ &\quad + 2M(x_1, x_1, \ldots, x_{i-1}, x_{i-1}, x_i', x_i', x_{i+1}, \\ &\qquad x_{i+1}, \ldots, x_n, x_n, x_{n+1}, x_{n+1}) \\ &= 2f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n, x_{n+1}) \\ &\quad + 2f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n, x_{n+1}) \end{aligned} \tag{9}$$

for all $x_1, x_2, \ldots, x_n, x_{n+1}, x_i' \in V$. Thus, f is multiquadratic.

Conversely, we assume that $f : V^{n+1} \to W$ is a multiquadratic function. We need to find the desired multiadditive function $M : V^{2(n+1)} \to W$. For this, we give the following notations.

For each fixed $z \in V$, define the mapping $g_z : V^n \to W$ by

$$g_z(x_1, \ldots, x_n) := f(x_1, \ldots, x_n, z) \quad \forall x_1, \ldots, x_n \in V. \tag{10}$$

Then g_z is a multiquadratic mapping (as $f : V^{n+1} \to W$ is multiquadratic). By induction, we let $M_z : V^{2n} \to W$ denote the corresponding multiadditive mapping for g_z; that is, M_z satisfies the symmetric condition (5) and

$$g_z(x_1, \ldots, x_n) = M_z(x_1, x_1, x_2, x_2, \ldots, x_n, x_n) \tag{11}$$

for all $x_1, \ldots, x_n \in V$. Moreover, the mapping $M_z : V^{2n} \to W$ is given by

$$M_z(x_{11}, x_{12}, x_{21}, x_{22}, \ldots, x_{n1}, x_{n2}) = \frac{1}{4^n} \sum_{i_1,\ldots,i_n \in \{1,-1\}} (i_1 \cdots i_n)\, g_z(x_{11} + i_1 x_{12}, \ldots, x_{n1} + i_n x_{n2}) \tag{12}$$

for all $x_{ij} \in V$, where $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2\}$.

On the other hand, for any fixed elements $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in V$, define $h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}} : V \to W$ by

$$h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x) = M_x(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) \tag{13}$$

for all $x \in V$. It can be verified that $h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}$ is a quadratic mapping. Thus, it follows from Proposition 1 that there exists a symmetric biadditive mapping $T_{x_{11},x_{12},\ldots,x_{n1},x_{n2}} : V \times V \to W$ such that

$$h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x) = T_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x, x) \tag{14}$$

for all $x \in V$. The mapping $T_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}$ is given by

$$T_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x, y) = \frac{1}{4}\left[h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x + y) - h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x - y)\right] \tag{15}$$

for all $x, y \in V$.

Now, we define the mapping $M : V^{2(n+1)} \to W$ by

$$\begin{aligned} M(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) &:= \frac{1}{4}\left[M_{x_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right. \\ &\quad \left. - M_{x_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right] \end{aligned} \tag{16}$$

for all $x_{ij} \in V$, $i \in \{1, 2, \ldots, n + 1\}$, $j \in \{1, 2\}$. In the following, we will show that M is the desired function for $f : V^{n+1} \to W$. First, we show that M is multiadditive. Indeed, by the definition of M (see (16)) and noting that for any $z \in V$ the function M_z is multiadditive, one can obtain that for each $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} &M(x_{11}, x_{12}, \ldots, x_{i1} + x'_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \\ &= \frac{1}{4}\left[M_{x_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{i1} + x'_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2})\right. \\ &\qquad - M_{x_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{i1} \\ &\qquad\qquad \left. + x'_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2})\right] \\ &= \frac{1}{4}\left[M_{x_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2})\right. \\ &\qquad + M_{x_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x'_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}) \\ &\qquad - M_{x_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}) \\ &\qquad \left. - M_{x_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x'_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2})\right] \\ &= \frac{1}{4}\left[M_{x_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2})\right. \\ &\qquad \left. - M_{x_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2})\right] \\ &\quad + \frac{1}{4}\left[M_{x_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x'_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2})\right. \\ &\qquad \left. - M_{x_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x'_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2})\right] \\ &= M(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \\ &\quad + M(x_{11}, x_{12}, \ldots, x'_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \end{aligned} \tag{17}$$

