Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 402793, 6 pages
http://dx.doi.org/10.1155/2013/402793

Research Article

Notes on the Global Well-Posedness for the Maxwell-Navier-Stokes System

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Received 28 March 2013; Accepted 26 June 2013

Academic Editor: Changxing Miao

Masmoudi (2010) obtained global well-posedness for 2D Maxwell-Navier-Stokes system. In this paper, we reprove global existence of regular solutions to the 2D system by using energy estimates and Brezis-Gallouet inequality. Also we obtain a blow-up criterion for solutions to 3D Maxwell-Navier-Stokes system.

1. Introduction

In this paper, we consider Maxwell-Navier-Stokes equations in $\mathbb{R}^d$ $(d = 2, 3)$ as follows:

$$\begin{gathered}
\frac{\partial v}{\partial t} + (v\cdot\nabla)v - \Delta v + \nabla p = j\times B \quad \text{in } \mathbb{R}^d\times(0,T),\\
\frac{\partial E}{\partial t} - \nabla\times B = -j \quad \text{in } \mathbb{R}^d\times(0,T),\\
\frac{\partial B}{\partial t} + \nabla\times E = 0 \quad \text{in } \mathbb{R}^d\times(0,T),\\
\nabla\cdot v = \nabla\cdot B = 0 \quad \text{in } \mathbb{R}^d\times(0,T),\\
j = E + v\times B,
\end{gathered} \tag{1}$$

subject to the initial data

$$\begin{gathered}
v(x,0) = v_0(x), \qquad E(x,0) = E_0(x),\\
B(x,0) = B_0(x).
\end{gathered} \tag{2}$$

Here v, E, and $B : \mathbb{R}^d\times(0,T) \to \mathbb{R}^3$ are vector fields defined on $\mathbb{R}^d$ $(d = 2$ or $3)$. Vector fields v, E, and B denote fluid velocity, electric fields and magnetic fields, respectively. p denotes the scalar pressure and j is the electric current given by Ohm's law. $j\times B$ represents the Lorentz force. Here we put the viscosity and the electric resistivity to be 1 for the simplification. Note that in 2D case, vector fields v, E, and B can be understood as $v(x,t) = (v_1(x_1,x_2,t), v_2(x_1,x_2,t), 0)$, and so forth.

For the compatibility of the initial data, we assume that

$$\nabla\cdot v_0 = \nabla\cdot B_0 = 0. \tag{3}$$

Since the divergence-free condition of the magnetic field is conserved, $\nabla\cdot B = 0$ in (1) is not necessary in general if we assume the divergence-free condition for the initial data of the magnetic field in $\mathbb{R}^d$. In many physical situations, current displacement term $\partial_t E$ is neglected because the physical coefficient for this term is very small ($\sim 1/c^2$, where c denotes the speed of light). But mathematically, the presence of the term $\partial_t E$ in the second equation (Ampere-Maxwell equation) preserves the hyperbolic nature of the Maxwell equation in the Maxwell-Navier-Stokes equations (see [1, 2] and references therein). Also we remark that full Maxwell-Navier-Stokes equations have been used for the accurate computation of electromagnetic hypersonics in aerothermodynamics (see [3, 4] and references therein). For further physical motivations, see [5].

Neglecting the current displacement term, Maxwell-Navier-Stokes system is reduced to the usual MHD system. There have been many extensive mathematical studies for the existence, blow-up criterion, and regularity criterion of MHD and related models (see [6–12] and references therein). Recently, Maxwell-Navier-Stokes system has been receiving much mathematical attention after pioneering work of Masmoudi [2]. In [2], global existence of regular solutions to (1) in $\mathbb{R}^2$ is proved by using the Besov-type $\widetilde{L}$ space technique developed by Chemin and Lerner [13]. In [1, 14], the local existence of mild solution and the global existence of (1) with small data have been studied. Duan [15] studied large time behaviour of solutions to (1). In [16], Ibrahim and Yoneda obtained local-in-time existence for nondecaying initial data in torus. Also Germain and Masmoudi [17] studied global existence of solutions to Euler-Maxwell equations with small data and Jang and Masmoudi [18] mathematically derived Ohm's law from the kinetic equation.

The aim of this paper is to study the global well-posedness for (1) using the standard energy estimates. We obtain the local-in-time existence of H^2 solution by using the standard mollifier technique (see Proposition 4) and re-prove the global existence of H^2 solution for 2D Maxwell-Navier-Stokes system (see Theorem 1) by using standard energy estimates and Brezis-Gallouet inequality, which was used to prove global existence of regular solution for the partial viscous Boussinesq equations by Chae [19]. Also we provide blow-up criterion of regular solutions to 3D Maxwell-Navier-Stokes equations (see Theorem 2).

We state our main results in the following.

Theorem 1. *Assume that* $(v_0, E_0, B_0) \in H^2(\mathbb{R}^2)$ *and* $\nabla \cdot v_0 = \nabla \cdot B_0 = 0$. *Then, for any* $T > 0$, *there exists a solution to 2D Maxwell-Navier-Stokes system* (1) *such that* $(v, E, B) \in C((0, T]; H^2)$ *and* $(\nabla v, j) \in L^2(0, T; H^2)$.

