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Research Article

Nodal Solutions of the p-Laplacian with Sign-Changing Weight

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We are concerned with determining values of γ, for which there exist nodal solutions of the boundary value problem $(|u'|^{p-2}u')' + \gamma m(t) f(u) = 0, t \in (0,1), u(0) = u(1) = 0$, where $m \in C[0,1]$ is a sign-changing function, $f : \mathbb{R} \to \mathbb{R}$ with $f(s)s > 0$. The proof of our main results is based upon global bifurcation techniques.

1. Introduction

In [1], Ma and Thompson considered determining values of r, for which there exist nodal solutions of the boundary value problem

$$u'' + rm(t) f(u) = 0, \quad t \in (0,1), \qquad u(0) = u(1) = 0, \tag{1}$$

under the following assumptions:

(H_1) $f \in C(\mathbb{R}, \mathbb{R})$ with $sf(s) > 0$ for $s \neq 0$;

($\widetilde{H}_2$) $m : [0,1] \to [0, +\infty)$ is continuous and does not vanish identically on any subinterval of $[0,1]$;

($\widetilde{H}_3$) there exist $f_0, f_\infty \in (0, +\infty)$ such that

$$f_0 = \lim_{|s| \to 0} \frac{f(s)}{s}, \qquad f_\infty = \lim_{|s| \to +\infty} \frac{f(s)}{s}. \tag{2}$$

Using the bifurcation theory of Rabinowitz [2, 3], they proved the following.

Theorem 1. *Let (H_1), ($\widetilde{H}_2$), and ($\widetilde{H}_3$) hold. Assume that, for some $k \in \mathbb{N}$, either*

$$\frac{\lambda_k}{f_\infty} < r < \frac{\lambda_k}{f_0} \ \text{or} \ \frac{\lambda_k}{f_0} < r < \frac{\lambda_k}{f_\infty}. \tag{3}$$

Then (1) has two solutions u_k^+ and u_k^- such that u_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0 and u_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.

The results of Theorem 1 have been extended to the case that the weight function changes its sign by Ma and Han [4]. Bifurcation methods have been applied to study the existence of nodal solutions of nonlinear two-point, multipoint, and periodic boundary value problems; see [5–9] and the references therein. The results they obtained extend some well-known theorems of the existence of positive solutions for the related problems [10].

However, no results on the existence of nodal solutions, even positive solutions, have been established for one-dimensional p-Laplacian equation with sign-changing weight $m(t)$. It is the purpose of this paper to establish a similar result to Theorem 1 for one-dimensional p-Laplacian equation with sign-changing weight. Problem with sign-changing weight arises from the selection-migration model in population genetics. In this model, $m(t)$ changes sign corresponding to the fact that an allele A_1 holds an advantage over a rival allele A_2 at the same points and is at a disadvantage at others; the parameter r corresponds to the reciprocal of diffusion; for details see [11].

If $m(t) \equiv 1$, Del Pino et al. [12] established the global bifurcation theory for one-dimensional p-Laplacian eigenvalue problem. Peral [13] got the global bifurcation theory for p-Laplacian eigenvalue problem on the unite ball. In [14], Del Pino and Manásevich obtained the global bifurcation from the principal eigenvalue for p-Laplacian eigenvalue problem on the general domain. If $m(t) \geq 0$ and is singular at $t = 0$ or $t = 1$, Lee and Sim [15] also established the bifurcation theory for one-dimensional p-Laplacian eigenvalue problem. However, if $m(t)$ changes sign, there are a few papers dealing with the p-Laplacian eigenvalue problem via bifurcation techniques. In [16], Drábek and Huang established the global bifurcation from the principal eigenvalue for p-Laplacian eigenvalue problem in $\mathbb{R}^N$.

The purpose of this paper is to study the bifurcation behavior of one-dimensional p-Laplacian eigenvalue problem as follows:

$$\varphi_p(u')' + \gamma m(t) f(u) = 0, \quad t \in (0,1), \qquad u(0) = u(1) = 0, \tag{4}$$

under the condition (H_1) and

(H_2) $m(t) \in C[0,1]$ changes sign and

$$\text{meas}\{x \in [0,1] \mid m(t) = 0\} = 0; \tag{5}$$

(H_3) there exists $f_0 \in (0,\infty)$ such that

$$f_0 = \lim_{|s|\to 0} \frac{f(s)}{\varphi_p(s)}, \tag{6}$$

where $\varphi_p(s) = |s|^{p-2}s$ with $1 < p < +\infty$;

(H_4) there exists $f_\infty \in (0,+\infty)$ such that

$$f_\infty = \lim_{|s|\to +\infty} \frac{f(s)}{\varphi_p(s)}. \tag{7}$$

Moreover, based on our global bifurcation theorem, we will prove the existence of nodal solutions for the corresponding nonlinear problem with a parameter (see Theorem 11).

The main tool is the global bifurcation techniques in [17].

The rest of this paper is arranged as follows. In Section 2, we establish the global bifurcation theory for one-dimensional p-Laplacian eigenvalue problem with sign-changing weight. In Section 3, we state and prove the main results of this paper.