for all $x'_{i1}, x_{11}, x_{12}, \ldots, x_{n+1,1}, x_{n+1,2} \in V$. Moreover, by the definition of M in (16) and the notations we gave in (13) and (15), we have that

$$\begin{aligned} &M(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x_{n+1,1} + x'_{n+1,1}, x_{n+1,2}) \\ &= \frac{1}{4}\left[M_{x_{n+1,1}+x'_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right. \\ &\qquad \left. - M_{x_{n+1,1}+x'_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right] \\ &= \frac{1}{4}\left[h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x_{n+1,1} + x'_{n+1,1} + x_{n+1,2})\right. \\ &\qquad \left. - h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x_{n+1,1} + x'_{n+1,1} - x_{n+1,2})\right] \\ &= T_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x_{n+1,1} + x'_{n+1,1}, x_{n+1,2}) \\ &= T_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x_{n+1,1}, x_{n+1,2}) \\ &\quad + T_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x'_{n+1,1}, x_{n+1,2}) \\ &= \frac{1}{4}\left[h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x_{n+1,1} + x_{n+1,2})\right. \\ &\qquad \left. - h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x_{n+1,1} - x_{n+1,2})\right] \\ &\quad + \frac{1}{4}\left[h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x'_{n+1,1} + x_{n+1,2})\right. \\ &\qquad \left. - h_{x_{11},x_{12},\ldots,x_{n1},x_{n2}}(x'_{n+1,1} - x_{n+1,2})\right] \\ &= \frac{1}{4}\left[M_{x_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right. \\ &\qquad \left. - M_{x_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right] \\ &\quad + \frac{1}{4}\left[M_{x'_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right. \\ &\qquad \left. - M_{x'_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right] \\ &= M(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \\ &\quad + M(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x'_{n+1,1}, x_{n+1,2}) \end{aligned} \tag{18}$$

for all $x'_{n+1,1}, x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2} \in V$. Similarly, we can see that M is additive in the other variables. Thus, we have shown that M is multiadditive.

Furthermore, since f is multiquadratic, we obtain that

$$f(x_1, x_2, \ldots, x_n, 0) = 0, \quad f(x_1, \ldots, x_n, 2x_{n+1}) = 4f(x_1, \ldots, x_n, x_{n+1}) \tag{19}$$

for all $x_1, \ldots, x_n, x_{n+1} \in V$. Thus, by the definition of M in (16) and the notations we gave in (10) and (11), one has

$$\begin{aligned} f(x_1, x_2, \ldots, x_n, x_{n+1}) &= \frac{1}{4}\left[f(x_1, x_2, \ldots, x_n, 2x_{n+1}) - f(x_1, x_2, \ldots, x_n, 0)\right] \\ &= \frac{1}{4}\left[g_{2x_{n+1}}(x_1, x_2, \ldots, x_n) - g_0(x_1, x_2, \ldots, x_n)\right] \\ &= \frac{1}{4}\left[M_{2x_{n+1}}(x_1, x_1, x_2, x_2, \ldots, x_n, x_n) - M_0(x_1, x_1, x_2, x_2, \ldots, x_n, x_n)\right] \\ &= M(x_1, x_1, x_2, x_2, \ldots, x_n, x_n, x_{n+1}, x_{n+1}) \end{aligned} \tag{20}$$

for all $x_1, \ldots, x_n, x_{n+1} \in V$.