Theorem 2. *Suppose that* $(v_0, E_0, B_0) \in H^2(\mathbb{R}^3)$ *and* $\nabla \cdot v_0 = \nabla \cdot B_0 = 0$. *If* T^*, *the maximal existence time of the local existence of regular solution to 3D Maxwell-Navier-Stokes system* (1), *is finite, then*

$$\int_0^{T^*} \|v(t)\|_{L^\infty}^2 + \|B(t)\|_{L^\infty}^{8/3} dt = \infty. \tag{4}$$

Remark 3. (1) As logarithmic inequality has been used in [2], Brezis-Gallouet inequality gives logarithmic-type estimates. But it provides double exponential bound compared with exponential bound in [2].

(2) The presence of the current displacement term $\partial_t E$ makes Maxwell-Navier-Stokes system do not enjoy the scaling invariance property of the usual Navier-Stokes system, $v_\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t)$. In Theorem 2, $\int_0^T \|v(t)\|_{L^\infty}^2 dt$ is concurrent with the usual scaling invariant norm of solutions to 3D Navier-Stokes equations.

The rest of this paper is organized as follows. In Section 2, we provide the local-in-time existence of regular solution to 2D and 3D Maxwell-Navier-Stokes systems and global existence of 2D Maxwell-Navier-Stokes system with large data. In Section 3, we provide the blow-up criterion for H^2 solution to 3D Maxwell-Navier-Stokes system.

2. Local Existence and Global Well-Posedness

At first, we note that one can have the energy identity in two or three dimensions:

$$\frac{1}{2}\frac{d}{dt}\left(\|v\|_{L^2}^2 + \|B\|_{L^2}^2 + \|E\|_{L^2}^2\right) + \|j\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = 0. \tag{5}$$

The previously energy inequality can be justified for local in time regular solution in the following proposition. In the following, C denotes a harmless constant which may change from one line to the other. We prove local-in-time existence of H^2 solution using the standard energy estimates.

Proposition 4. *Let* $(u_0, E_0, B_0) \in H^2(\mathbb{R}^d)$ $(d = 2$ *or* $3)$ *with* $\nabla \cdot u_0 = \nabla \cdot B_0 = 0$. *Then there exists* $T = T(\|u_0\|_{H^2}, \|E_0\|_{H^2}, \|B_0\|_{H^2})$ *such that there exists a unique solution* $(u, E, B) \in L^\infty(0, T; H^2(\mathbb{R}^d)) \cap Lip(0, T; L^2)$.

Proof. We use the mollifier method as described in [20]. Although the details are similar to [20], we provide some a priori estimates for the reader's sake. We consider the standard mollifier operator

$$\mathcal{J}_\epsilon f = \rho_\epsilon * f, \qquad \rho_\epsilon(\cdot) = \frac{1}{\epsilon^d}\rho\left(\frac{\cdot}{\epsilon^d}\right), \tag{6}$$

where $\rho \in C_0^\infty(\mathbb{R}^d)$, and $\rho \geq 0$, $\int_{\mathbb{R}^d} \rho dx = 1$.

We introduce the following regularized system of (1):

$$\begin{aligned}
&\partial_t v^\epsilon + \mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon v^\epsilon \cdot \nabla\right)\mathcal{J}_\epsilon v^\epsilon - \Delta \mathcal{J}_\epsilon^2 v^\epsilon + \nabla p^\epsilon \\
&\quad = \mathcal{J}_\epsilon\left(\mathcal{J}_\epsilon^2 j^\epsilon \times \mathcal{J}_\epsilon B^\epsilon\right) \quad \text{in } \mathbb{R}^d \times (0, T), \\
&\partial_t E^\epsilon - \nabla \times \mathcal{J}_\epsilon^2 B^\epsilon = -\mathcal{J}_\epsilon^2 j^\epsilon \quad \text{in } \mathbb{R}^d \times (0, T), \\
&\partial_t B^\epsilon + \nabla \times \mathcal{J}_\epsilon^2 E^\epsilon = 0 \quad \text{in } \mathbb{R}^d \times (0, T), \\
&\nabla \cdot v^\epsilon = \nabla \cdot B^\epsilon = 0 \quad \text{in } \mathbb{R}^d \times (0, T), \\
&j^\epsilon = E^\epsilon + \mathcal{J}_\epsilon v^\epsilon \times \mathcal{J}_\epsilon B^\epsilon,
\end{aligned} \tag{7}$$

with initial data $(v_0^\epsilon, E_0^\epsilon, B_0^\epsilon) = (\mathcal{J}_\epsilon v_0, \mathcal{J}_\epsilon E_0, \mathcal{J}_\epsilon B_0)$.

