Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 401756, 13 pages
http://dx.doi.org/10.1155/2013/401756

Review Article
Hyperstability and Superstability

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Received 31 August 2013; Accepted 26 October 2013

Academic Editor: Bing Xu

This is a survey paper concerning the notions of hyperstability and superstability, which are connected to the issue of Ulam's type stability. We present the recent results on those subjects.

1. Introduction

In this paper we provide some recent results concerning hyperstability and superstability of functional equations. Those two notions are very similar but somewhat different. They are connected with the issue of Ulam's type stability.

Let us mention that various aspects of Ulam's type stability, motivated by a problem raised by Ulam (cf. [1, 2]) in 1940 in his talk at the University of Wisconsin, have been a very popular subject of investigations for the last nearly fifty years (see, e.g., [3–11]). For example the following definition somehow describes the main ideas of such stability notion for equations in n variables ($\mathbb{R}_+$ stands for the set of all nonnegative reals).

Definition 1. Let A be a nonempty set, (X, d) be a metric space, $\mathscr{E} \subset \mathscr{C} \subset {\mathbb{R}_+}^{A^n}$ be nonempty, $\mathscr{T}$ be an operator mapping $\mathscr{C}$ into ${\mathbb{R}_+}^{A}$, and $\mathscr{F}_1, \mathscr{F}_2$ be operators mapping a nonempty set $\mathscr{D} \subset X^A$ into X^{A^n}. We say that the operator equation

$$\mathscr{F}_1\varphi(x_1, \dots, x_n) = \mathscr{F}_2\varphi(x_1, \dots, x_n) \tag{1}$$

is $(\mathscr{E}, \mathscr{T})$-stable provided for any $\varepsilon \in \mathscr{E}$ and $\varphi_0 \in \mathscr{D}$ with

$$\begin{aligned} d(\mathscr{F}_1\varphi_0(x_1, \dots, x_n), \mathscr{F}_2\varphi_0(x_1, \dots, x_n)) \\ \le \varepsilon(x_1, \dots, x_n), \quad x_1, \dots, x_n \in A, \end{aligned} \tag{2}$$

there exists a solution $\varphi \in \mathscr{D}$ of (1) such that

$$d(\varphi(x), \varphi_0(x)) \le \mathscr{T}\varepsilon(x), \quad x \in A. \tag{3}$$

(As usual, C^D denotes the family of all functions mapping a set $D \neq \emptyset$ into a set $C \neq \emptyset$.) Roughly speaking, $(\mathscr{E}, \mathscr{T})$-stability of (1) means that every approximate (in the sense of (2)) solution of (1) is always close (in the sense of (3)) to an exact solution of (1).

The next theorem has been considered to be one of the most classical results on Ulam's type stability.

Theorem 2. *Let E_1 and E_2 be normed spaces, E_2 complete, and $c \ge 0$ and $p \neq 1$ fixed real numbers. If $f : E_1 \to E_2$ is a mapping satisfying*

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| \\ \le c(\|x\|^p + \|y\|^p), \quad x, y \in E_1 \setminus \{0\}, \end{aligned} \tag{4}$$

then there exists a unique function $T : E_1 \to E_2$ such that

$$\begin{aligned} T(x+y) = T(x) + T(y), \quad x, y \in E_1, \\ \|f(x) - T(x)\| \le \frac{c\|x\|^p}{|2^{p-1} - 1|}, \quad x \in E_1 \setminus \{0\}. \end{aligned} \tag{5}$$

This theorem is composed of the outcomes from [1, 12–14] and it is known (see [13]; cf. also [15, 16]) that for $p = 1$ an analogous result is not valid. Moreover, it has been shown in [17] that estimation (5) is optimal for $p \ge 0$ in the general case.

Theorem 2 has a very nice simple form, but it has been improved in [18], where it has been shown that, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (4) must actually be additive (and the completeness of E_2 is not necessary in such a situation). Namely, we have the following result ($\mathbb{N}$ stands for the set of all positive integers).