Now, we verify the expression of the mapping M. By the definition of M again and the notations we gave in (10) and (12), also noting that f is multiquadratic, one can obtain that

$$\begin{aligned} &M(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \\ &= \frac{1}{4}\left[M_{x_{n+1,1}+x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}) - M_{x_{n+1,1}-x_{n+1,2}}(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2})\right] \\ &= \frac{1}{4}\left[\frac{1}{4^n}\sum_{i_1,\ldots,i_n\in\{1,-1\}}(i_1\cdots i_n)\, g_{x_{n+1,1}+x_{n+1,2}}(x_{11}+i_1x_{12}, \ldots, x_{n1}+i_nx_{n2}) \right.\\ &\quad \left. - \frac{1}{4^n}\sum_{i_1,\ldots,i_n\in\{1,-1\}}(i_1\cdots i_n)\, g_{x_{n+1,1}-x_{n+1,2}}(x_{11}+i_1x_{12}, \ldots, x_{n1}+i_nx_{n2})\right] \\ &= \frac{1}{4^{n+1}}\left[\sum_{i_1,\ldots,i_n\in\{1,-1\}}(i_1\cdots i_n) \times f(x_{11}+i_1x_{12}, \ldots, x_{n1}+i_nx_{n2}, x_{n+1,1}+x_{n+1,2}) \right.\\ &\quad \left. - \sum_{i_1,\ldots,i_n\in\{1,-1\}}(i_1\cdots i_n) \times f(x_{11}+i_1x_{12}, \ldots, x_{n1}+i_nx_{n2}, x_{n+1,1}-x_{n+1,2})\right] \\ &= \frac{1}{4^{n+1}}\sum_{i_1,\ldots,i_n,i_{n+1}\in\{1,-1\}}(i_1\cdots i_n\cdot i_{n+1}) \times f(x_{11}+i_1x_{12}, \ldots, x_{n1}+i_nx_{n2}, x_{n+1,1}+i_{n+1}x_{n+1,2}) \end{aligned} \tag{21}$$

for all $x_{ij} \in V$, $i \in \{1, 2, \ldots, n+1\}$, $j \in \{1, 2\}$.

Finally, we check the symmetric property of M. Fix any $x_{ij} \in V$, where $i \in \{1, 2, \ldots, n+1\}$ and $j \in \{1, 2\}$. Since f is multiquadratic, it follows that f is an even mapping in each variable. Then by (21), it is easy to verify that

$$\begin{aligned} &M(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \\ &\quad = M(x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}, x_{n+1,2}, x_{n+1,1}). \end{aligned} \tag{22}$$

Moreover, due to the symmetric property of $M_{x_{n+1,1}+x_{n+1,2}}$ and $M_{x_{n+1,1}-x_{n+1,2}}$ and from the definition of M (see (16)) we can get

$$\begin{aligned} &M(x_{11}, x_{12}, \ldots, x_{i1}, x_{i2}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \\ &\quad = M(x_{11}, x_{12}, \ldots, x_{i2}, x_{i1}, \ldots, x_{n1}, x_{n2}, x_{n+1,1}, x_{n+1,2}) \end{aligned} \tag{23}$$

for each $i \in \{1, 2, \ldots, n\}$. So the desired symmetric property of M is proved. Thus, we have shown that $M : V^{2(n+1)} \to W$ is the desired multiadditive mapping for the multiquadratic mapping $f : V^n \to W$. The proof is complete. □

3. A Characterization for the Multiquadratic Functional Equation

The following theorem provides a sufficient and necessary condition for a mapping to be multiquadratic.

Theorem 3. *Let V be a commutative semigroup with the identity element 0, and let W be a linear space. A mapping $f : V^n \to W$ is multiquadratic if and only if*

$$\sum_{i_1,\ldots,i_n\in\{1,-1\}} f(x_{11}+i_1x_{12}, \ldots, x_{n1}+i_nx_{n2}) = 2^n \sum_{j_1,\ldots,j_n\in\{1,2\}} f(x_{1j_1}, \ldots, x_{nj_n}) \tag{24}$$

for all $(x_{11}, \ldots, x_{n1}), (x_{12}, \ldots, x_{n2}) \in V^n$.