Taking the L^2 inner product of $(7)_1$, $(7)_2$, and $(7)_3$ with v^ϵ, E^ϵ, B^ϵ, respectively, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v^\epsilon\|_{L^2}^2 + \|E^\epsilon\|_{L^2}^2 + \|B^\epsilon\|_{L^2}^2\right) \\
&\quad + \|\nabla \mathcal{J}_\epsilon v^\epsilon\|_{L^2}^2 + \|\mathcal{J}_\epsilon j^\epsilon\|_{L^2}^2 \\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^d} \left(\mathcal{J}_\epsilon v^\epsilon\right) \cdot \nabla\left(\mathcal{J}_\epsilon v^\epsilon\right)^2 dx \\
&\quad + \int_{\mathbb{R}^d} \left(\nabla \times \mathcal{J}_\epsilon B^\epsilon\right) \cdot \mathcal{J}_\epsilon E^\epsilon dx \\
&\quad - \int_{\mathbb{R}^d} \left(\nabla \times \mathcal{J}_\epsilon E^\epsilon\right) \cdot \mathcal{J}_\epsilon B^\epsilon dx \\
&\quad + \int_{\mathbb{R}^d} \left(\mathcal{J}_\epsilon^2 j^\epsilon \times \mathcal{J}_\epsilon B^\epsilon\right) \cdot \mathcal{J}_\epsilon v^\epsilon dx \\
&\quad + \int_{\mathbb{R}^d} \mathcal{J}_\epsilon^2 j^\epsilon \cdot \left(\mathcal{J}_\epsilon v^\epsilon \times \mathcal{J}_\epsilon B^\epsilon\right) dx = 0.
\end{aligned} \tag{8}$$

We compute the derivative D^α, α is a multi-index such that $|\alpha| \le 2$, of (7), multiply them by $D^\alpha v^\epsilon$, $D^\alpha E^\epsilon$, and $D^\alpha B^\epsilon$, respectively, and integrate them over $\mathbb{R}^d$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v^\epsilon\|_{H^2}^2 + \|E^\epsilon\|_{H^2}^2 + \|B^\epsilon\|_{H^2}^2\right) \\
&\quad + \|\nabla \mathcal{J}_\epsilon v^\epsilon\|_{H^2}^2 + \|\mathcal{J}_\epsilon j^\epsilon\|_{H^2}^2 \\
&\le C\|\mathcal{J}_\epsilon v^\epsilon \otimes \mathcal{J}_\epsilon v^\epsilon\|_{H^2}\|\nabla \mathcal{J}_\epsilon v^\epsilon\|_{H^2} \\
&\quad + C\left\|\mathcal{J}_\epsilon^2 j^\epsilon \times \mathcal{J}_\epsilon B^\epsilon\right\|_{H^2}\|\mathcal{J}_\epsilon v^\epsilon\|_{H^2} \\
&\quad + C\left\|\mathcal{J}_\epsilon^2 j^\epsilon\right\|_{H^2}\|\mathcal{J}_\epsilon v^\epsilon \times \mathcal{J}_\epsilon B^\epsilon\|_{H^2} \\
&\le C\left(\|\mathcal{J}_\epsilon v^\epsilon\|_{H^2}^4 + \|\mathcal{J}_\epsilon B^\epsilon\|_{H^2}^4\right) \\
&\quad + \frac{1}{2}\left(\|\mathcal{J}_\epsilon \nabla v^\epsilon\|_{H^2}^2 + \|\mathcal{J}_\epsilon j^\epsilon\|_{H^2}^2\right).
\end{aligned} \tag{9}$$

In the previously mentioned, $\mathcal{J}_\epsilon v^\epsilon \otimes \mathcal{J}_\epsilon v^\epsilon$ denotes a tensor $(\mathcal{J}_\epsilon v_i^\epsilon \mathcal{J}_\epsilon v_j^\epsilon)_{1\le j\le d}$.

Using Picard's theorem, these estimates imply local existence of solution. □

The main ingredient of the proof of Theorem 1 is the following Brezis-Gallouet inequality (logarithmic Sobolev inequality):

$$\begin{aligned}
\|f\|_{L^\infty} &\le C\left(1 + \|f\|_{L^2} + \|\nabla f\|_{L^2}\left(\log^+\|\Delta f\|_{L^2}\right)^{1/2}\right), \\
&\qquad f \in H^2\left(\mathbb{R}^2\right).
\end{aligned} \tag{10}$$

Here $\log^+ a$ denotes $\log(e + a)$.

Proof of Theorem 1. We provide a priori estimates on the regular solutions. Let T be a finite maximal time of existence in Proposition 4. By obtaining H^2 bound on $(0, T]$ of solution, we can continue solution beyond T by using Proposition 4.

