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Research Article

On Caristi Type Maps and Generalized Distances with Applications

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We prove some new existence theorems of fixed points for Caristi type maps and some suitable generalized distances without lower semicontinuity assumptions on dominated functions. As applications of our results, some new fixed point theorems and new generalizations of the Banach contraction principle are given.

1. Introduction

In 1972, Caristi proved the following famous fixed point theorem.

Theorem 1 (Caristi [1]). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R}$ a lower semicontinuous and bounded below function. Suppose that T is a Caristi type map on X dominated by f; that is, T satisfies*

$$d(x, Tx) \leq f(x) - f(Tx) \quad \textit{for each } x \in X. \tag{1}$$

Then T has a fixed point in X.

It is well-known that the Caristi's fixed point theorem is one of the most valuable generalization of the Banach contraction principle [2], and it is equivalent to the Ekeland's variational principle, to the Takahashi's nonconvex minimization theorem, to the Daneš' drop theorem, to the petal theorem, and to the Oettli-Théra's theorem; see [3–26] and references therein for more details. A number of generalizations in various different directions of the Caristi's fixed point theorem have been investigated by several authors; see, for example, [4–30] and references therein. An interesting direction of research is the extension of Caristi's fixed point theorem, Ekeland's variational principle, and Takahashi's nonconvex minimization theorem to generalized distances, for example, w-distances [5, 10, 14, 19], τ-distances [11, 12, 22], τ-functions [13, 15, 18, 22–25, 31–36], weak τ-functions [24, 25], P-distances [26], Q-functions [21], generalized pseudodistances [22, 23], and others. For more details on these generalizations, one can refer to [5, 10–26] and references therein.

Let us recall how we can exploit Caristi's fixed point theorem to prove the Banach contraction principle. A selfmap T on a metric space (X, d) is called *contractive or Banach type* if there exists a real number $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \text{for any } x, y \in X. \tag{2}$$

It is obvious that if T is a contractive map on X, then T is continuous on X and (2) will deduce the following inequality:

$$d(x, Tx) \leq \frac{1}{1-\lambda} d(x, Tx) - \frac{1}{1-\lambda} d(Tx, T^2 x) \quad \text{for any } x \in X. \tag{3}$$

The inequality (3) admits that T is a Caristi type map on X dominated by f defined by $f(x) := (1/(1-\lambda)) d(x, Tx)$. From the continuity of T, the function f is continuous on X, and therefore the Caristi's fixed point theorem is applicable to prove the Banach contraction principle. It is quite obvious that for any map T and any generalized distances p, the function $x \mapsto p(x, Tx)$ is not necessarily to be continuous even lower semicontinuous, so such well-known generalized versions of Caristi's fixed point theorem with lower semicontinuity are not easily applicable to any generalized version of Banach contraction principle for generalized distances. Motivated by the reason, in the recently paper [20], the author established some new versions of Caristi type fixed point theorem such that they can be applicable to prove generalized versions of Banach contraction principle for suitable generalized distances.

This work can be considered as a continuation of the paper [20]. In this paper, we first establish some new fixed point theorems for Caristi type maps and some suitable generalized distances without assuming that the dominated functions possess lower semicontinuity property. As applications of our results, some new fixed point theorems and new generalizations of the Banach contraction principle are given. We have already succeeded in utilizing our new versions of Caristi type fixed point theorem to deal with the existence results for any map T satisfying

$$d(Tx,Ty) \le \alpha(d(x,y))\, d(x,y) \quad \forall x,y \in X, \tag{4}$$

where $\alpha : [0,+\infty) \to [0,1)$ is a function satisfying $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0,+\infty)$.

2. Preliminaries

We recall in this section the notations, definitions, and results needed. Let (X,d) be a metric space. An extended real valued function $\phi : X \to (-\infty,+\infty]$ is said to be *lower semicontinuous* (*l.s.c.*, for short) at $w \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to w$ as $n \to \infty$, we have $\phi(w) \le \liminf_{n\to\infty}\phi(x_n)$. The function ϕ is called to be l.s.c. on X if ϕ is l.s.c. at every point of X. The function ϕ is said to be proper if $\phi \not\equiv +\infty$. Let $T : X \to X$ be a selfmap. T is said to be *closed* if $GrT = \{(x,y) \in X\times X : y = Tx\}$, the graph of T, is closed in $X \times X$. A point v in X is a fixed point of T if $Tv = v$. The set of fixed points of T is denoted by $\mathcal{F}(T)$. Throughout this paper we denote by $\mathbb{N}$ and $\mathbb{R}$, the set of positive integers and nonnegative real numbers, respectively.