Proof. Assume that $f : V^n \to W$ satisfies (24). Putting

$$(x_{11}, \ldots, x_{n1}) = (x_{12}, \ldots, x_{n2}) = (0, \ldots, 0) \tag{25}$$

in (24) we get $2^n f(0,\dots,0) = 2^{2n} f(0,\dots,0)$, and consequently, we have $f(0,\dots,0)=0$. Next, fix $j\in\{1,\dots,n\}$, $x_{j1}\in V$, and put $x_{j2}=x_{ki_k}=0$, where $i_k\in\{1,2\}$, for $k\in\{1,\dots,n\}\setminus\{j\}$. Then, by (24),

$$2^n f\left(0,\dots,0,x_{j1},0,\dots,0\right) = 2^n 2^{n-1} f\left(0,\dots,0,x_{j1},0,\dots,0\right), \quad (26)$$

and thus $f(0,\dots,0,x_{j1},0,\dots,0)=0$. Continuing in this fashion, we obtain that $f(x)=0$ for any $x\in V^n$ with at least one component which is equal to 0.

Now, fix $j\in\{1,\dots,n\}$, $x_{11},\dots,x_{n1},x_{j2}\in V$ and put $x_{k2}=0$ for $k\in\{1,\dots,n\}\setminus\{j\}$ in (24). Then

$$\begin{aligned}2^{n-1} f\left(x_{11},\dots,x_{j1}+x_{j2},\dots,x_{n1}\right) &+ 2^{n-1} f\left(x_{11},\dots,x_{j1}-x_{j2},\dots,x_{n1}\right)\\ &= 2^n\left[f\left(x_{11},\dots,x_{j1},\dots,x_{n1}\right) + f\left(x_{11},\dots,x_{j2},\dots,x_{n1}\right)\right],\end{aligned} \quad (27)$$

and thus

$$\begin{aligned}f\left(x_{11},\dots,x_{j1}+x_{j2},\dots,x_{n1}\right) &+ f\left(x_{11},\dots,x_{j1}-x_{j2},\dots,x_{n1}\right)\\ &= 2f\left(x_{11},\dots,x_{j1},\dots,x_{n1}\right) + 2f\left(x_{11},\dots,x_{j2},\dots,x_{n1}\right),\end{aligned} \quad (28)$$

which proves that f is multiquadratic.

Conversely, we assume that f is multiquadratic, and we prove (24) by mathematical induction. If $f: V \to W$ is quadratic, then $f(x_{11}+x_{12})+f(x_{11}-x_{12}) = 2f(x_{11})+2f(x_{12})$ for all $x_{11},x_{12}\in V$. So, (24) holds for $n=1$. It is easy to verify that (24) holds for $n=2$. Indeed,

$$\begin{aligned}f&\left(x_{11}+x_{12},x_{21}+x_{22}\right) + f\left(x_{11}+x_{12},x_{21}-x_{22}\right) + f\left(x_{11}-x_{12},x_{21}+x_{22}\right) + f\left(x_{11}-x_{12},x_{21}-x_{22}\right)\\ &= 2f\left(x_{11}+x_{12},x_{21}\right) + 2f\left(x_{11}+x_{12},x_{22}\right) + 2f\left(x_{11}-x_{12},x_{21}\right) + 2f\left(x_{11}-x_{12},x_{22}\right)\\ &= 4\left[f\left(x_{11},x_{21}\right) + f\left(x_{12},x_{21}\right) + f\left(x_{11},x_{22}\right) + f\left(x_{12},x_{22}\right)\right]\end{aligned} \quad (29)$$

for all $x_{11},x_{12},x_{21},x_{22}\in V$. Assume that (24) holds for some positive integer $n>2$. Then,