Taking curl operator on $(1)_1$ and $\partial_i = \partial/\partial x_i$ $(i = 1, 2)$ operator on $(1)_2$ and $(1)_3$, we have

$$\begin{aligned}
&\frac{\partial \omega}{\partial t} + (v\cdot\nabla)\,\omega - \Delta\omega = \nabla \times (j \times B), \quad \text{in } \mathbb{R}^2 \times (0, T), \\
&\frac{\partial(\partial_i E)}{\partial t} - \nabla \times \partial_i B = -\partial_i j, \quad \text{in } \mathbb{R}^2 \times (0, T), \\
&\frac{\partial(\partial_i B)}{\partial t} + \nabla \times \partial_i E = 0, \quad \text{in } \mathbb{R}^2 \times (0, T).
\end{aligned} \tag{11}$$

(i) H^1 Estimates. Taking scalar product (11) with ω, $\partial_i E$, and $\partial_i B$, respectively, and summing over $i = 1, 2$, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 = \int_{\mathbb{R}^2} \nabla \times (j \times B)\cdot\omega\,dx, \tag{12}$$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla E\|_{L^2}^2 &= \sum_i \int_{\mathbb{R}^2} \nabla \times \partial_i B \cdot \partial_i E\,dx - \int_{\mathbb{R}^2} \nabla j \cdot \nabla E\,dx, \\
\frac{1}{2}\frac{d}{dt}\|\nabla B\|_{L^2}^2 &= -\sum_i \int_{\mathbb{R}^2} \nabla \times \partial_i E \cdot \partial_i B\,dx.
\end{aligned} \tag{13}$$

Using the identity

$$\int_{\mathbb{R}^2} \nabla \times \partial_i B \cdot \partial_i E\,dx = \int_{\mathbb{R}^2} \nabla \times \partial_i E \cdot \partial_i B\,dx \tag{14}$$

and $E = j - v \times B$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla E\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right) + \|\nabla j\|_{L^2}^2 \\
&\quad = \int_{\mathbb{R}^2} \nabla j \cdot \nabla(v \times B)\,dx.
\end{aligned} \tag{15}$$

In the following, ϵ denotes a sufficiently small positive number. Since it holds that $\nabla \times (j \times B) = (B \cdot \nabla) j$, we estimate the right-hand side of (12) using Young's inequality and interpolation inequality:

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \nabla \times (j \times B)\cdot\omega\,dx\right| \\
&\quad \le \|B\|_{L^4}\|\nabla j\|_{L^2}\|\omega\|_{L^4} \\
&\quad \le C\|B\|_{L^2}^{1/2}\|\nabla B\|_{L^2}^{1/2}\|\omega\|_{L^2}^{1/2}\|\nabla\omega\|_{L^2}^{1/2}\|\nabla j\|_{L^2} \\
&\quad \le C\|\omega\|_{L^2}^2\|\nabla B\|_{L^2}^2 + \epsilon\|\nabla\omega\|_{L^2}^2 + \epsilon\|\nabla j\|_{L^2}^2,
\end{aligned} \tag{16}$$

where ϵ is a small positive number. Also we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} \nabla j \cdot \nabla(v \times B)\,dx\right| \\
&\quad \le \int_{\mathbb{R}^2} |\nabla j|\,|v|\,|\nabla B|\,dx \\
&\quad + \int_{\mathbb{R}^2} |\nabla j|\,|B|\,|\nabla v|\,dx = I + II.
\end{aligned} \tag{17}$$

We estimate

$$\begin{aligned}
I &\le C\|v\|_{L^\infty}^2\|\nabla B\|_{L^2}^2 + \epsilon\|\nabla j\|_{L^2}^2, \\
II &\le \|B\|_{L^4}\|\nabla v\|_{L^4}\|\nabla j\|_{L^2} \le C\|B\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\nabla B\|_{L^2}^2 \\
&\quad + \epsilon\|\Delta v\|_{L^2}^2 + \epsilon\|\nabla j\|_{L^2}^2.
\end{aligned} \tag{18}$$

Collecting previous estimates, we have

$$\begin{aligned}
&\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|\nabla E\|_{L^2}^2 + \|\nabla B\|_{L^2}^2\right) \\
&\quad + \|\nabla\omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 \le C\|\omega\|_{L^2}^2\|\nabla B\|_{L^2}^2 \\
&\quad + C\|v\|_{L^\infty}^2\|\nabla B\|_{L^2}^2 + C\|\nabla B\|_{L^2}^2\|\nabla v\|_{L^2}^2\|\nabla B\|_{L^2}^2.
\end{aligned} \tag{19}$$

(ii) H^2 Estimates. Taking Δ operator on $(1)_1$, $(1)_2$, and $(1)_3$ and L^2 scalar product with Δv, ΔE, and ΔB, respectively, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Delta v\|_{L^2}^2 + \|\nabla\Delta v\|_{L^2}^2 &\le C\int_{\mathbb{R}^2}|\nabla v|\left|D^2 v\right|dx \\ &\quad + \int_{\mathbb{R}^2}\left|\Delta\left(j\times B\right)\right||\Delta v|\,dx := I_1 + I_2, \end{aligned} \tag{20}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right) + \left\|\Delta j\right\|_{L^2}^2 \le \int_{\mathbb{R}^2}\left|\Delta j\right||\Delta(v\times B)|\,dx := I_3.$$