Recall that a function $p : X\times X \to [0,+\infty)$ is called a *w-distance* [5, 10–20, 31], first introduced and defined by Kada, Suzuki, and Takahashi, if the following are satisfied:

($w1$) $p(x,z) \le p(x,y) + p(y,z)$ for any $x,y,z \in X$;

($w2$) for any $x \in X$, $p(x,\cdot) : X \to [0,+\infty)$ is l.s.c.;

($w3$) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$.

A function $p : X\times X \to [0,+\infty)$ is said to be a *τ-function* [13, 15, 18, 22–25, 31–36], first introduced and studied by Lin and Du, if the following conditions hold:

($\tau1$) $p(x,z) \le p(x,y) + p(y,z)$ for all $x,y,z \in X$;

($\tau2$) if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x,y_n) \le M$ for some $M = M(x) > 0$, then $p(x,y) \le M$;

($\tau3$) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty}\sup\{p(x_n, x_m) : m > n\} = 0$, if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;

($\tau4$) for $x,y,z \in X$, $p(x,y) = 0$ and $p(x,z) = 0$ imply $y = z$.

Note that not either of the implications $p(x,y) = 0 \Leftrightarrow x = y$ necessarily holds and p is nonsymmetric in general. It is well known that the metric d is a w-distance and any w-distance is a τ-function, but the converse is not true; see [13, 31] for more detail.

Example 2 (see [31, Example A]). Let $X = \mathbb{R}$ with the metric $d(x,y) = |x-y|$ for $x,y \in X$, and $0 < a < b$. Define the function $p : X\times X \to [0,+\infty)$ by

$$p(x,y) = \max\{a(y-x), b(x-y)\}. \tag{5}$$

Then p is a τ-function.

The following result is crucial in this paper.

Theorem 3 (see [18, Lemma 2.1]). *Let (X,d) be a metric space and $p : X\times X \to [0,+\infty)$ a function. Assume that p satisfies the condition ($\tau3$). If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty}\sup\{p(x_n,x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

Recently, the concepts of weak τ-function and generalized pseudodistance were introduced and studied by Khanh and Quy [24, 25] and Włodarczyk and Plebaniak [22] as follows.

Definition 4. Let (X,d) be a metric space. A function $p : X\times X \to [0,+\infty)$ is called

(i) *a weak τ-function* [24, 25] on X if conditions ($\tau1$), ($\tau3$), and ($\tau4$) hold;

(ii) *a generalized pseudodistance* [22] on X if conditions ($\tau1$) and ($\tau3$) hold.

It is obvious that any τ-function is a weak τ-function and every weak τ-function is a generalized pseudodistance, but the converse parts are not always true. The first observation is that there exists a weak τ-function which is not a τ-function.

Example 5 (see [24, Example 2.5]). Let $X = [0,+\infty)$, $\gamma > 0$, and $p : X\times X \to [0,+\infty)$ be defined by

$$p(x,y) = \begin{cases} |x-y| + \gamma, & \text{if } x \ne y, \\ \dfrac{3}{2}\gamma, & \text{if } x = y. \end{cases} \tag{6}$$

Then p is a weak τ-function which is neither a τ-function nor a w-distance.

The following example shows that there exists a generalized pseudodistance which is not a weak τ-function.

Example 6 (see [22, Example 1.3]). Define a function $p : [0,2]\times[0,2] \to [0,+\infty)$ by

$$p(x,y) = \begin{cases} 0, & \text{if } x-y = -2; \\ |x-y|, & \text{if } -2 < x-y \le 0; \\ x-y+2, & \text{if } 0 < x-y \le 2. \end{cases} \tag{7}$$

Then p is a generalized pseudodistance but not a weak τ-function.

Very recently, the author first introduced the following concepts.