Theorem 3. *Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ nonempty, $c \geq 0$, and $p < 0$. Assume also that*

$$-X = X, \tag{6}$$

where $-X := \{-x : x \in X\}$, and there exists a positive integer m_0 with

$$-x, nx \in X, \quad x \in X,\ n \in \mathbb{N},\ n \geq m_0. \tag{7}$$

Then every operator $g : E_1 \to E_2$ such that

$$\begin{aligned} &\| g(x+y) - g(x) - g(y) \| \\ &\quad \leq c\left(\|x\|^p + \|y\|^p\right), \quad x, y \in X,\ x+y \in X \end{aligned} \tag{8}$$

is additive on X; that is,

$$g(x+y) = g(x) + g(y), \quad x, y \in X,\ x+y \in X. \tag{9}$$

Clearly, since (5) gives the best possible estimation for $p \geq 0$ in the general case, a result analogous to Theorem 3 is not true for $p \geq 0$.

On account of Theorem 3, we can reformulate Theorem 2 as follows.

Theorem 4. *Let E_1 and E_2 be normed spaces and let $c \geq 0$ and $p \neq 1$ be fixed real numbers. Assume also that $f : E_1 \to E_2$ is a mapping satisfying (4). If $p \geq 0$ and E_2 is complete, then there exists a unique additive function $T : E_1 \to E_2$ such that (5) holds. If $p < 0$, then f is additive.*

Following the terminology introduced in [19] and next used in, for example, [20] (see also [3, pages 27–29]), we can describe the second statement of Theorem 4, for $p < 0$, as the φ-hyperstability of the additive Cauchy equation for $\varphi(x, y) \equiv c(\|x\|^p + \|y\|^p)$.

It is interesting that the hyperstability result, described in Theorem 3, does not remain valid without condition (6), which is shown in the following remark ($\mathbb{R}$ denotes the set of all reals).

Remark 5. Let $p < 0$, $a \geq 0$, $I = (a, \infty)$, and $f, T : I \to \mathbb{R}$ be given by $T(x) = 0$ and $f(x) = x^p$ for $x \in I$. Then clearly

$$|f(x) - T(x)| = x^p, \quad x \in I. \tag{10}$$

Note that also

$$|f(x+y) - f(x) - f(y)| \leq x^p + y^p, \quad x, y \in I. \tag{11}$$

In fact, fix $x, y \in I$ and suppose, for instance, that $x \leq y$. Then

$$(x+y)^p \leq (2x)^p = 2^p x^p \leq x^p \leq x^p + y^p \tag{12}$$

and consequently

$$|f(x+y) - f(x) - f(y)| = x^p + y^p - (x+y)^p \leq x^p + y^p. \tag{13}$$

However, with a somewhat different (though still natural) form of the function φ, φ-hyperstability still holds even without (6). Namely, in [21, Theorem 1.3] the subsequent result has been proved.

Theorem 6. *Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ nonempty, $c \geq 0$, and p, q real numbers with $p+q < 0$. Assume also that there is an $m_0 \in \mathbb{N}$ such that*

$$nx \in X, \quad x \in X,\ n \in \mathbb{N},\ n \geq m_0. \tag{14}$$

Then every operator $g : E_1 \to E_2$ satisfying the inequality

$$\begin{aligned} &\| g(x+y) - g(x) - g(y) \| \\ &\quad \leq c\|x\|^p \|y\|^q, \quad x, y \in X,\ x+y \in X \end{aligned} \tag{15}$$

is additive on X.

We refer the reader to, for example, [22, Theorem 1.1, Chapter XVIII], [23, Chapter 4], [24, pages 143-144], and [25, Proposition 3.8] for some information on the following natural issue: when for an operator $T_0 : E_1 \to E_2$ that is additive on $X \subset E_1$, there is an additive $T : E_1 \to E_2$ with $T(x) = T_0(x)$ for $x \in X$.

2. Hyperstability Results for the Cauchy Equation

Formally, we can introduce the following definition.

Definition 7. Let A be a nonempty set, (X, d) a metric space, $\varepsilon \in \mathbb{R}_+^{A^n}$, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $\mathcal{D} \subset X^A$ into X^{A^n}. We say that operator equation (1) is ε-hyperstable provided every $\varphi_0 \in \mathcal{D}$ satisfying inequality (2) fulfils (1).

The hyperstability results have various interesting consequences. For instance, note that we deduce at once from Theorem 6 a bit surprising conclusion that each function $f : E_1 \to E_2$ is either additive on X or satisfies the condition

$$\sup_{x,y \in X, x+y \in X} \| f(x+y) - f(x) - f(y) \| \|x\|^r \|y\|^s = \infty \tag{16}$$

for any real numbers $r, s, r+s > 0$, where E_1 and E_2 are normed spaces and X is a nonempty subset of $E_1 \setminus \{0\}$ fulfilling condition (14) for some $m_0 \in \mathbb{N}$.