$$\begin{aligned}\sum_{i_1,\dots,i_{n+1}\in\{1,-1\}} & f\left(x_{11}+i_1x_{12},\dots,x_{n1}+i_nx_{n2},x_{n+1,1}+i_{n+1}x_{n+1,2}\right)\\ &= \sum_{i_1,\dots,i_n\in\{1,-1\}}\left[f\left(x_{11}+i_1x_{12},\dots,x_{n1}+i_nx_{n2},x_{n+1,1}+x_{n+1,2}\right) + f\left(x_{11}+i_1x_{12},\dots,x_{n1}+i_nx_{n2},x_{n+1,1}-x_{n+1,2}\right)\right]\\ &= 2^n\sum_{j_1,\dots,j_n\in\{1,2\}}\left[f\left(x_{1j_1},\dots,x_{nj_n},x_{n+1,1}+x_{n+1,2}\right) + f\left(x_{1j_1},\dots,x_{nj_n},x_{n+1,1}-x_{n+1,2}\right)\right]\\ &= 2^n\sum_{j_1,\dots,j_n\in\{1,2\}}\left[2f\left(x_{1j_1},\dots,x_{nj_n},x_{n+1,1}\right) + 2f\left(x_{1j_1},\dots,x_{nj_n},x_{n+1,2}\right)\right]\\ &= 2^{n+1}\sum_{j_1,\dots,j_n,j_{n+1}\in\{1,2\}} f\left(x_{1j_1},\dots,x_{nj_n},x_{n+1j_{n+1}}\right).\end{aligned} \quad (30)$$

Thus, (24) holds for $n+1$, and this completes the proof. □

4. Stability

In this section, we give two results on the stability of the multiquadratic functional equation. Throughout this section, let V be a commutative semigroup with the identity element 0, and let W be a Banach space.

Theorem 4. *Assume that for every $i\in\{1,\dots,n\}$, $\varphi_i: V^{n+1}\to[0,\infty)$ is a mapping such that for any $(x_1,\dots,x_{n+1})\in V^{n+1}$*

$$\begin{aligned}\widetilde{\varphi}_i&\left(x_1,\dots,x_{n+1}\right)\\ &:= \sum_{j=0}^{\infty}\frac{1}{4^j}\left[\varphi_i\left(2^jx_1,x_2,\dots,x_{n+1}\right) + \cdots + \varphi_i\left(x_1,\dots,x_{i-2},2^jx_{i-1},x_i,\dots,x_{n+1}\right)\right.\\ &\quad + \frac{1}{4}\varphi_i\left(x_1,\dots,x_{i-1},2^jx_i,2^jx_{i+1},x_{i+2},\dots,x_{n+1}\right) + \varphi_i\left(x_1,\dots,x_i,x_{i+1},2^jx_{i+2},x_{i+3},\dots,x_{n+1}\right)\\ &\quad \left. + \cdots + \varphi_i\left(x_1,\dots,x_n,2^jx_{n+1}\right)\right] < \infty.\end{aligned} \quad (31)$$

If $f : V^n \to W$ *is a function satisfying*

$$\begin{aligned}\| f(x_1,\ldots,x_{i-1},x_i+x_i',x_{i+1},\ldots,x_n) \\ + f(x_1,\ldots,x_{i-1},x_i-x_i',x_{i+1},\ldots,x_n) \\ - 2f(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) \\ -2f(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n)\| \\ \le \varphi_i(x_1,\ldots,x_i,x_i',x_{i+1},\ldots,x_n)\end{aligned} \tag{32}$$

for all $(x_1,\ldots,x_i,x_i',x_{i+1},\ldots,x_n) \in V^{n+1}$, $i \in \{1,\ldots,n\}$, *then for every* $i \in \{1,\ldots,n\}$ *there exists a multiquadratic mapping* $F_i : V^n \to W$ *such that for any* $(x_1,\ldots,x_n) \in V^n$ *one has*

$$\begin{aligned}\Big\| f(x_1,\ldots,x_n) - \frac{1}{3} f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) \\ -F_i(x_1,\ldots,x_n)\Big\| \\ \le \widetilde{\varphi}_i(x_1,\ldots,x_i,x_i,x_{i+1},\ldots,x_n).\end{aligned} \tag{33}$$

For every $i \in \{1,\ldots,n\}$ *the function* F_i *is given by*

$$F_i(x_1,\ldots,x_n) := \lim_{j\to\infty} \frac{1}{4^j} f(x_1,\ldots,x_{i-1},2^j x_i,x_{i+1},\ldots,x_n) \tag{34}$$

for all $(x_1,\ldots,x_n) \in V^n$.