Definition 7 (see [20]). Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$, $\varphi : \mathbb{R} \to (0, +\infty)$, and $p : X \times X \to [0, +\infty)$ be functions. A single-valued selfmap $T : X \to X$ is called

(i) Caristi type on X dominated by p, φ, and f (abbreviated as (p, φ, f)-Caristi type on X) if

$$p(x, Tx) \le \varphi(f(x))(f(x) - f(Tx)) \quad \text{for each } x \in X; \tag{8}$$

(ii) Caristi type on X dominated by p and f (abbreviated as (p, f)-Caristi type on X) if

$$p(x, Tx) \le f(x) - f(Tx) \quad \text{for each } x \in X; \tag{9}$$

(iii) Caristi type on X dominated by φ and f (abbreviated as (φ, f)-*Caristi type* on X) if

$$d(x, Tx) \le \varphi(f(x))(f(x) - f(Tx)) \quad \text{for each } x \in X; \tag{10}$$

(iv) Caristi type on X dominated by f (abbreviated as (f)-Caristi type on X) if

$$d(x, Tx) \le f(x) - f(Tx) \quad \text{for each } x \in X. \tag{11}$$

Clearly, if T is (p, f)-Caristi type (resp. (f)-Caristi type) on X, then T is (p, φ, f)-Caristi type (resp. (φ, f)-Caristi type) on X with $\varphi(t) = 1$ for all t. The following example illustrates that their converse are not always true.

Example 8. Let $X = [0, +\infty)$ with the usual metric $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Let $p : X \times X \to [0, +\infty)$ be defined by

$$p(x, y) = \max\{20(x - y), 40(y - x)\}, \tag{12}$$

for all $x, y \in X$. By Example 2, we know that p is a τ-function. Let $T : X \to X$ be defined by $Tx = x^2$, $x \in X$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 4x - 12, & \text{if } x \in [0, 1) \\ 15 - 8x, & \text{if } x \in [1, +\infty). \end{cases} \tag{13}$$

Then f is not lower semicontinuous at $x = 1$. For $i = 1, 2$, let $\varphi_i : \mathbb{R} \to (0, +\infty)$ be defined by

$$\varphi_1(t) = 2, \quad \varphi_2(t) = 6 \quad \forall t \in \mathbb{R}, \tag{14}$$

respectively. For $x \in [0, 1)$, we have

$$\begin{aligned} d(x, Tx) &= x - x^2 < 4(x - x^2) = f(x) - f(Tx), \\ p(x, Tx) &= \max\{20(x - Tx), 40(Tx - x)\} \\ &= 20(x - x^2) \\ &< \varphi_2(f(x))(f(x) - f(Tx)). \end{aligned} \tag{15}$$

For $x \in [1, +\infty)$, we have

$$\begin{aligned} d(x, Tx) &= x^2 - x < 8(x^2 - x) = f(x) - f(Tx), \\ p(x, Tx) &= 40(x^2 - x) < \varphi_2(f(x))(f(x) - f(Tx)). \end{aligned} \tag{16}$$

Hence, for any $x \in X$, we show

$$\begin{aligned} d(x, Tx) &\le f(x) - f(Tx) \\ &\le \varphi_i(f(x))(f(x) - f(Tx)) \quad \text{for each } i \in \{1, 2\}, \\ p(x, Tx) &\le \varphi_2(f(x))(f(x) - f(Tx)). \end{aligned} \tag{17}$$

So, T is (f)-Caristi type on X as well as (φ_i, f)-Caristi type on X for all $i \in \{1, 2\}$. Moreover, we know that T is (p, φ_2, f)-Caristi type on X, but it is neither (p, f)-Caristi type nor (p, φ_1, f)-Caristi type on X based on the following fact

$$p(x, Tx) > \varphi_1(f(x))(f(x) - f(Tx)) > f(x) - f(Tx) \quad \forall x \in X. \tag{18}$$

Definition 9 (see [31–36]). A function $\alpha : [0, +\infty) \to [0, 1)$ is said to be an $\mathcal{MT}$*-function* (or $\mathcal{R}$*-function*) if $\limsup_{s \to t^+} \alpha(s) < 1$ for all $t \in [0, +\infty)$.

It is obvious that if $\alpha : [0, +\infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then α is an $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class. But it is worth to mention that there exist functions which are not $\mathcal{MT}$-functions.