Theorem 6 yields also the following two simple corollaries (see [21]), which correspond to some results from [26–33] on inhomogeneous Cauchy equation (18) and cocycle equation (19).

Corollary 8. *Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ nonempty, $G : X^2 \to E_2$, and $G(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X$ with $x_0 + y_0 \in X$. Assume also that (14) holds with an $m_0 \in \mathbb{N}$ and there are $p, q \in \mathbb{R}$ and $c > 0$ such that $p+q < 0$ and*

$$\|G(x, y)\| \leq c\|x\|^p \|y\|^q, \quad x, y \in X,\ x+y \in X. \tag{17}$$

Then the functional equation

$$g_0(x+y) = g_0(x) + g_0(y) + G(x,y), \quad x, y \in X,\ x+y \in X \tag{18}$$

has no solution in the class of functions $g_0 : X \to E_2$.

Corollary 9. *Let* E_1 *and* E_2 *be normed spaces,* $X \subset E_1 \setminus \{0\}$ *nonempty,* $G : {E_1}^2 \to E_2$ *satisfy the cocycle functional equation*

$$G(x,y) + G(x+y,z) = G(x,y+z) + G(y,z), \quad x, y, z \in E_1, \tag{19}$$

and $G(x,y) = G(y,x)$ *for* $x, y \in E_1$. *Assume also that* (14) *holds with an* $m_0 \in \mathbb{N}$ *and there are* $p, q \in \mathbb{R}$ *and* $c > 0$ *such that* $p+q < 0$ *and* (17) *holds. Then* $G(x,y) = 0$ *for any* $x, y \in X$ *with* $x + y \in X$.

The hyperstability results that we have presented so far have been obtained through the fixed point theorem from [34] (see also [35, 36]; cf. [4] for a survey on similar methods using the fixed point results). Now, we provide some further φ-hyperstability results (with functions φ of some other natural forms) for the Cauchy additive equation, proved in [25] by some other methods.

Theorem 10. *Let* $(X, \langle\cdot \mid \cdot\rangle)$ *be a real inner product space with* $\dim X > 1$, *Y a normed space, and* $g : X \to Y$. *If there are positive real numbers* $p \neq 1$ *and* L *such that*

$$\|g(x+y) - g(x) - g(y)\| \le L\left|\langle x \mid y\rangle\right|^p, \quad x, y \in X, \tag{20}$$

then g is additive.

If $p = 1$, then g does not need to be additive (see [25]).

Theorem 11. *Let X and Y be normed spaces,* $\dim X > 2$, *and* $g : X \to Y$. *Suppose also that there are positive real numbers p and* L_0 *with*

$$\|g(x+y) - g(x) - g(y)\| \le L_0\left|\|x+y\|^2 - \|x-y\|^2\right|^p, \quad x, y \in X. \tag{21}$$

If $p \neq 1$ *or X is not a real inner product space, then g is additive.*

If X is a real inner product space and $p = 1$, then g does not need to be additive (see [25]).

Given a normed space X and $A, B : X \to X$, we simplify the notations writing $AB := A \circ B$ and defining the mapping $A + B : X \to X$ by

$$(A+B)(x) := A(x) + B(x), \quad x \in X. \tag{22}$$

Moreover, if $\emptyset \neq U \subset V \subset X$ and $C : V \to X$, then we put

$$\|C\|_U := \inf\{\xi \in \mathbb{R} : \|C(x) - C(y)\| \le \xi\|x-y\| \text{ for } x, y \in U\}. \tag{23}$$

Clearly, if $A : X \to X$ is additive, then we have (with $U = X$)

$$\|A\|_X = \inf\{\xi \in \mathbb{R} : \|A(x)\| \le \xi\|x\| \text{ for } x \in X\}. \tag{24}$$

Now, we are in a position to present another result from [25].