2. Some Preliminaries

Let E be the Banach space $C_0^1[0,1]$ with the norm

$$\|u\| = \max\left\{\|u\|_\infty, \|u'\|_\infty\right\}. \tag{8}$$

Let $Y = L^1(0,1)$ with its usual normal $\|\cdot\|_{L^1}$.

We start by considering the following auxiliary problem:

$$\varphi_p(u')' = h, \quad t \in (0,1), \qquad u(0) = u(1) = 0, \tag{9}$$

for a given $h \in L^1(0,1)$. By a solution of problem (9), we understand a function $u \in E$ with $\varphi_p(u')$ absolutely continuous which satisfies (9). Problem (9) is equivalently written to

$$u(t) = G_p(h)(t) := \int_0^t \varphi_p^{-1}\left(a(h) + \int_0^s h(\tau)\,d\tau\right) ds, \tag{10}$$

where $a : Y \to \mathbb{R}$ is a continuous function satisfying

$$\int_0^1 \varphi_p^{-1}\left(a(h) + \int_0^s h(\tau)\,d\tau\right) ds = 0. \tag{11}$$

It is known that $G_p : Y \to E$ is continuous and maps equi-integrable sets of Y into relatively compacts of E. One may refer to Lee and Sim [15] for details.

Since the bifurcation points of

$$\varphi_p(u'(t))' + \lambda m(t) f(u(t)) = 0 \quad \text{a.e. in } (0,1), \qquad u(0) = u(1) = 0 \tag{12}$$

is related to the eigenvalues of the problem

$$\varphi_p(u'(t))' + \lambda m(t) \varphi_p(u(t)) = 0 \quad \text{a.e. in } (0,1), \qquad u(0) = u(1) = 0. \tag{13}$$

We define the operator $T_\lambda^p : E \to E$ by

$$\begin{aligned} T_\lambda^p(u)(t) &= \int_0^t \varphi_p^{-1}\left(a\left(-\lambda m \varphi_p(u(\tau))\right) - \int_0^s \lambda m(\tau) \varphi_p(u(\tau))\,d\tau\right) ds \\ &=: G_p\left(-\lambda m \varphi_p(u)\right)(t). \end{aligned} \tag{14}$$

Then $T_\lambda^p : E \to E$ is completely continuous and problem (13) is equivalent to

$$u = T_\lambda^p(u). \tag{15}$$

The following spectrum result plays a fundamental role in our study.

Lemma 2 (see [18, 19]). *Let (H_2) hold. Then*

(i) *the set of all eigenvalues of the problem* (13) *is two infinite sequences of simple eigenvalues as follows:*

$$\begin{gathered} 0 < \mu_1^+(p) < \mu_2^+(p) < \cdots < \mu_k^+(p) < \cdots, \\ \lim_{k\to+\infty} \mu_k^+(p) = +\infty, \\ 0 > \mu_1^-(p) > \mu_2^-(p) > \cdots > \mu_k^-(p) > \cdots, \\ \lim_{k\to+\infty} \mu_k^-(p) = -\infty; \end{gathered} \tag{16}$$

(ii) *for $k \in \mathbb{N}$ and $\nu \in \{+,-\}$,* Ker $(I - T^p_{\mu_k^\nu(p)})$ *is a space of E with dimensional* 1*;*

(iii) *the eigenfunction corresponding to* $\mu_k^\nu(p)$ *has exactly* $k-1$ *simple zeros in* $(0,1)$.

Remark 3. Using the Gronwall inequality, we can easily show that all zeros of eigenfunction corresponding to eigenvalue $\mu_k^\nu(p)$ are simple.

It is very known that T_λ^2 is completely continuous in $C^1[0,1]$. Thus, the Leray-Schauder degree $d_{LS}(I-T_\lambda^2, B_r(0), 0)$ is well-defined for arbitrary r-ball $B_r(0)$ and $\lambda \neq \mu_k^\nu$, $k \in \mathbb{Z}$ and $\nu \in \{+,-\}$.

Lemma 4. *For* $r>0$, *we have*

$$d_{LS}\left(I-T_\lambda^2, B_r(0), 0\right) = \begin{cases} 1, & if \ \ \lambda \in \left(\mu_1^-(2), \mu_1^+(2)\right), \\ (-1)^k, & if \ \ \lambda \in \left(\mu_k^+(2), \mu_{k+1}^+(2)\right), k \in \mathbb{N}, \\ (-1)^k, & if \ \ \lambda \in \left(\mu_{k+1}^-(2), \mu_k^-(2)\right), k \in \mathbb{N}. \end{cases} \tag{17}$$

Proof. We divide the proof into two cases.

Case 1. $\lambda \geq 0$. Since T_λ^2 is compact and linear, by [20, Theorem 8.10] and Lemma 2 (ii) with $p=2$,

$$d_{LS}\left(I-T_\lambda^2, B_r(0), 0\right) = (-1)^{m(\lambda)}, \tag{18}$$

where $m(\lambda)$ is the sum of algebraic multiplicity of the eigenvalues μ of (13) satisfying $\mu^{-1}\lambda > 1$.