Example 10 (see [32]). Let $\alpha : [0, +\infty) \to [0, 1)$ be defined by

$$\alpha(t) := \begin{cases} \dfrac{\sin t}{t}, & \text{if } t \in \left(0, \dfrac{\pi}{2}\right] \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Since $\limsup_{s \to 0^+} \alpha(s) = 1$, φ is not an $\mathcal{MT}$-function.

Recently, Du [32] first proved the following characterizations of $\mathcal{MT}$-functions.

Theorem 11 (see [32]). *Let $\alpha : [0, +\infty) \to [0, 1)$ be a function. Then the following statements are equivalent.*

(a) *α is an $\mathcal{MT}$-function.*

(b) *For each $t \in [0, +\infty)$, there exist $r_t^{(1)} \in [0, 1)$ and $\varepsilon_t^{(1)} > 0$ such that $\alpha(s) \le r_t^{(1)}$ for all $s \in (t, t + \varepsilon_t^{(1)})$.*

(c) *For each $t \in [0, +\infty)$, there exist $r_t^{(2)} \in [0, 1)$ and $\varepsilon_t^{(2)} > 0$ such that $\alpha(s) \le r_t^{(2)}$ for all $s \in [t, t + \varepsilon_t^{(2)}]$.*

(d) *For each $t \in [0, +\infty)$, there exist $r_t^{(3)} \in [0, 1)$ and $\varepsilon_t^{(3)} > 0$ such that $\alpha(s) \le r_t^{(3)}$ for all $s \in (t, t + \varepsilon_t^{(3)}]$.*

(e) *For each $t \in [0, +\infty)$, there exist $r_t^{(4)} \in [0, 1)$ and $\varepsilon_t^{(4)} > 0$ such that $\alpha(s) \le r_t^{(4)}$ for all $s \in [t, t + \varepsilon_t^{(4)})$.*

(f) *For any nonincreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $[0,+\infty)$, *one has* $0 \le \sup_{n\in\mathbb{N}} \alpha(x_n) < 1$.

(g) α *is a function of contractive factor; that is, for any strictly decreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $[0,+\infty)$, *one has* $0 \le \sup_{n\in\mathbb{N}}\alpha(x_n) < 1$.

3. New Results for Caristi Type Maps and Their Applications

We start with the following useful auxiliary result.

Theorem 12. *Let* (X,d) *be a metric space,* $f : X \to (-\infty, +\infty]$ *a proper and bounded below function,* $\varphi : \mathbb{R} \to (0,+\infty)$ *a nondecreasing function,* $p : X \times X \to [0,\infty)$ *a function, and* $T : X \to X$ *a selfmap on* X. *Let* $u \in X$ *with* $f(u) < +\infty$. *Define* $x_1 = u$ *and* $x_{n+1} = Tx_n$ *for each* $n \in \mathbb{N}$. *If* p *satisfies* $(\tau1)$ *and* T *is* (p,φ,f)*-Caristi type on* X, *then*

$$\lim_{n\to\infty} \sup\{p(x_n,x_m) : m > n\} = 0. \tag{20}$$

Moreover, if we futher assume that p *satisfies* $(\tau3)$, *then* $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence in* X.

Proof. For $x_1 = u$, $f(x_1) < +\infty$. Since T is (p,φ,f)-Caristi type on X, we get

$$\begin{aligned} p(x_1,x_2) &= p(x_1,Tx_1) \\ &\le \varphi(f(x_1))(f(x_1) - f(Tx_1)) \\ &= \varphi(f(x_1))(f(x_1) - f(x_2)), \end{aligned} \tag{21}$$

which implies

$$f(x_2) \le f(x_1) < +\infty. \tag{22}$$

Similarly, we have

$$\begin{gathered} p(x_2,x_3) = p(x_2,Tx_2) \le \varphi(f(x_2))(f(x_2) - f(x_3)), \\ f(x_3) \le f(x_2) \le f(x_1) < +\infty. \end{gathered} \tag{23}$$

Hence, by induction, we can obtain the following inequalities

$$p(x_n,x_{n+1}) \le \varphi(f(x_n))(f(x_n) - f(x_{n+1})), \tag{24}$$

$$f(x_{n+1}) \le f(x_n) < +\infty \quad \text{for each } n \in \mathbb{N}. \tag{25}$$

Since f is bounded below,

$$r := \lim_{n\to\infty} f(x_n) = \inf_{n\in\mathbb{N}} f(x_n) \ \text{ exists.} \tag{26}$$