We estimate I_1, I_2, and I_3 using interpolation inequality, Young's inequality, and Hölder's inequality:

$$\begin{aligned} I_1 &\le C\|\nabla v\|_{L^4}\|\Delta v\|_{L^4}\|\Delta v\|_{L^2} \\ &\le C\|\nabla v\|_{L^2}^{1/2}\|\Delta v\|_{L^2}^{3/2}\|\nabla\Delta v\|_{L^2}^{1/2} \\ &\le C\|\nabla v\|_{L^2}^{2/3}\|\Delta v\|_{L^2}^2 + \epsilon\|\nabla\Delta v\|_{L^2}^2, \end{aligned} \tag{21}$$

$$\begin{aligned} I_2 &\le C\int_{\mathbb{R}^2}\left|\nabla j\right||\nabla B||\Delta v|\,dx \\ &\quad + C\int_{\mathbb{R}^2}\left|\Delta j\right||B||\Delta v|\,dx \\ &\quad + C\int_{\mathbb{R}^2}\left|j\right||\Delta B||\Delta v|\,dx := I_{21} + I_{22} + I_{23}. \end{aligned} \tag{22}$$

Each term can be estimated by the standard interpolation inequality and Young's inequality as follows:

$$\begin{aligned} I_{21} &\le C\left\|\nabla j\right\|_{L^4}\|\nabla B\|_{L^4}\|\Delta v\|_{L^2} \\ &\le C\left\|j\right\|_{L^2}^{1/4}\left\|\Delta j\right\|_{L^2}^{3/4}\|B\|_{L^2}^{1/4}\|\Delta B\|_{L^2}^{3/4}\|\nabla v\|_{L^2}^{1/2}\|\nabla\Delta v\|_{L^2}^{1/2} \\ &\le C\left\|j\right\|_{L^2}^{2/3}\|B\|_{L^2}^{2/3}\|\nabla v\|_{L^2}^{4/3}\|\Delta B\|_{L^2}^2 + \epsilon\left\|\Delta j\right\|_{L^2}^2 \\ &\le C\left(\left\|j\right\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right)\|\Delta B\|_{L^2}^2 + \epsilon\left\|\Delta j\right\|_{L^2}^2, \\ I_{22} &\le C\left\|\Delta j\right\|_{L^2}\|B\|_{L^4}\|\Delta v\|_{L^4} \\ &\le C\left\|\Delta j\right\|_{L^2}\|B\|_{L^2}^{3/4}\|\Delta B\|_{L^2}^{1/4}\|\Delta v\|_{L^2}^{1/2}\|\nabla\Delta v\|_{L^2}^{1/2} \\ &\le \epsilon\left\|\Delta j\right\|_{L^2}^2 + C\|\Delta B\|_{L^2}^{1/2}\|\Delta v\|_{L^2}^{1/2}\|\nabla\Delta v\|_{L^2} \\ &\le \epsilon\left\|\Delta j\right\|_{L^2}^2 + \epsilon\|\nabla\Delta v\|_{L^2}^2 + C\left\|j\right\|_{L^2}\|\nabla v\|_{L^2}\|\Delta B\|_{L^2}^2, \\ I_{23} &\le C\left\|j\right\|_{L^\infty}\|\Delta B\|_{L^2}\|\Delta v\|_{L^2} \\ &\le C\left\|j\right\|_{L^2}\left\|\Delta j\right\|_{L^2}^{1/2}\|\Delta B\|_{L^2}\|\nabla v\|_{L^2}^{1/2}\|\nabla\Delta v\|_{L^2}^{1/2} \\ &\le \epsilon\left\|\Delta j\right\|_{L^2}^2 + C\left\|j\right\|_{L^2}^{2/3}\|\Delta B\|_{L^2}^{4/3}\|\nabla v\|_{L^2}^{2/3}\|\nabla\Delta v\|_{L^2}^{2/3} \\ &\le \epsilon\left\|\Delta j\right\|_{L^2}^2 + \epsilon\|\nabla\Delta v\|_{L^2}^2 + C\left\|j\right\|_{L^2}\|\nabla v\|_{L^2}\|\Delta B\|_{L^2}^2. \end{aligned} \tag{23}$$

I_3 can be written as

$$\begin{aligned} I_3 &\le C\int_{\mathbb{R}^2}\left|\Delta j\right||\nabla v||\nabla B|\,dx \\ &\quad + C\int_{\mathbb{R}^2}\left|\Delta j\right||\Delta v||B|\,dx \\ &\quad + C\int_{\mathbb{R}^2}\left|\Delta j\right||v||\Delta B|\,dx := I_{31} + I_{32} + I_{33}, \end{aligned} \tag{24}$$

$$\begin{aligned} I_{31} &\le C\left\|\Delta j\right\|_{L^2}\|\nabla v\|_{L^4}\|\nabla B\|_{L^4} \\ &\le C\left\|\Delta j\right\|_{L^2}\|\nabla v\|_{L^2}^{3/4}\|\nabla\Delta v\|_{L^2}^{1/4}\|B\|_{L^2}^{1/4}\|\Delta B\|_{L^2}^{3/4} \\ &\le \epsilon\left\|\Delta j\right\|_{L^2}^2 + \epsilon\|\nabla\Delta v\|_{L^2}^2 + C\|\nabla v\|_{L^2}^2\|\Delta B\|_{L^2}^2. \end{aligned}$$