If $\lambda \in [0, \mu_1^+(2))$, then there are no such μ at all; then

$$d_{LS}\left(I-T_\lambda^2, B_r(0), 0\right) = (-1)^{m(\lambda)} = (-1)^0 = 1. \tag{19}$$

If $\lambda \in (\mu_k^+(2), \mu_{k+1}^+(2))$ for some $k \in \mathbb{N}$, then

$$\left(\mu_j^+(2)\right)^{-1}\lambda > 1, \quad j \in \{1, \cdots, k\}. \tag{20}$$

This together with Lemma 2 (ii) implies the following:

$$d_{LS}\left(I-T_\lambda^2, B_r(0), 0\right) = (-1)^k. \tag{21}$$

Case 2. $\lambda < 0$. In this case, we consider a new sign-changing eigenvalue problem as follows

$$u'' + \widehat{\lambda}\widehat{m}(t)u = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \tag{22}$$

where $\lambda = -\lambda$, $\widehat{m}(t) = -m(t)$. It is easy to check that

$$\widehat{\mu}_k^+(2) = -\mu_k^-(2), \quad k \in \mathbb{N}. \tag{23}$$

Thus, we may use the result obtained in Case 1 to deduce the desired result. □

We first show that the principle eigenvalue function $\mu_1^\nu : (1,+\infty) \to \mathbb{R}$ is continuous.

Proposition 5. *The eigenvalue function* $\mu_1^\nu : (1,+\infty) \to \mathbb{R}$ *is continuous.*

Proof. We only show that $\mu_1^+ : (1,+\infty) \to \mathbb{R}$ is continuous since the case of μ_1^- is similar. In the following proof, we will shorten μ_1^+ to μ_1. From the variational characterization of $\mu_1(p)$, it follows that

$$\mu_1(p) = \sup\left\{\mu > 0 \mid \mu \int_0^1 m(t)|u|^p\, dt \leq \int_0^1 \left|u'\right|^p dt, \forall u \in C_c^\infty(0,1)\right\}. \tag{24}$$

Let $\{p_j\}_{j=1}^\infty$ be a sequence in $(1,+\infty)$ convergent to $p>1$. We will show that

$$\lim_{j\to+\infty} \mu_1\left(p_j\right) = \mu_1(p). \tag{25}$$

To do this, let $u \in C_c^\infty(0,1)$. Then, from (24),

$$\mu_1\left(p_j\right)\int_0^1 m(t)|u|^{p_j}dt \leq \int_0^1 \left|u'\right|^{p_j}dt. \tag{26}$$

On applying the Dominated Convergence Theorem, we find that

$$\limsup_{j\to+\infty} \mu_1\left(p_j\right)\int_0^1 m(t)|u|^p dt \leq \int_0^1 \left|u'\right|^p dt. \tag{27}$$

Relation (27), the fact that u is arbitrary and (24) yield

$$\limsup_{j\to+\infty} \mu_1\left(p_j\right) \leq \mu_1(p). \tag{28}$$

Thus, to prove (25), it suffices to show that

$$\liminf_{j\to+\infty} \mu_1\left(p_j\right) \geq \mu_1(p). \tag{29}$$

Let $\{p_k\}_{k=1}^\infty$ be a subsequence of $\{p_j\}_{j=1}^\infty$ such that $\lim_{k\to+\infty}\mu_1(p_k) = \liminf_{j\to+\infty}\mu_1(p_j)$.

Let us fix $\varepsilon_0 > 0$ so that $p-\varepsilon_0 > 1$ and, for each $0<\varepsilon<\varepsilon_0$, $W_0^{1,p-\varepsilon}(0,1)$ is compactly embedded into $L^{p+\varepsilon}(0,1)$. For $k \in \mathbb{N}$, let us choose $u_k \in W_0^{1,p_k}(0,1)$ such that

$$\int_0^1 \left|u_k'\right|^{p_k} dt = 1, \tag{30}$$

$$\int_0^1 \left|u_k'\right|^{p_k} dt = \mu_1\left(p_k\right)\int_0^1 m(t)\left|u_k\right|^{p_k} dt. \tag{31}$$

For $0<\varepsilon<\varepsilon_0$, there exists $k_0 \in \mathbb{N}$ such that $p-\varepsilon < p_k < p+\varepsilon$ for any $k \geq k_0$. Thus, for $k \geq k_0$, (30) and Hölder's inequality imply that

$$\int_0^1 \left|u_k'\right|^{p-\varepsilon} dt \leq 1. \tag{32}$$

This shows that $\{u_k\}_{k=k_0}^\infty$ is a bounded sequence in $W_0^{1,p-\varepsilon}(0,1)$. Passing to a subsequence if necessary, we can assume that $u_k \rightharpoonup u$ in $W_0^{1,p-\varepsilon}(0,1)$ and hence that $u_k \to u$ in $L^{p+\varepsilon}(0,1)$. Furthermore, $u \in L^p(0,1)$ and $u_k \to u$ in $L^{p_k}(0,1)$ for $k \ge k_0$. It follows that

$$\begin{aligned}
&\left|\int_0^1 |u_k|^{p_k}\,dt - \int_0^1 |u|^{p_k}dt\right| \\
&\le \int_0^1 p_k |u+\theta u_k|^{p_k-1} |u_k - u|\,dt \\
&\le (p+\varepsilon)\left(\int_0^1 |u+\theta u_k|^{p_k}\,dt\right)^{(p_k-1)/p_k} \\
&\quad\times \left(\int_0^1 |u_k - u|^{p_k}dt\right)^{1/p_k} \\
&\le (p+\varepsilon)\left(\|u\|_{p_k} + \|u_k\|_{p_k}\right)^{p_k-1}\left(\int_0^1 |u_k-u|^{p_k}dt\right)^{1/p_k} \\
&\longrightarrow 0
\end{aligned} \tag{33}$$

as $k \to +\infty$. It is clear that

$$\int_0^1 |u|^{p_k}dt - \int_0^1 |u|^p dt \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty. \tag{34}$$