By (25), since φ is nondecreasing, we have

$$\varphi(f(x_n)) \le \varphi(f(x_1)) \quad \forall n \in \mathbb{N}. \tag{27}$$

For $m > n$ with $m,n \in \mathbb{N}$, taking into account $(\tau1)$, (24), (26), and (27), we get

$$p(x_n,x_m) \le \sum_{j=n}^{m-1} p(x_j,x_{j+1}) \le \varphi(f(x_1))(f(x_n) - r). \tag{28}$$

Let $\alpha_n = \varphi(f(x_1))(f(x_n) - r)$, $n \in \mathbb{N}$. Then $\sup\{p(x_n,x_m) : m > n\} \le \alpha_n$ for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} f(x_n) = r$, we obtain $\lim_{n\to\infty}\alpha_n = 0$, and hence $\lim_{n\to\infty} \sup\{p(x_n,x_m) : m > n\} = 0$. Moreover, if p satisfies $(\tau3)$, then the desired conclusion follows from Theorem 3 immediately. The proof is completed. □

Applying Theorem 12, we prove a new fixed point theorem for Caristi type maps and generalized pseudodistances. It is worth to mention that in Theorem 13 we pose some suitable assumptions on the map T without assuming that the domi-nated functions possess lower semicontinuity property.

Theorem 13. *Let* (X,d) *be a complete metric space,* $f : X \to (-\infty,+\infty]$ *a proper and bounded below function,* $\varphi : \mathbb{R} \to (0,+\infty)$ *a nondecreasing function, and* p *a generalized pseudodistance on* X. *Suppose that* $T : X \to X$ *is a* (p,φ,f)*-Caristi type selfmap on* X *and one of the following conditions is satisfied:*

(H1) T *is continuous;*

(H2) T *is closed;*

(H3) $p(x,y) = 0$ *implies* $x = y$ *for all* $x,y \in X$ *and the map* $g : X \to [0,\infty)$ *defined by* $g(x) = p(x,Tx)$ *is l.s.c.;*

(H4) *the map* $h : X \to [0,\infty)$ *defined by* $h(x) = d(x,Tx)$ *is l.s.c.;*

(H5) *for any sequence* $\{z_n\}$ *in* X *with* $z_{n+1} = Tz_n$, $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} z_n = a$, *we have* $\lim_{n\to\infty} p(z_n,Ta) = 0$.

Then T admits a fixed point in X. Moreover, for any $w \in X$ with $f(w) < +\infty$, the sequence $\{T^n w\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof. Let $S = \{x \in X : f(x) < +\infty\}$. Since f is proper, $S \ne \emptyset$. Let $w \in S$. Define $x_1 = w$ and $x_{n+1} = Tx_n = T^n w$ for each $n \in \mathbb{N}$. Since p is a generalized pseudodistance on X, by applying Theorem 12, we know that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and

$$\lim_{n\to\infty} \sup\{p(x_n,x_m) : m > n\} = 0. \tag{29}$$

The last equality implies

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{30}$$

By the completeness of X, there exists $v_w \in X$ such that $x_n \to v_w$ as $n \to \infty$.

Now, we verify $v_w \in \mathcal{F}(T)$. If (H1) holds, since T is continuous on X, $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}$ and $x_n \to v_w$ as $n \to \infty$, we get

$$v_w = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\left(\lim_{n\to\infty} x_n\right) = Tv_w, \tag{31}$$

which means $v_w \in \mathcal{F}(T)$. If (H2) holds, since T is closed, $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}$ and $x_n \to v_w$ as $n \to \infty$, we have $Tv_w = v_w$. Suppose that (H3) holds. By the lower semicontinuity of g, $x_n \to v_w$ as $n \to \infty$ and (30), we obtain

$$\begin{aligned} p(v_w, Tv_w) &= g(v_w) \le \liminf_{n\to\infty} g(x_n) \\ &= \lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \end{aligned} \tag{32}$$

which implies $p(v_w, Tv_w) = 0$. By the hypothesis in (H3), we get $v_w \in \mathcal{F}(T)$. Suppose that (H4) holds. Since $\{x_n\}_{n\in\mathbb{N}}$ is convergent in X,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{33}$$