The same as the estimate of I_{22}, we obtain

$$\begin{aligned} I_{32} &\le \epsilon\left\|\Delta j\right\|_{L^2}^2 + \epsilon\|\nabla\Delta v\|_{L^2}^2 \\ &\quad + C\left\|j\right\|_{L^2}\|\nabla v\|_{L^2}\|\Delta B\|_{L^2}^2. \end{aligned} \tag{25}$$

Also we have

$$I_{33} \le C\left\|\Delta j\right\|_{L^2}\|v\|_{L^\infty}\|\Delta B\|_{L^2} \le \epsilon\left\|\Delta j\right\|_{L^2}^2 + C\|v\|_{L^\infty}^2\|\Delta B\|_{L^2}^2. \tag{26}$$

Therefore, we have

$$\begin{aligned} &\frac{d}{dt}\left(\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right) \\ &\quad + \|\nabla\Delta v\|_{L^2}^2 + \left\|\Delta j\right\|_{L^2}^2 \\ &\quad \le C\left(1 + \|\nabla v\|_{L^2}^2 + \left\|j\right\|_{L^2}^2 + \|v\|_{L^\infty}^2\right) \\ &\quad \times\left(\|\Delta v\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right). \end{aligned} \tag{27}$$

(iii) Use of Brezis-Gallouet Inequality. Using Brezis-Gallouet inequality, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right) \\ &\quad + \|\nabla\Delta v\|_{L^2}^2 + \left\|\Delta j\right\|_{L^2}^2 \\ &\le C\left(1 + \|\nabla v\|_{L^2}^2 + \left\|j\right\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2\right) \\ &\quad \times\left(\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right) \\ &\quad \times\log^+\left(\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right). \end{aligned} \tag{28}$$

Let $y(t) = \|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2$, and let $z(t) = 1 + \|\Delta v\|_{L^2}^2 + \|j\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2$. Hence one has

$$\frac{d}{dt}y(t) \le Cz(t)\,y(t)\log^+ y(t). \tag{29}$$

Since

$$\int_0^T z(t)\,dt \le C(1+T), \tag{30}$$

the bound of $y(t)$ is immediate as follows:

$$\begin{aligned}\sup_{0\le t\le T}&\left(\|\Delta v\|_{L^2}^2+\|\Delta E\|_{L^2}^2+\|\Delta B\|_{L^2}^2\right)\\ &\le\left(\|\Delta v_0\|_{L^2}^2+\|\Delta E_0\|_{L^2}^2+\|\Delta B_0\|_{L^2}^2\right)\\ &\quad\times\exp\left(\exp\left(C\left(T+1\right)\right)\right).\end{aligned}\tag{31}$$

This completes the proof of Theorem 1. □

3. Blow-Up Criterion for 3D Maxwell-Navier-Stokes System

In this section, we provide a blow-up criterion for H^2 solution in Proposition 4 to 3D Maxwell-Navier-Stokes system.

Proof of Theorem 2. Assume that

$$\int_0^{T^*}\|v(t)\|_{L^\infty}^2+\|B(t)\|_{L^\infty}^{8/3}dt<\infty,\tag{32}$$

where T^* is the finite maximal existence time of a classical solution.

Similar to the computation in Section 2, one has H^1 estimates of E and B as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|\nabla E\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+\left\|\nabla j\right\|_{L^2}^2\\ &=\int_{\mathbb{R}^3}\nabla j\cdot\nabla\left(v\times B\right)dx\\ &\le C\|\nabla(v\times B)\|_{L^2}^2+\epsilon\left\|\nabla j\right\|_{L^2}^2\\ &\le C\|B\|_{L^\infty}^2\|\nabla v\|_{L^2}^2+C\|v\|_{L^\infty}^2\|\nabla B\|_{L^2}^2+\epsilon\left\|\nabla j\right\|_{L^2}^2.\end{aligned}\tag{33}$$

H^1 estimates of v are as follows:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2+\|\Delta v\|_{L^2}^2 &\le\int_{\mathbb{R}^3}|v|\,|\nabla v|\,|\Delta v|\,dx\\ &\quad+\int_{\mathbb{R}^3}\left|j\times B\right|\,|\Delta v|\,dx\\ &\le C\|v\|_{L^\infty}^2\|\nabla v\|_{L^2}^2\\ &\quad+C\left\|j\times B\right\|_{L^2}^2+\epsilon\|\Delta v\|_{L^2}^2.\end{aligned}\tag{34}$$