Thus,

$$\int_0^1 |u_k|^{p_k}dt \longrightarrow \int_0^1 |u|^p dt. \tag{35}$$

Similarly, we can also obtain that

$$\begin{aligned}
&\int_0^1 m^+(t)|u_k|^{p_k}dt \longrightarrow \int_0^1 m^+(t)|u|^p dt, \\
&\int_0^1 m^-(t)|u_k|^{p_k}dt \longrightarrow \int_0^1 m^-(t)|u|^p dt,
\end{aligned} \tag{36}$$

where $m^+(t) = \max\{m(t),0\}$ and $m^-(t) = -\min\{m(t),0\}$. Therefore,

$$\begin{aligned}
&\int_0^1 m(t)|u_k|^{p_k}dt \\
&= \int_0^1 m^+(t)|u_k|^{p_k}dt - \int_0^1 m^-(t)|u_k|^{p_k}dt \\
&\longrightarrow \int_0^1 m^+(t)|u|^p dt - \int_0^1 m^-(t)|u|^p dt \\
&= \int_0^1 m(t)|u|^p dt.
\end{aligned} \tag{37}$$

We note that (30) and (31) imply that

$$\mu_1(p_k)\int_0^1 m(t)|u_k|^{p_k}\,dt = 1 \tag{38}$$

for all $k \in \mathbb{N}$. Thus, letting k go to $+\infty$ in (38) and using (37), we find that

$$\liminf_{j\to+\infty} \mu_1(p_k)\int_0^1 m(t)|u|^p dt = 1. \tag{39}$$

On the other hand, since $u_k \rightharpoonup u$ in $W_0^{1,p-\varepsilon}(0,1)$, from (32) we obtain that

$$\|u'\|_{p-\varepsilon}^{p-\varepsilon} \le \liminf_{k\to+\infty} \|u_k'\|_{p-\varepsilon}^{p-\varepsilon} \le 1^{\varepsilon/p}. \tag{40}$$

Now, letting $\varepsilon \to 0^+$ and applying Fatou's Lemma, we find that

$$\|u'\|_p^p \le 1. \tag{41}$$

Hence, $u \in W^{1,p}(0,1)$; here $W^{1,p}(0,1)$ denotes the radially symmetric subspace of $W^{1,p}(0,1)$. We claim that actually $u \in W_0^{1,p}(0,1)$. Indeed, we know that $u \in W_0^{1,p-\varepsilon}(0,1)$ for each $0<\varepsilon<\varepsilon_0$. For $\phi \in C_c^\infty(\mathbb{R})$, it is easy to see that

$$\left|\int_0^1 u\phi' dt\right| \le \|u'\|_{p-\varepsilon}\|\phi\|_{(p-\varepsilon)'}, \quad i = 1,\dots,N. \tag{42}$$

Then, letting $\varepsilon \to 0^+$, we obtain that

$$\left|\int_0^1 u\phi' dt\right| \le \|u'\|_p\|\phi\|_{p'}, \quad i = 1,\dots,N, \tag{43}$$

where $p' = p/(p-1)$. Since ϕ is arbitrary, from Proposition IX-18 of [21], we find that $u \in W_0^{1,p}(0,1)$, as desired.

Finally, combining (39) and (41), we obtain that

$$\liminf_{j\to+\infty} \mu_1(p_k)\int_0^1 m(t)|u|^p dt \ge \int_0^1 |u'|^p dt. \tag{44}$$

This and the variational characterization of $\mu_1(p)$ imply (29) and hence (25). This concludes the proof of the lemma. □

Using Remark 3, Lemma 2, and Proposition 5, we will show that all eigenvalue functions $\mu_k^{\pm} : (1,+\infty) \to \mathbb{R}$, $2 \le k \in \mathbb{N}$ are continuous.

Lemma 6. *For fixed $2 \le k \in \mathbb{N}$ and $\nu \in \{+,-\}$, $\mu_k^\nu(p)$ as a function of $p \in (1,+\infty)$ is continuous.*

Proof. Let u_k^ν be an eigenfunction corresponding to $\mu_k^\nu(p)$. By Lemma 2 and Remark 3, we know that u has exactly $k-1$ simple zeros in I; that is, there exist $c_{k,1},\dots,c_{k,k-1} \in I$ such that $u(c_{k,1}) = \cdots = u(c_{k,k-1}) = 0$. For convenience, we set $c_{k,0} = 0$, $c_{k,k} = 1$, and $J_i = (c_{k,i-1}, c_{k,i})$ for $i = 1,\dots,k$. Let $\mu_1^\nu(p, m/J_i, J_i)$ denote the first positive or negative eigenvalue of the restriction of problem (13) on J_i for $i = 1,\dots,k$. Lemma 3 of [18] follows that $\mu_k^\nu(p) = \mu_1^\nu(p, m/J_i, J_i)$ for $i = 1,\dots,k$. Using a similar proof to Proposition 5, we can show that $\mu_1^\nu(p, m/J_i, J_i)$ is continuous with respect to p for $i = 1,\dots,k$. Therefore, $\mu_k^\nu(p)$ is also continuous with respect to p. □