Since

$$d(v_w, Tv_w) = h(v_w) \le \liminf_{n\to\infty} d(x_n, x_{n+1}) = 0, \tag{34}$$

we obtain $d(v_w, Tv_w) = 0$, and hence $v_w \in \mathcal{F}(T)$. Finally, assume (H5) holds. Since $\lim_{n\to\infty}\sup\{p(x_n, x_m) : m > n\} = 0$ and $\lim_{n\to\infty} p(x_n, Tv_w) = 0$, there exists $\{a_n\} \subset \{x_n\}$ with $\lim_{n\to\infty}\sup\{p(a_n, a_m) : m > n\} = 0$ and $b_n = Tv_w$ for all $n \in \mathbb{N}$, such that $\lim_{n\to\infty} p(a_n, b_n) = 0$. By ($\tau$3), $\lim_{n\to\infty} d(a_n, b_n) = 0$. Since $a_n \to v_w$ as $n \to \infty$ and $d(b_n, v_w) \le d(b_n, a_n) + d(a_n, v_w)$, we get $b_n \to v_w$ as $n \to \infty$. So $Tv_w = v_w$ or $v_w \in \mathcal{F}(T)$. Therefore, in any case, we prove $v_w \in \mathcal{F}(T)$. Since $w \in S$ is arbitrary, the sequence $\{T^n w\}_{n\in\mathbb{N}}$ converges to a fixed point v_w of T. This completes the proof. □

Here, we give an example illustrating Theorem 13. This example also gives a negative answer to the uniqueness of fixed point.

Example 14. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define $p : X \times X \to [0, +\infty)$ by

$$p(x, y) = \max\{2(x - y), 3(y - x)\}, \tag{35}$$

for all $x, y \in X$. Then p is a generalized pseudodistance on X. Let $f : X \to \mathbb{R}$ and $\varphi : \mathbb{R} \to (0, +\infty)$ be defined by

$$\begin{gathered} f(x) = \begin{cases} \dfrac{1}{3}x - 25, & \text{if } x \in \left[0, \dfrac{1}{2}\right) \\ \dfrac{1}{2}x - 3, & \text{if } x \in \left[\dfrac{1}{2}, 1\right], \end{cases} \\ \varphi(t) = 10 \quad \forall t \in \mathbb{R}, \end{gathered} \tag{36}$$

respectively. So $f(x) < +\infty$ for all $x \in X$. Note that f is not lower semicontinuous at $x = 1/2$, so f is not lower semicontinuous on X. Since $f(x) \ge -25$ for all $x \in X$, f is a bounded below function on X. Let $T : X \to X$ be defined by

$$Tx = x^2 \quad \forall x \in X. \tag{37}$$

Then T is continuous on X and $\mathcal{F}(T) = \{0, 1\}$. It is also easy to see that T is closed and the map $x \mapsto d(x, Tx)$ is l.s.c. Hence (H1), (H2), and (H4) as in Theorem 13 hold. We deduce that for any $x \in X$,

$$\begin{aligned} p(x, Tx) &= \max\{2(x - Tx), 3(Tx - x)\} \\ &= 2\left(x - x^2\right) \\ &< \varphi(f(x))(f(x) - f(Tx)), \end{aligned} \tag{38}$$

so T is (p, φ, f)-Caristi type on X. Let $w \in X$. Since

$$\lim_{n\to\infty} T^n w = \lim_{n\to\infty} w^{2n} = \begin{cases} 0, & \text{if } w \in [0, 1), \\ 1, & \text{if } w = 1, \end{cases} \tag{39}$$

we know that $\{T^n w\}_{n\in\mathbb{N}}$ converges and the limit of $\{T^n w\}_{n\in\mathbb{N}}$ belongs to $\mathcal{F}(T) = \{0, 1\}$. On the other hand, since all the assumptions of Theorem 13 are satisfied, by applying Theorem 13, we also prove that T has a fixed point in X and for any $w \in X$, the sequence $\{T^n w\}_{n\in\mathbb{N}}$ converges to a fixed point of T. It is worth noticing that any well-known generalized version of Caristi's fixed point theorem is not applicable here.

The following conclusions are immediate from Theorem 13.