The estimate of $\|j\times B\|_{L^2}^2$ is provided in the following:

$$\begin{aligned}\left\|j\times B\right\|_{L^2}^2&\le C\|E\times B\|_{L^2}^2+C\|(v\times B)\times B\|_{L^2}^2\\ &\le C\|E\|_{L^2}^2\|B\|_{L^\infty}^2+C\|v\|_{L^6}^2\|B\|_{L^6}^4\\ &\le C\|E\|_{L^2}^2\|B\|_{L^\infty}^2+C\|\nabla v\|_{L^2}^2\|B\|_{L^2}^{4/3}\|B\|_{L^\infty}^{8/3}.\end{aligned}\tag{35}$$

Thus we have

$$\begin{aligned}\frac{d}{dt}&\left(\|\nabla v\|_{L^2}^2+\|\nabla E\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)\\ &\quad+\|\Delta v\|_{L^2}^2+\left\|\nabla j\right\|_{L^2}^2\\ &\le C\left(1+\|v\|_{L^\infty}^2+\|B\|_{L^\infty}^{8/3}\right)\\ &\quad\times\left(\|\nabla v\|_{L^2}^2+\|\nabla E\|_{L^2}^2+\|\nabla B\|_{L^2}^2\right)+C\|B\|_{L^\infty}^2.\end{aligned}\tag{36}$$

Gronwall's inequality gives us that

$$\|(\nabla v,\nabla E,\nabla B)\|_{L^\infty(0,T^*;L^2)}^2+\left\|(\Delta v,\nabla j)\right\|_{L^2(0,T^*;L^2)}^2\le C<\infty.\tag{37}$$

Next, we consider H^2 estimates.

Integrating by parts and using Young's inequality, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|\Delta E\|_{L^2}^2+\|\Delta B\|_{L^2}^2\right)\\ &\quad+\left\|\Delta j\right\|_{L^2}^2\le C\|\Delta\left(v\times B\right)\|_{L^2}^2+\epsilon\left\|\Delta j\right\|_{L^2}^2\\ &\le C\|\Delta v\|_{L^2}^2\|B\|_{L^\infty}^2+C\|v\|_{L^\infty}^2\|\Delta B\|_{L^2}^2\\ &\quad+C\|\nabla v\|_{L^4}^2\|\nabla B\|_{L^4}^2+\epsilon\left\|\Delta j\right\|_{L^2}^2\\ &\le C\|\Delta v\|_{L^2}^2\|B\|_{L^\infty}^2+C\|v\|_{L^\infty}^2\|\Delta B\|_{L^2}^2\\ &\quad+C\|v\|_{L^\infty}\|B\|_{L^\infty}\|\Delta v\|_{L^2}\|\Delta B\|_{L^2}+\epsilon\left\|\Delta j\right\|_{L^2}^2.\end{aligned}\tag{38}$$

Similarly, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta v\|_{L^2}^2&\le C\|\nabla\left(v\cdot\nabla v\right)\|_{L^2}^2\\ &\quad+C\left\|\nabla\left(j\times B\right)\right\|_{L^2}^2+\epsilon\|\nabla\Delta v\|_{L^2}^2\\ &\le C\|\nabla v\|_{L^4}^4+C\|v\|_{L^\infty}^2\|\Delta v\|_{L^2}^2\\ &\quad+C\|\nabla E\|_{L^6}^2\|B\|_{L^3}^2+C\|E\|_{L^6}^2\|\nabla B\|_{L^3}^2\\ &\quad+C\|\nabla\left(v\times B\right)\|_{L^2}^2\|B\|_{L^\infty}^2+C\|v\times B\|_{L^3}^2\,\|\nabla B\|_{L^6}^2.\end{aligned}\tag{39}$$

Using the interpolation inequality, one has

$$\|\nabla v\|_{L^4}^4\le C\|v\|_{L^\infty}^2\|\Delta v\|_{L^2}^2.\tag{40}$$

Interpolation inequality and Young's inequality produce that

$$\begin{aligned}\|E\|_{L^6}^2\|\nabla B\|_{L^3}^2&\le C\|\nabla E\|_{L^2}^2\|\nabla B\|_{L^2}\|\Delta B\|_{L^2}\\ &\le C\|\nabla E\|_{L^2}^2\left(\|\nabla B\|_{L^2}^2+\|\Delta B\|_{L^2}^2\right).\end{aligned}\tag{41}$$

Similarly, we estimate that

$$\begin{aligned}\|\nabla\left(v\times B\right)\|_{L^2}^2\|B\|_{L^\infty}^2&\le C\left(\|\nabla v\|_{L^3}^2\|\nabla B\|_{L^2}^2+\|\nabla v\|_{L^2}^2\|\nabla B\|_{L^3}^2\right),\\ \|v\times B\|_{L^3}^2\|\nabla B\|_{L^6}^2&\le C\|v\|_{L^\infty}^2\|B\|_{L^3}^2\|\Delta B\|_{L^2}^2.\end{aligned}\tag{42}$$