Theorem 12. *Let X and Y be normed spaces and* $\emptyset \neq U \subset X$. *Assume that* $C, D : X \to X$ *are additive,*

$$CD = DC, \tag{25}$$

$$C(x), D(x), E(x) \in U, \quad x \in U, \tag{26}$$

where $E := C + D$. *Let, moreover,* $p \in \mathbb{R}_+$ *be such that one of the following two conditions is valid:*

(a) *E is injective,* $U \subset E(U)$ *and*

$$\left(\|D\|_U^p + \|C\|_U^p\right)\left\|E^{-1}\right\|_U^p < 1; \tag{27}$$

(b) $U \subset D(U)$, *D is injective and*

$$\left(\|E\|_U^p + \|C\|_U^p\right)\left\|D^{-1}\right\|_U^p < 1. \tag{28}$$

Then every function $g : U \to Y$ *for which there exists an* $L \in \mathbb{R}_+$ *such that*

$$\|g(x+y) - g(x) - g(y)\| \le L\|C(x) - D(y)\|^p, \quad x, y \in U,\ x+y \in U \tag{29}$$

is additive on U.

Remark 13. Observe that condition (25) holds when $D = C^n$ with a nonnegative integer n or $D(x) = \gamma x$ for $x \in X$ with a rational number γ (because C is assumed to be additive).

Remark 14. For instance, the inequality in (a) holds for $p > 1$, $U = X$, and

$$C(x) = D(x) = \lambda x, \quad x \in X \tag{30}$$

with a $\lambda \in \mathbb{R}$. Analogously, the inequality in (b) is valid when $p > 1$, $U = X$,

$$C(x) = -\lambda x, \quad D(x) = 2\lambda x, \quad x \in X \tag{31}$$

with a $\lambda \in \mathbb{R}$.

For similar hyperstability results in some situations where neither condition (a) nor (b) is fulfilled we refer the reader to [25, Corollaries 3.5 and 3.6].

We end this part of the paper with one more hyperstability result (on a restricted domain) from [25]. To do this, let us recall some notions.

Given nonempty sets $X, Y, \mathcal{J} \subset 2^X$ and $f, g : X \to Y$, we say that $f = g$ $\mathcal{J}$-almost everywhere (abbreviated to $\mathcal{J}$-a.e.) in X if there is a set $T \in \mathcal{J}$ such that $f(x) = g(x)$ for every $x \in X \setminus T$. If, moreover, X is a normed space, then we also write $\alpha T := \{\alpha x : x \in T\}$ for $T \subset X$ and $\alpha \in \mathbb{R}$.

Now we are in a position to present [25, Theorem 4.1] (which actually is a consequence of some previous results).

Theorem 15. *Let X and Y be normed spaces, $g : X \to Y$, and $\mathcal{I} \subset 2^X$ a σ-ideal such that*

$$x + \alpha T \in \mathcal{I}, \quad T \in \mathcal{I},\ \alpha \in \mathbb{R},\ x \in X. \tag{32}$$

Assume also that one of the following two conditions is fulfilled:

(i) *there exist $T \in \mathcal{I}$, $c, d \in \mathbb{R}$, $cd(c+d) \neq 0$, $L > 0$ and $p > 1$ such that*

$$\|g(x+y) - g(x) - g(y)\| \le L\|cx - dy\|^p, \quad x, y \in X \setminus T; \tag{33}$$

(ii) *there exist $T \in \mathcal{I}$, $C : X \to X$ with $C(2x) = 2C(x)$ for $x \in X$ and positive reals L and $p \neq 1$ such that*

$$\begin{aligned} &\|g(x+y) - g(x) - g(y)\| \\ &\quad \le L\|C(x) - C(y)\|^p, \quad x, y \in X \setminus T. \end{aligned} \tag{34}$$

Then there is a unique additive operator $f : X \to Y$ with $f = g\mathcal{I}$-a.e. in X.

A hyperstability result for the multi-Cauchy equation (which actually is a system of Cauchy equations) can be found in [37, Corollary 4].

Finally, we would like to call the reader's attention to a general theorem in [38] which yields numerous other hyperstability results for the Cauchy additive equation.

3. Hyperstability of the Linear Functional Equation

Now, we present the hyperstability results for the linear functional equation of the form

$$f(Ax + By) = af(x) + bf(y) + z_0 \tag{35}$$

in the class of functions $f : X \to Y$, where X is a linear space over a field $\mathbb{F}$, Y is a linear space over a field $\mathbb{K}$, $a, b \in \mathbb{K}$, $z_0 \in Y$, and $A, B \in \mathbb{F}$. Clearly, for $a = b = 1$, $z_0 = 0$, and $A = B = 1$ (35) is the well-known (additive) Cauchy functional equation

$$f(x+y) = f(x) + f(y), \tag{36}$$

and with $a = b = 1/2$, $z_0 = 0$, and $A = B = 1/2$ it is the Jensen equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}. \tag{37}$$

If $t \in (0,1)$, $a = t$, $b = 1-t$, $z_0 = 0$, $A = t$, and $B = 1-t$, then (35) has the form

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) \tag{38}$$

and its solution is called a t-affine function. For further information and references on (36)–(38) we refer the reader to [22, 39].