Proof. Fix $x_1,\ldots,x_n \in V$, $j \in N \cup \{0\}$ (where N denotes the set of the positive integers) and $i \in \{1,\ldots,n\}$. Putting $x_i' := x_i$ in (32), we get

$$\begin{aligned}\| f(x_1,\ldots,x_{i-1},2x_i,x_{i+1},\ldots,x_n) \\ +f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) - 4f(x_1,\ldots,x_n)\| \\ \le \varphi_i(x_1,\ldots,x_i,x_i,x_{i+1},\ldots,x_n).\end{aligned} \tag{35}$$

Hence

$$\begin{aligned}\Big\| f(x_1,\ldots,x_n) - \frac{1}{4} f(x_1,\ldots,x_{i-1},2x_i,x_{i+1},\ldots,x_n) \\ -\frac{1}{4} f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\Big\| \\ \le \frac{1}{4}\varphi_i(x_1,\ldots,x_i,x_i,x_{i+1},\ldots,x_n).\end{aligned} \tag{36}$$

Dividing both sides of the above inequality by 4^j and replacing x_i by $2^j x_i$, we obtain

$$\begin{aligned}\Big\| \frac{1}{4^j} f(x_1,\ldots,x_{i-1},2^j x_i,x_{i+1},\ldots,x_n) \\ - \frac{1}{4^{j+1}} f(x_1,\ldots,x_{i-1},2^{j+1} x_i,x_{i+1},\ldots,x_n) \\ -\frac{1}{4^{j+1}} f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\Big\| \\ \le \frac{1}{4^{j+1}}\varphi_i(x_1,\ldots,2^j x_i,2^j x_i,x_{i+1},\ldots,x_n),\end{aligned} \tag{37}$$

and consequently for any nonnegative integers l and m with $l < m$, we obtain

$$\begin{aligned}\Big\| \frac{1}{4^l} f(x_1,\ldots,x_{i-1},2^l x_i,x_{i+1},\ldots,x_n) \\ - \frac{1}{4^m} f(x_1,\ldots,x_{i-1},2^m x_i,x_{i+1},\ldots,x_n) \\ -\sum_{j=l}^{m-1} \frac{1}{4^{j+1}} f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)\Big\| \\ \le \sum_{j=l}^{m-1} \frac{1}{4^{j+1}}\varphi_i(x_1,\ldots,2^j x_i,2^j x_i,x_{i+1},\ldots,x_n).\end{aligned} \tag{38}$$

Therefore, it follows from (31) that $\{1/4^j f(x_1,\ldots,x_{i-1},2^j x_i,x_{i+1},\ldots,x_n)\}_{j\in N}$ is a Cauchy sequence. Since the space W is complete, this sequence is convergent, and we define $F_i : V^n \to W$ by (34). Putting $l = 0$, letting $m \to \infty$ in (38), and using (31), we see that (33) holds.

Finally, fix also $x_i' \in V$, $j \in N$, and notice that according to (32) we have

$$\begin{aligned}\Big\| \frac{1}{4^j} f(x_1,\ldots,x_{i-1},2^j(x_i+x_i'),x_{i+1},\ldots,x_n) \\ + \frac{1}{4^j} f(x_1,\ldots,x_{i-1},2^j(x_i-x_i'),x_{i+1},\ldots,x_n) \\ - \frac{2}{4^j} f(x_1,\ldots,x_{i-1},2^j x_i,x_{i+1},\ldots,x_n) \\ -\frac{2}{4^j} f(x_1,\ldots,x_{i-1},2^j x_i',x_{i+1},\ldots,x_n)\Big\| \\ \le \frac{1}{4^j}\varphi_i(x_1,\ldots,2^j x_i,2^j x_i',x_{i+1},\ldots,x_n).\end{aligned} \tag{39}$$