Lemma 7. *(i) Let $\{\mu_k^+(p)\}_{k\in\mathbb{N}}$ be the sequence of positive eigenvalues of* (13). *Let λ be a constant with $\lambda \neq \mu_k^+(p)$ for all $k \in \mathbb{N}$. Then, for arbitrary $r > 0$,*

$$\deg\left(T_\lambda^p, B_r(0), 0\right) = (-1)^\beta, \tag{45}$$

where β is the number of eigenvalues $\mu_n^+(p)$ of problem (13) *less than λ.*

(ii) Let $\{\mu_k^-(p)\}_{k\in\mathbb{N}}$ be the sequence of negative eigenvalues of (13). *Consider $\lambda \neq \mu_k^-(p)$, $k \in \mathbb{N}$; then*

$$\deg\left(T_\lambda^p, B_r(0), 0\right) = (-1)^\beta, \quad \forall r > 0, \tag{46}$$

where β is the number of eigenvalues $\mu_k^-(p)$ of problem (25) *larger than λ.*

Proof. We will only prove the case $\lambda > \mu_1^+(p)$ since the proof for the other cases is similar. We also only give the proof for the case $p > 2$. Proof for the case $1 < p < 2$ is similar. Assume that $\mu_k^+(p) < \lambda < \mu_{k+1}^+(p)$ for some $k \in \mathbb{N}$. Since the eigenvalues depend continuously on p, there exists a continuous function $\chi : [2, p] \to \mathbb{R}$ and $q \in [2, p]$ such that $\mu_k^+(q) < \chi(q) < \mu_{k+1}^+(q)$ and $\lambda = \chi(p)$. Define

$$\Phi(q, u) = u - G_q\left(-\chi(q) m(t) \varphi_q(u)\right). \tag{47}$$

It is easy to show that $\Phi(q, u)$ is a compact perturbation of the identity such that, for all $u \neq 0$, by definition of $\chi(q)$, $\Phi(q, u) \neq 0$, for all $q \in [2, p]$. Hence, the invariance of the degree under homotopology and the classical result for $p = 2$ imply

$$\deg\left(T_\lambda^p, B_r(0), 0\right) = \deg\left(T_\lambda^2, B_r(0), 0\right) = (-1)^k. \tag{48}$$

□

For the existence of bifurcation branches for (12), we will make use of the following global bifurcation theorem results.

Lemma 8 (see [17]). *Let X be a Banach space. Let $F : \mathbb{R}\times X \to X$ be completely continuous such that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Suppose that there exist constants $\rho, \eta \in \mathbb{R}$, with $\rho < \eta$, such that $(\rho, 0)$ and $(\eta, 0)$ are not bifurcation points for the equation*

$$u - F(\lambda, u) = 0. \tag{49}$$

Furthermore, assume that

$$\deg\left(I - F(\rho, \cdot), B_r(0), 0\right) \neq \deg\left(I - F(\eta, \cdot), B_r(0), 0\right), \tag{50}$$

where $B_r(0) = \{u \in X : \|u\| < r\}$ is an isolating neighborhood of the trivial solution for both constants ρ and η. Let

$$\begin{aligned} \mathcal{S} = {} & \overline{\{(\lambda, u) : (\lambda, u) \text{ is a solution of } (49) \text{ with } u \neq 0\}} \\ & \cup \left([\rho, \eta] \times \{0\}\right), \end{aligned} \tag{51}$$

and let $\mathcal{C}$ be the component of $\mathcal{S}$ containing $[\rho, \eta] \times \{0\}$. Then, either

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times X$ or*

(ii) *$\mathcal{C} \cap [(\mathbb{R} \setminus [\rho, \eta]) \times \{0\}] \neq \emptyset$.*

Define the Nemytskii operators $H : \mathbb{R} \times E \to Y$ by

$$H(\lambda, u)(t) := -\lambda m(t) f(u(t)). \tag{52}$$

Then, it is clear that H is continuous operator which sends bounded sets of $\mathbb{R} \times E$ into an equi-integrable sets of Y and problem (12) can be equivalently written as

$$u = G_p \circ H(\lambda, u) := F(\lambda, u). \tag{53}$$

F is completely continuous in $\mathbb{R} \times E \to E$ and $F(\lambda, 0) = 0$, for all $\lambda \in \mathbb{R}$.

Notice that (12) with $\lambda = 0$ has only the trivial solution. Applying this fact and Lemma 8 and the same method to prove [15, Theorem 2.1] with obvious changes, we may obtain the following.