The subsequent theorem has been proved in [40].

Theorem 16. *Let X be a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Y be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $A, B \in \mathbb{F} \setminus \{0\}$, $a, b \in \mathbb{K}$, $c \le 0$, $p < 0$, and $f : X \to Y$ satisfy*

$$\begin{aligned} &\|f(Ax + By) - af(x) - bf(y)\| \\ &\quad \le c\left(\|x\|^p + \|y\|^p\right), \quad x, y \in X \setminus \{0\}. \end{aligned} \tag{39}$$

Then

$$f(Ax + By) = af(x) + bf(y), \quad x, y \in X \setminus \{0\}. \tag{40}$$

Similar results, for Jensen equation (37), but on a restricted domain, have been obtained in [41]. Namely, we have the following three theorems.

Theorem 17. *Let X be a normed space, U be a nonempty subset of $X \setminus \{0\}$ such that there exists a positive integer n_0 with*

$$nx \in U, \quad x \in U,\ n \in \mathbb{N},\ n \ge n_0, \tag{41}$$

let Y be a Banach space, $c \ge 0$, $p, q \in \mathbb{R}$, $p + q < 0$, and $f : U \to Y$ satisfy

$$\begin{aligned} &\left\| f\left(\frac{x+y}{2}\right) - \frac{f(x) + f(y)}{2} \right\| \\ &\quad \le c\|x\|^p\|y\|^q, \quad x, y \in U,\ \frac{x+y}{2} \in U. \end{aligned} \tag{42}$$

Then f is Jensen on U; that is,

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}, \quad x, y \in U,\ \frac{x+y}{2} \in U. \tag{43}$$

Theorem 18. *Let X be a normed space, U a nonempty subset of $X \setminus \{0\}$ such that there exists a positive integer n_0 with*

$$-\frac{1}{n}x, \frac{1}{2}\left(1 - \frac{1}{n}\right)x \in U, \quad x \in U,\ n \in \mathbb{N},\ n \ge n_0, \tag{44}$$

Y a Banach space, $c \ge 0$, $p, q \in \mathbb{R}$, $p + q > 1$, and $f : U \to Y$ satisfy (42). Then f is Jensen on U.

Theorem 19. *Let X be a normed space, U a nonempty subset of $X \setminus \{0\}$ such that there exists a positive integer n_0 with*

$$\frac{1}{n}x, \left(2 - \frac{1}{n}\right)x \in U, \quad x \in U,\ n \in \mathbb{N},\ n \ge n_0, \tag{45}$$

Y a Banach space, $c \ge 0$, $p, q \in \mathbb{R}$, $0 < p + q < 1$, and $f : U \to Y$ satisfy (42). Then f is Jensen on U.

We finish this section of the paper by proving one more very simple hyperstability result for (35).

Theorem 20. *Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Y a normed space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $a, b \in \mathbb{K}$, $z_0 \in Y$, $A, B \in \mathbb{F}$, $L, p, q \in \mathbb{R}_+$, $p + q > 0$, and let one of the following two conditions be valid:*

(i) *$q \neq 0$ and $|A|^{p+q} \neq |a|$;*

(ii) *$p \neq 0$ and $|B|^{p+q} \neq |b|$.*

Then every function $g_0 : X \to Y$ satisfying the inequality

$$\begin{aligned} & \left\| g_0(Ax+By) - a g_0(x) - b g_0(y) - z_0 \right\| \\ & \quad \leq L \|x\|^p \|y\|^q, \quad x, y \in X \end{aligned} \tag{46}$$

is a solution of the equation

$$g_0(Ax+By) = a g_0(x) + b g_0(y) + z_0, \quad x, y \in X. \tag{47}$$

Proof. First, observe that in the case when $a+b=1$, inequality (46) with $x=y=0$ implies $z_0=0$.