Next, fix $k \in \{1,\ldots,n\} \setminus \{i\}$, $x_k' \in V$, and assume that $k < i$ (the same arguments apply to the case where $k > i$). Then, it follows from (32) that

$$\begin{aligned}\Big\| \frac{1}{4^j} f(x_1,\ldots,x_{k-1},x_k \\ +x_k',x_{k+1},\ldots,x_{i-1},2^j x_i,x_{i+1},\ldots,x_n) \\ + \frac{1}{4^j} f(x_1,\ldots,x_{k-1},x_k \\ -x_k',x_{k+1},\ldots,x_{i-1},2^j x_i,x_{i+1},\ldots,x_n) \\ - \frac{2}{4^j} f(x_1,\ldots,x_{i-1},2^j x_i,x_{i+1},\ldots,x_n) \\ - \frac{2}{4^j} f(x_1,\ldots,x_{k-1},x_k',x_{k+1},\ldots, \\ x_{i-1},2^j x_i,x_{i+1},\ldots,x_n)\Big\| \\ \le \frac{1}{4^j}\varphi_k(x_1,\ldots,x_k,x_k',x_{k+1},\ldots,x_{i-1},2^j x_i,x_{i+1},\ldots,x_n).\end{aligned} \tag{40}$$

Letting $j \to \infty$ in the above two inequalities and using (31), we see that the mapping F_i is multiquadratic. □

Theorem 5. *Assume that* $\varphi : V^{2n} \to [0, \infty)$ *is a mapping such that*

$$\widetilde{\varphi}(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) := \sum_{j=0}^{\infty} \frac{1}{4^{n(j+1)}} \varphi\left(2^j x_{11}, 2^j x_{12}, \dots, 2^j x_{n1}, 2^j x_{n2}\right) < \infty \tag{41}$$

for all $(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in V^{2n}$. *If* $f : V^n \to W$ *is a function satisfying*

$$\left\| \sum_{i_1,\dots,i_n \in \{1,-1\}} f(x_{11} + i_1 x_{12}, \dots, x_{n1} + i_n x_{n2}) - 2^n \sum_{j_1,\dots,j_n \in \{1,2\}} f\left(x_{1j_1}, \dots, x_{nj_n}\right) \right\| \le \varphi(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \tag{42}$$

for all $(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in V^{2n}$ *and letting* $f(x) = 0$ *for any* $x \in V^n$ *with one component which is equal to 0, then there exists a unique multiquadratic mapping* $F : V^n \to W$ *such that*

$$\| f(x_{11}, \dots, x_{n1}) - F(x_{11}, \dots, x_{n1}) \| \le \widetilde{\varphi}(x_{11}, x_{11}, \dots, x_{n1}, x_{n1}) \tag{43}$$

for all $(x_{11}, \dots, x_{n1}) \in V^n$. *The function* F *is given by*

$$F(x_{11}, \dots, x_{n1}) := \lim_{j\to\infty} \frac{1}{4^{nj}} f\left(2^j x_{11}, \dots, 2^j x_{n1}\right) \tag{44}$$

for all $(x_{11}, \dots, x_{n1}) \in V^n$.

Proof. Fix $(x_{11}, \dots, x_{n1}) \in V^n$ and $j \in N \cup \{0\}$. Putting $x_{i2} := x_{i1}$ for $i \in \{1, \dots, n\}$ in (42), we get

$$\| f(2x_{11}, \dots, 2x_{n1}) - 4^n f(x_{11}, \dots, x_{n1}) \| \le \varphi(x_{11}, x_{11}, \dots, x_{n1}, x_{n1}). \tag{45}$$