Corollary 15. *Let (X, d) be a complete metric space, $f : X \to (-\infty, +\infty]$ a proper and bounded below function, and p a generalized pseudodistance on X. Suppose that $T : X \to X$ is a (p, f)-Caristi type selfmap on X and one of the conditions (H1), (H2), (H3), (H4), and (H5) as in Theorem 13 holds. Then T admits a fixed point in X. Moreover, for any $w \in X$ with $f(w) < +\infty$, the sequence $\{T^n w\}_{n\in\mathbb{N}}$ converges to a fixed point of T.*

Corollary 16. *Let (X, d) be a complete metric space, $f : X \to (-\infty, +\infty]$ a proper and bounded below function, and $\varphi : \mathbb{R} \to (0, +\infty)$ a nondecreasing function. Suppose that $T : X \to X$ is a (φ, f)-Caristi type selfmap on X and one of the following conditions is satisfied:*

(D1) *T is continuous;*

(D2) *T is closed;*

(D3) *the map $h : X \to [0, \infty)$ defined by $h(x) = d(x, Tx)$ is l.s.c.*

Then T admits a fixed point in X. Moreover, for any $w \in X$ with $f(w) < +\infty$, the sequence $\{T^n w\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Corollary 17. *Let (X, d) be a complete metric space and $f : X \to (-\infty, +\infty]$ a proper and bounded below function. Suppose that $T : X \to X$ is a (f)-Caristi type selfmap on X and one of the conditions (D1), (D2), and (D3) as in Corollary 16 holds. Then T admits a fixed point in X. Moreover, for any $w \in X$ with $f(w) < +\infty$, the sequence $\{T^n w\}_{n\in\mathbb{N}}$ converges to a fixed point of T.*

Now, we give another quite useful auxiliary theorem for our applications.

Theorem 18. *Let (X,d) be a metric space, $p : X \times X \to [0,+\infty)$ a function, and $T : X \to X$ a selfmap. Suppose that there exists an $\mathcal{MT}$-function $\alpha : [0,+\infty) \to [0,1)$ such that*

$$p(Tx,Ty) \leq \alpha(p(x,y))\, p(x,y) \quad \forall x,y \in X. \tag{40}$$

Then there exists a function $\beta : X \to [0,1)$ such that for each $x \in X$,

$$\begin{gathered} \beta(Tx) \leq \beta(x), \\ \alpha\left(p\left(T^{n-1}x, T^n x\right)\right) \leq \beta(x) \quad \forall n \in \mathbb{N}. \end{gathered} \tag{41}$$

Here, we denote $T^0 = I$ (the identity map).

Proof. Let $x \in X$ be given. From our hypothesis, we have

$$\begin{aligned} p\left(T^n x, T^{n+1}x\right) &\leq \alpha\left(p\left(T^{n-1}x, T^n x\right)\right) p\left(T^{n-1}x, T^n x\right) \\ &< p\left(T^{n-1}x, T^n x\right) \end{aligned} \tag{42}$$

for each $n \in \mathbb{N}$. So the sequence $\{p(T^{n-1}x, T^n x)\}_{n\in\mathbb{N}}$ is strictly decreasing in $[0,+\infty)$. Since α is an $\mathcal{MT}$-function, by (g) of Theorem 11, we obtain

$$0 \leq \sup_{n\in\mathbb{N}} \alpha\left(p\left(T^{n-1}x, T^n x\right)\right) < 1. \tag{43}$$

Since $x \in X$ is arbitrary, we can define a new function $\beta : X \to [0,1)$ by

$$\beta(x) := \sup_{n\in\mathbb{N}} \alpha\left(p\left(T^{n-1}x, T^n x\right)\right) \quad \text{for } x \in X. \tag{44}$$

It is obvious that for each $x \in X$, we have

$$\begin{gathered} \beta(Tx) \leq \beta(x), \\ \alpha\left(p\left(T^{n-1}x, T^n x\right)\right) \leq \beta(x) \quad \forall n \in \mathbb{N}. \end{gathered} \tag{45}$$

□

As an interesting application of Theorem 13, we prove the following new fixed point theorems for Banach type maps.

Theorem 19. *Let (X,d) be a complete metric space, p a generalized pseudodistance on X with $p(x,y) = 0$ implies $x = y$ for all $x, y \in X$, and $T : X \to X$ a selfmap. Suppose that*

(a) *there exists an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ such that*

$$p(Tx,Ty) \leq \alpha(p(x,y))\, p(x,y) \quad \forall x,y \in X; \tag{46}$$

(b) *one of the conditions (H1), (H2), (H3), (H4), and (H5) as in Theorem 13 holds.*

Then T admits a unique fixed point in X. Moreover, for each $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to the unique fixed point of T.