Lemma 9. *Assume that (H_1), (H_2), and (H_3) hold. Then, for fixed $p > 1$ and for fixed $\sigma \in \{+, -\}$, each $(\mu_k^\nu(p)/f_0, 0)$ is a bifurcation point of* (12) *and the associated bifurcation branch $(\mathcal{C}_k^\nu)^\sigma$ satisfies the following;*

(1) *$(\mathcal{C}_k^\nu)^\sigma$ is unbounded in E;*

(2) *$(\mathcal{C}_k^\nu)^\sigma \subset (\mathbb{R} \times \Phi_k^\sigma) \cup \{(\mu_k^\nu(p), 0)\}$, where Φ_k^σ is the set of function $u \in C_0^1[0, 1]$ which has exact $k - 1$ simple zeros in $(0, 1)$, and σu is positive near* 0.

Finally, we give a key lemma that will be used in Section 3. Let

$$\begin{aligned} I^+ &:= \{t \in [0, 1] \mid m(t) > 0\}, \\ I^- &:= \{t \in [0, 1] \mid m(t) < 0\}. \end{aligned} \tag{54}$$

Lemma 10. *Let (H_2) hold. Let $I = [a, b]$ be such that $I \subset I_+$ and*

$$\text{meas } I > 0. \tag{55}$$

Let $g_n : [0, 1] \to (0, +\infty)$ be such that

$$\lim_{n\to+\infty} g_n(t) = +\infty, \quad \text{uniformly on } I. \tag{56}$$

Let $y_n \in E$ be a solution of the equation

$$\varphi_p\left(y_n'\right)' + m(t) g_n(t) \varphi_p\left(y_n\right) = 0, \quad t \in (0, 1). \tag{57}$$

Then, the number of zeros of $y_n|_I$ goes to infinity as $n \to +\infty$.

Proof. After taking a subsequence if necessary, we may assume that

$$m(t) g_{n_j}(t) \geq j, \quad t \in I, \tag{58}$$

as $j \to +\infty$. It is easy to check that the distance between any two consecutive zeros of any nontrivial solution of the equation

$$\varphi_p\left(u'(t)\right)' + j\varphi_p(u(t)) = 0, \quad t \in I, \tag{59}$$

goes to zero as $j \to +\infty$. Using this with [21, Lemma 2.5], it follows the desired results. □

3. Main Results and Its Proof

Let $\mu_k^{\pm}$ be the kth positive or negative eigenvalue of (13). By applying Lemma 9, we will establish the main results as follows.

Theorem 11. *Let* (H_1), (H_2), (H_3), *and* (H_4) *hold. Assume that, for some* $k \in \mathbb{N}$, *either*

$$\gamma \in \left(\frac{\mu_k^+(p)}{f_\infty}, \frac{\mu_k^+}{f_0}\right) \cup \left(\frac{\mu_k^-(p)}{f_0}, \frac{\mu_k^-(p)}{f_\infty}\right) \tag{60}$$

or

$$\gamma \in \left(\frac{\mu_k^+(p)}{f_0}, \frac{\mu_k^+(p)}{f_\infty}\right) \cup \left(\frac{\mu_k^-(p)}{f_\infty}, \frac{\mu_k^-(p)}{f_0}\right). \tag{61}$$

Then, (4) *has two solutions* u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near 0 and* u_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near 0.*

Proof. We only prove the case of $\gamma > 0$. The case of $\gamma < 0$ is similar. Consider the problem

$$\varphi_p\left(u'\right)' + \lambda\gamma m(t) f(u) = 0, \quad t \in (0,1), \qquad u(0) = 0, \quad u(1) = 0. \tag{62}$$

Considering the results of Lemma 9, we have that, for each integer $k \geq 1$, $\sigma \in \{+,-\}$, there exists a continuum $(C_k^+)^\sigma \subseteq \Phi_k^\sigma$ of solutions of (62) joining $(\mu_k^+(p)/\gamma f_0, 0)$ to infinity in $(0,\infty)\times\Phi_k^\sigma$. Moreover, $(C_k^+)^\sigma \setminus \{(\mu_k^+(p)/\gamma f_0, 0)\} \subset (0,\infty)\times\Phi_k^\sigma$.

It is clear that any solution of (62) of the form $(1,u)$ yields a solution u of (4). We will show that $(C_k^+)^\sigma$ crosses the hyperplane $\{1\}\times E$ in $\mathbb{R}\times E$. To this end, it will be enough to show that $(C_k^+)^\sigma$ joins $(\mu_k^+(p)/\gamma f_0, 0)$ to $(\mu_k^+(p)/\gamma f_\infty, +\infty)$. Let $(\eta_n, y_n) \in (C_k^+)^\sigma$ satisfy

$$\mu_n + \|y_n\| \longrightarrow +\infty. \tag{63}$$

We note that $\eta_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (62) for $\lambda = 0$ and $(C_k^+)^\sigma \cap (\{0\}\times E) = \emptyset$.

Case 1. $\mu_k^+(p)/f_\infty < \gamma < \mu_k^+(p)/f_0$. In this case, we only need to show that

$$\left(\frac{\mu_k^+(p)}{\gamma f_\infty}, \frac{\mu_k^+(p)}{\gamma f_0}\right) \subseteq \left\{\mu \in \mathbb{R} : (\mu, u) \in \left(C_k^+\right)^\sigma\right\}. \tag{64}$$

We divide the proof into two steps.