Dividing both sides of the above inequality by $4^{n(j+1)}$ and replacing x_{i1} by $2^j x_{i1}$ for $i \in \{1, \dots, n\}$, we see that

$$\left\| \frac{1}{4^{n(j+1)}} f\left(2^{j+1} x_{11}, \dots, 2^{j+1} x_{n1}\right) - \frac{1}{4^{nj}} f\left(2^j x_{11}, \dots, 2^j x_{n1}\right) \right\| \le \frac{1}{4^{n(j+1)}} \varphi\left(2^j x_{11}, 2^j x_{11}, \dots, 2^j x_{n1}, 2^j x_{n1}\right), \tag{46}$$

and consequently for any nonnegative integers l and m with $l < m$ we obtain

$$\left\| \frac{1}{4^{nm}} f(2^m x_{11}, \dots, 2^m x_{n1}) - \frac{1}{4^{nl}} f\left(2^l x_{11}, \dots, 2^l x_{n1}\right) \right\| \le \sum_{j=l}^{m-1} \frac{1}{4^{n(j+1)}} \varphi\left(2^j x_{11}, 2^j x_{11}, \dots, 2^j x_{n1}, 2^j x_{n1}\right). \tag{47}$$

Therefore, it follows from (41) that $\{1/4^{nj} f(2^j x_{11}, \dots, 2^j x_{n1})\}_{j\in N}$ is a Cauchy sequence. Since the space W is complete, this sequence is convergent, and we define $F : V^n \to W$ by (44). Putting $l = 0$, taking $m \to \infty$ in (47), and using (41), we can see that the inequality (43) holds.

Next, fix also $(x_{12}, \dots, x_{n2}) \in V^n$, and note that according to (42) we have

$$\left\| \frac{1}{4^{nj}} \sum_{i_1,\dots,i_n \in \{1,-1\}} f\left(2^j (x_{11} + i_1 x_{12}), \dots, 2^j (x_{n1} + i_n x_{n2})\right) - 2^n \sum_{j_1,\dots,j_n \in \{1,2\}} \frac{1}{4^{nj}} f\left(2^j x_{1j_1}, \dots, 2^j x_{nj_n}\right) \right\| \le \frac{1}{4^{nj}} \varphi\left(2^j x_{11}, 2^j x_{12}, \dots, 2^j x_{n1}, 2^j x_{n2}\right). \tag{48}$$

Letting $j \to \infty$ in the above inequality and using (41), we see that F satisfies (24). By Theorem 3, we obtain that F is multiquadratic.

Finally, assume that $F' : V^n \to W$ is another multiquadratic mapping satisfying (43). Fix $k \in N \cup \{0\}$. Since F and F' are multiquadratic mappings, it is easy to verify that

$$\begin{aligned} F\left(2^k x_{11}, \dots, 2^k x_{n1}\right) &= 4^{nk} F(x_{11}, \dots, x_{n1}), \\ F'\left(2^k x_{11}, \dots, 2^k x_{n1}\right) &= 4^{nk} F'(x_{11}, \dots, x_{n1}). \end{aligned} \tag{49}$$

Then, using (41) and (43), we have

$$\begin{aligned} &\| F(x_{11}, \dots, x_{n1}) - F'(x_{11}, \dots, x_{n1}) \| \\ &= \left\| \frac{1}{4^{nk}} F\left(2^k x_{11}, \dots, 2^k x_{n1}\right) - \frac{1}{4^{nk}} F'\left(2^k x_{11}, \dots, 2^k x_{n1}\right) \right\| \\ &\le \left\| \frac{1}{4^{nk}} F\left(2^k x_{11}, \dots, 2^k x_{n1}\right) - \frac{1}{4^{nk}} f\left(2^k x_{11}, \dots, 2^k x_{n1}\right) \right\| \\ &\quad + \left\| \frac{1}{4^{nk}} f\left(2^k x_{11}, \dots, 2^k x_{n1}\right) - \frac{1}{4^{nk}} F'\left(2^k x_{11}, \dots, 2^k x_{n1}\right) \right\| \end{aligned}$$

$$\leq \frac{2}{4^{nk}} \tilde{\varphi}\left(2^k x_{11}, 2^k x_{11}, \ldots, 2^k x_{n1}, 2^k x_{n1}\right)$$
$$= 2\sum_{j=k}^{\infty} \frac{1}{4^{n(j+1)}} \varphi\left(2^j x_{11}, 2^j x_{11}, \ldots, 2^j x_{n1}, 2^j x_{n1}\right); \tag{50}$$

hence letting $k \to \infty$ we obtain $F = F'$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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