Proof. Denote $T^0 = I$ (the identity map). Applying Theorem 18, there exists a function $\beta : X \to [0,1)$ such that for each $x \in X$,

$$\begin{gathered} \beta(Tx) \leq \beta(x), \\ \alpha\left(p\left(T^{n-1}x, T^n x\right)\right) \leq \beta(x) \quad \forall n \in \mathbb{N}. \end{gathered} \tag{47}$$

For each $x \in X$, by (46), we get

$$\begin{aligned} p(x,Tx) &- \alpha(p(x,Tx))\, p(x,Tx) \\ &\leq p(x,Tx) - p\left(Tx, T^2x\right). \end{aligned} \tag{48}$$

By exploiting the inequalities (47) and (48), we obtain

$$\begin{aligned} p(x,Tx) &\leq \frac{1}{1-\alpha(p(x,Tx))} p(x,Tx) \\ &\quad - \frac{1}{1-\alpha(p(x,Tx))} p\left(Tx,T^2x\right) \\ &\leq \frac{1}{1-\beta(x)} p(x,Tx) \\ &\quad - \frac{1}{1-\beta(Tx)} p\left(Tx,T^2x\right). \end{aligned} \tag{49}$$

Let $\varphi : \mathbb{R} \to (0,+\infty)$ and $f : X \to \mathbb{R}$ be defined by

$$\begin{gathered} \varphi(t) = 1 \quad \text{for } t \in \mathbb{R}, \\ f(x) = \frac{1}{1-\beta(x)} p(x,Tx) \quad \text{for } x \in X, \end{gathered} \tag{50}$$

respectively. Then φ is a nondecreasing function, and f is a bounded below function. Clearly, $f(x) < +\infty$ for all $x \in X$. By (49), we obtain

$$p(x,Tx) \leq \varphi(f(x))\,(f(x) - f(Tx)) \quad \forall x \in X. \tag{51}$$

Hence we prove that $T : X \to X$ is a (p,φ,f)-Caristi type selfmap on X. Applying Theorem 13, we know $\mathcal{F}(T) \neq \emptyset$. We claim that $\mathcal{F}(T)$ is a singleton set. Let $u, v \in \mathcal{F}(T)$. Then $Tu = u$ and $Tv = v$. From (46), we have

$$p(u,v) = p(Tu,Tv) \leq \alpha(p(u,v))\, p(u,v) \tag{52}$$

or

$$(1 - \alpha(p(u,v)))\, p(u,v) \leq 0, \tag{53}$$

which implies $p(u,v) = 0$. By our hypothesis, we get $u = v$ and our claim is proved. By the uniqueness of fixed point of T and applying Theorem 13 again, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to the unique fixed point of T for any $x \in X$. □

As a direct consequence of Theorem 19, we obtain the following result.

Corollary 20. *Let (X,d) be a complete metric space, p be a generalized pseudodistance on X with $p(x,y) = 0$ implies $x = y$ for all $x, y \in X$, and $T : X \to X$ a selfmap. Suppose that*

(a) *there exists* $\gamma \in [0, 1)$ *such that*

$$p(Tx, Ty) \leq \gamma p(x, y) \quad \forall x, y \in X; \tag{54}$$

(b) *one of the conditions (H1), (H2), (H3), (H4), and (H5) as in Theorem 13 holds.*

Then T admits a unique fixed point in X. Moreover, for each $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to the unique fixed point of T.

Applying Theorem 19, we obtain a generalization of the celebrated Banach contraction principle.

Corollary 21. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *a selfmap. Suppose that there exists an* $\mathcal{MT}$*-function* $\alpha : [0, \infty) \to [0, 1)$ *such that*

$$d(Tx, Ty) \leq \alpha(d(x, y))\, d(x, y) \quad \forall x, y \in X. \tag{55}$$

Then T admits a unique fixed point in X. Moreover, for each $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to the unique fixed point of T.

Proof. By (55), we know that T is continuous on X. Hence the conclusion follows from Theorem 19 immediately. □

Remark 22. Theorems 13 and 19, Corollaries 15–21 all generalize and improve the celebrated Banach contraction principle.

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