Step 1. We show that, if there exists a constant number $M > 0$ such that

$$\eta_n \subset (0, M] \tag{65}$$

for $n \in \mathbb{N}$ large enough, then $(C_k^+)^\sigma$ joins $(\mu_k^+(p)/\gamma f_0, 0)$ to $(\mu_k^+(p)/\gamma f_\infty, +\infty)$.

In this case, it follows that

$$\|y_n\| \longrightarrow +\infty. \tag{66}$$

Let $\xi \in C(\mathbb{R})$ be such that

$$f(u) = f_\infty \varphi_p(u) + \xi(u). \tag{67}$$

Then,

$$\lim_{|u|\to+\infty} \frac{\xi(u)}{\varphi_p(u)} = 0. \tag{68}$$

Let

$$\tilde{\xi}(u) = \max_{0\leq|s|\leq u} |\xi(s)|. \tag{69}$$

Then, $\tilde{\xi}$ is nondecreasing and

$$\lim_{u\to+\infty} \frac{\tilde{\xi}(u)}{|u|^{p-1}} = 0. \tag{70}$$

We divide the equation

$$\varphi_p\left(y_n'\right)' - \mu_n\gamma m(t) f_\infty \varphi_p(y_n) = \mu_n\gamma m(t)\xi(y_n) \tag{71}$$

by $\|y_n\|$ and set $\overline{y}_n = y_n/\|y_n\|$. Since $\overline{y}_n$ is bounded in E, after taking a subsequence if necessary, we have $\overline{y}_n \rightharpoonup \overline{y}$ for some $\overline{y} \in E$ and $\overline{y}_n \to \overline{y}$ in Y with $\|\overline{y}\| = 1$. Moreover, from (70) and the fact that $\tilde{\xi}$ is nondecreasing, we have

$$\lim_{n\to+\infty} \frac{\xi(y_n(t))}{\|y_n\|^{p-1}} = 0, \tag{72}$$

since

$$\frac{\xi(y_n(t))}{\|y_n\|^{p-1}} \leq \frac{\tilde{\xi}(|y_n(t)|)}{\|y_n\|^{p-1}} \leq \frac{\tilde{\xi}(\|y_n(t)\|_\infty)}{\|y_n\|^{p-1}} \leq \frac{\tilde{\xi}(\|y_n(t)\|)}{\|y_n\|^{p-1}}. \tag{73}$$

By the continuity and compactness of G_p, it follows that

$$\overline{y} = G_p\left(\overline{\mu}\gamma m(t) f_\infty \varphi_p(\overline{y})\right), \tag{74}$$

where $\overline{\mu} = \lim_{n\to+\infty}\mu_n$, again choosing a subsequence and relabeling if necessary.

We claim that

$$\overline{y} \in \left(C_k^+\right)^\sigma. \tag{75}$$

Suppose on the contrary that $\overline{y} \in (C_k^+)^\sigma$. Since $\overline{y} \neq 0$ is a solution of (74) and all zeros of $\overline{y}$ in $[0,1]$ are simple, it follows that $\overline{y} \in (C_h^+)^\iota \neq (C_k^+)^\sigma$ for some $h \in \mathbb{N}$ and $\iota \in \{+,-\}$.

By the openness of $E \setminus (C_k^+)^\sigma$, we have that there exists a neighborhood $U(\overline{y}, \rho_0)$ such that

$$U(\overline{y}, \rho_0) \subset E \setminus \left(C_k^+\right)^\sigma, \tag{76}$$

which contradicts the facts that $\overline{y}_n \to \overline{y}$ in E and $\overline{y}_n \in (C_k^+)^\sigma$. Therefore, $\overline{y} \in C_k^\nu$. Moreover, by Lemma 2, $\overline{\mu}\gamma f_\infty = \mu_k^+(p)$, so that

$$\overline{\mu} = \frac{\lambda_k}{\gamma f_\infty}. \tag{77}$$

Therefore, $(C_k^+)^\sigma$ joins $(\mu_k^+(p)/\gamma f_0, 0)$ to $(\mu_k^+(p)/\gamma f_\infty, +\infty)$.

Step 2. We show that there exists a constant M such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that

$$\lim_{n \to +\infty} \mu_n = +\infty. \tag{78}$$

Since $(\eta_n, y_n) \in (C_k^+)^\sigma$, it follows that

$$\varphi\left(y_n'\right)' + \gamma \eta_n m(t) \frac{f(y_n)}{\varphi(y_n)} \varphi(y_n) = 0. \tag{79}$$

Let

$$0 = \tau(0, n) < \tau(1, n) < \cdots < \tau(k, n) = 1 \tag{80}$$

be the zeros of y_n in $[0, 1]$. Then, after taking a subsequence if necessary,

$$\lim_{n \to +\infty} \tau(l, n) := \tau(l, \infty), \qquad l \in \{0, 1, \cdots, k-1\}. \tag{81}$$

Notice that Lemma 10 and the fact that y_n has exactly $k-1$ simple zeros in $[0, 1]$ yield

$$\left[\cup_{l=0}^{k-1} \left(\tau(l, \infty), \tau(l+1, \infty)\right)\right] \cap I^+ = \emptyset, \tag{82}$$

which implies that

$$\text{meas}\left\{\left[\cup_{l=0}^{k-1} \left(\tau(l, \infty), \tau(l+1, \infty)\right)\right] \cap I^-\right\} = 1. \tag{83}$$

However, this contradicts (H_2): $0 < \text{meas}\ I^- < 1$.