We already know that

$$\|(\nabla v, \nabla E, \nabla B)\|_{L^\infty(0,T^*;L^2)}^2 < C. \quad (43)$$

Gathering all the estimates, we achieve

$$\begin{aligned} &\frac{d}{dt}\left(\|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right) \\ &\quad + \|\nabla \Delta v\|_{L^2}^2 + \left\|\Delta j\right\|_{L^2}^2 \\ &\leq C\left(1 + \|v\|_{L^\infty}^2 + \|B\|_{L^\infty}^2\right) \\ &\quad \times \left(1 + \|\Delta v\|_{L^2}^2 + \|\Delta E\|_{L^2}^2 + \|\Delta B\|_{L^2}^2\right). \end{aligned} \quad (44)$$

Using Gronwall's inequality, we conclude that

$$\|(\Delta v, \Delta E, \Delta B)\|_{L^\infty(0,T^*;L^2)}^2 + \left\|(\nabla \Delta v, \Delta j)\right\|_{L^2(0,T^*;L^2)}^2 \leq C < \infty. \quad (45)$$

This completes the proof of Theorem 2. □

Acknowledgments

Ensil Kang's work was supported by research fund from Chosun University, 2009, and Jihoom Lee's work was partly supported by the National Research Foundation of Korea (NRF-2011-0006697) and Chung-Ang University Research Grants in 2013.

References

[1] P. Germain, S. Ibrahim, and N. Masmoudi, "On the well-posedness of the Navier-Stokes-Maxwell system," http://arxiv.org/abs/1207.6187.

[2] N. Masmoudi, "Global well posedness for the Maxwell-Navier-Stokes system in 2D," *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, vol. 93, no. 6, pp. 559–571, 2010.

[3] D. D'Ambrosio and D. Giordano, "Electromagnetic fluid dynamics for aerospace applications. Part I: classification and critical review of physical models," in *Proceedings of the 35th AIAA Plasmadynamics and Lasers Conference*, Portland, Ore, USA, June 2004, AIAA Paper 2004-2165.

[4] D. D'Ambrosio, M. Pandol, and D. Giordano, "Electromagnetic fluid dynamics for aerospace applications. Part II: numerical simulations using different physical models," in *Proceedings of the 35th AIAA Plasmadynamics and Lasers Conference*, Portland, Ore, USA, June 2004, AIAA Paper 2004-2362.

[5] P. A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, UK, 2001.

[6] C. Cao and J. Wu, "Two regularity criteria for the 3D MHD equations," *Journal of Differential Equations*, vol. 248, no. 9, pp. 2263–2274, 2010.

[7] C. Cao and J. Wu, "Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion," *Advances in Mathematics*, vol. 226, no. 2, pp. 1803–1822, 2011.

[8] G. Duvaut and J.-L. Lions, "Inéquations en thermoélasticité et magnétohydrodynamique," *Archive for Rational Mechanics and Analysis*, vol. 46, pp. 241–279, 1972.

[9] Z. Lei and Y. Zhou, "BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity," *Discrete and Continuous Dynamical Systems. Series A*, vol. 25, no. 2, pp. 575–583, 2009.

[10] M. Sermange and R. Temam, "Some mathematical questions related to the MHD equations," *Communications on Pure and Applied Mathematics*, vol. 36, no. 5, pp. 635–664, 1983.

[11] J. Wu, "Regularity criteria for the generalized MHD equations," *Communications in Partial Differential Equations*, vol. 33, no. 1–3, pp. 285–306, 2008.

[12] Y. Zhou, "Regularity criteria for the generalized viscous MHD equations," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 24, no. 3, pp. 491–505, 2007.

[13] J.-Y. Chemin and N. Lerner, "Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes," *Journal of Differential Equations*, vol. 121, no. 2, pp. 314–328, 1995.

[14] S. Ibrahim and S. Keraani, "Global small solutions for the Navier-Stokes-Maxwell system," *SIAM Journal on Mathematical Analysis*, vol. 43, no. 5, pp. 2275–2295, 2011.

[15] R. Duan, "Green's function and large time behavior of the Navier-Stokes-Maxwell system," *Analysis and Applications*, vol. 10, no. 2, pp. 133–197, 2012.

[16] S. Ibrahim and T. Yoneda, "Local solvability and loss of smoothness of the Navier-Stokes-Maxwell equations with large initial data," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 2, pp. 555–561, 2012.

[17] P. Germain and N. Masmoudi, "Global existence for the Euler-Maxwell system," http://arxiv.org/abs/1107.1595.

[18] J. Jang and N. Masmoudi, "Derivation of Ohm's law from the kinetic equations," *SIAM Journal on Mathematical Analysis*, vol. 44, no. 5, pp. 3649–3669, 2012.

[19] D. Chae, "Global regularity for the 2D Boussinesq equations with partial viscosity terms," *Advances in Mathematics*, vol. 203, no. 2, pp. 497–513, 2006.

[20] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, vol. 27 of *Cambridge Texts in Applied Mathematics*, Cambridge University Press, Cambridge, UK, 2002.