Case 2. $\mu_k^+(p)/f_0 < \gamma < \mu_k^+(p)/f_\infty$. In this case, we have that

$$\frac{\mu_k^+(p)}{\gamma f_0} < 1 < \frac{\mu_k^+(p)}{\gamma f_\infty}. \tag{84}$$

Assume that $(\eta_n, y_n) \in (C_k^+)^\sigma$ is such that

$$\lim_{n \to +\infty} \left(\mu_n + \|y_n\|\right) = +\infty. \tag{85}$$

If $\eta_n \to +\infty$, then we are done!

If there exists $M > 0$, such that, for $n \in \mathbb{N}$ sufficiently large,

$$\eta_n \in (0, M]. \tag{86}$$

Applying the same method used in Step 1 of Case 1, after taking a subsequence and relabeling if necessary, it follows that

$$(\eta_n, y_n) \longrightarrow \left(\frac{\mu_k^+(p)}{\gamma f_\infty}, +\infty\right) \quad \text{as } n \longrightarrow +\infty. \tag{87}$$

Thus, $(C_k^+)^\sigma$ joins $(\mu_k^+(p)/\gamma f_0, 0)$ to $(\mu_k^+(p)/\gamma f_\infty, +\infty)$. □

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References

[1] R. Ma and B. Thompson, "Nodal solutions for nonlinear eigenvalue problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 59, no. 5, pp. 707–718, 2004.

[2] P. H. Rabinowitz, "Nonlinear Sturm-Liouville problems for second order ordinary differential equations," *Communications on Pure and Applied Mathematics*, vol. 23, pp. 939–961, 1970.

[3] P. H. Rabinowitz, "Some global results for nonlinear eigenvalue problems," *Journal of Functional Analysis*, vol. 7, pp. 487–513, 1971.

[4] R. Ma and X. Han, "Existence of nodal solutions of a nonlinear eigenvalue problem with indefinite weight function," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 2119–2125, 2009.

[5] Y. Cui, J. Sun, and Y. Zou, "Global bifurcation and multiple results for Sturm-Liouville problems," *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2185–2192, 2011.

[6] R. Ma and D. O'Regan, "Nodal solutions for second-order m-point boundary value problems with nonlinearities across several eigenvalues," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 7, pp. 1562–1577, 2006.

[7] B. P. Rynne, "Global bifurcation for $2m$th-order boundary value problems and infinitely many solutions of superlinear problems," *Journal of Differential Equations*, vol. 188, no. 2, pp. 461–472, 2003.

[8] R. Ma, "Bifurcation from infinity and multiple solutions for periodic boundary value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 42, no. 1, pp. 27–39, 2000.

[9] Y. Liu and D. O'Regan, "Multiplicity results using bifurcation techniques for a class of boundary value problems of impulsive differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 4, pp. 1769–1775, 2011.

[10] J. Henderson and H. Wang, "Positive solutions for nonlinear eigenvalue problems," *Journal of Mathematical Analysis and Applications*, vol. 208, no. 1, pp. 252–259, 1997.

[11] W. H. Fleming, "A selection-migration model in population genetics," *Journal of Mathematical Biology*, vol. 2, no. 3, pp. 219–233, 1975.

[12] M. del Pino, M. Elgueta, and R. Manásevich, "A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t, u) = 0, u(0) = u(T) = 0, p > 1$," *Journal of Differential Equations*, vol. 80, no. 1, pp. 1–13, 1989.

[13] I. Peral, "Multiplicity of solutions for the p-Laplacian," ICTP SMR 990/1, 1997.

[14] M. A. del Pino and R. F. Manásevich, "Global bifurcation from the eigenvalues of the p-Laplacian," *Journal of Differential Equations*, vol. 92, no. 2, pp. 226–251, 1991.

[15] Y.-H. Lee and I. Sim, "Global bifurcation phenomena for singular one-dimensional p-Laplacian," *Journal of Differential Equations*, vol. 229, no. 1, pp. 229–256, 2006.

[16] P. Drábek and Y. X. Huang, "Bifurcation problems for the p-Laplacian in $\mathbb{R}^N$," *Transactions of the American Mathematical Society*, vol. 349, no. 1, pp. 171–188, 1997.

[17] K. Schmitt and R. Thompson, *Nonlinear Analysis and Differential Equations: An Introduction*, University of Utah Lecture Notes, University of Utah Press, Salt Lake City, Utah, USA, 2004.

[18] A. Anane, O. Chakrone, and M. Moussa, "Spectrum of one dimensional p-Laplacian operator with indefinite weight," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 17, pp. 1–11, 2002.

[19] G. Meng, P. Yan, and M. Zhang, "Spectrum of one-dimensional p-Laplacian with an indefinite integrable weight," *Mediterranean Journal of Mathematics*, vol. 7, no. 2, pp. 225–248, 2010.

[20] K. Deimling, *Nonlinear Functional Analysis*, Springer, New-York, NY, USA, 1987.

[21] H. Brezis, *Analyse Fonctionnelle: Theéorie et Applications*, Masson, Paris, France, 1983.