Put

$$\begin{aligned} z_1 &:= \begin{cases} g_0(0), & a+b=1, \\ \dfrac{z_0}{1-(a+b)}, & a+b \neq 1, \end{cases} \\ g(x) &:= g_0(x) - z_1, \quad x \in X. \end{aligned} \tag{48}$$

It is easily seen that

$$\left\| g(Ax+By) - a g(x) - b g(y) \right\| \leq L \|x\|^p \|y\|^q, \quad x, y \in X, \tag{49}$$

whence with $x=y=0$ we get $g(0)=0$.

We consider only case (i) (case (ii) is analogous). First, assume that $|a| < |A|^{p+q}$. Then (49) with $y=0$ gives

$$g(x) = a g\left(A^{-1} x\right), \quad x \in X. \tag{50}$$

We show by induction that, for each $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$\begin{aligned} & \left\| g(Ax+By) - a g(x) - b g(y) \right\| \\ & \quad \leq \left(|A^{-1}|^{p+q} |a| \right)^n L \|x\|^p \|y\|^q, \quad x, y \in X. \end{aligned} \tag{51}$$

The case $n=0$ follows immediately from (49). So take an $l \in \mathbb{N}_0$ and assume that (51) holds true with $n=l$. Then, by (50),

$$\begin{aligned} & \left\| g(Ax+By) - a g(x) - b g(y) \right\| \\ & = \left\| a g\left(A^{-1}(Ax+By)\right) - a^2 g\left(A^{-1}x\right) - ab g\left(A^{-1}y\right) \right\| \\ & = |a| \left\| g\left(AA^{-1}x + BA^{-1}y\right) - a g\left(A^{-1}x\right) - b g\left(A^{-1}y\right) \right\| \\ & \leq \left(|A^{-1}|^{p+q} |a| \right)^l L |a| \left\| A^{-1}x \right\|^p \left\| A^{-1}y \right\|^q \\ & = \left(|A^{-1}|^{p+q} |a| \right)^{l+1} L \|x\|^p \|y\|^q, \quad x, y \in X. \end{aligned} \tag{52}$$

Thus we have proved that (51) is valid for each $n \in \mathbb{N}_0$.

Letting $n \to \infty$ in (51) we see that

$$g(Ax+By) = a g(x) + b g(y), \quad x, y \in X, \tag{53}$$

which implies (47).

If $|a| > |A|^{p+q}$, then $a \neq 0$ and from (49) with $y=0$ we obtain

$$g(x) = \frac{1}{a} g(Ax), \quad x \in X. \tag{54}$$

Analogously as before we show that, for each $n \in \mathbb{N}_0$,

$$\begin{aligned} & \left\| g(Ax+By) - a g(x) - b g(y) \right\| \\ & \quad \leq \left(\frac{|A|^{p+q}}{|a|} \right)^n L \|x\|^p \|y\|^q, \quad x, y \in X. \end{aligned} \tag{55}$$

Letting $n \to \infty$ in (55) we get (53), and consequently (47) holds. □

Remark 21. Let $g_0(x) = x$ for $x \in X = Y$, $\mathbb{F} = \mathbb{K}$, $a \neq A$ and $b = B$. Then

$$g_0(Ax+By) - a g_0(x) - b g_0(y) = (A-a)x, \quad x, y \in X, \tag{56}$$

whence

$$\begin{aligned} & \left\| g_0(Ax+By) - a g_0(x) - b g_0(y) \right\| \\ & \quad = |A-a| \, \|x\|, \quad x, y \in X. \end{aligned} \tag{57}$$

Thus g_0 is an example of a function which satisfies (46) with $z_0 = 0$, $L = |A-a|$, $q=0$ and $p=1$ but is not a solution of (47). This shows that the assumption $q \neq 0$ in (i) is not superfluous.

Remark 22. Let $X = Y = \mathbb{R}$, $A^2 = a$, $B^2 = b$, and $g_0(x) = x^2$ for $x \in X$. Then

$$\begin{aligned} & \left| g_0(Ax+By) - a g_0(x) - b g_0(y) \right| \\ & \quad = 2 |AB| \, |x| \, |y|, \quad x, y \in X. \end{aligned} \tag{58}$$

Thus g_0 is an example of a function which satisfies (46) with $z_0 = 0$, $L = 2|AB|$ and $p = q = 1$ but is not a solution of (47). This proves that assumptions (i) and (ii) of Theorem 20 are not superfluous.

4. Hyperstability of Some Other Equations

In this part of the paper we present the hyperstability results for some other equations. The first two theorems have been proved in [19].

Theorem 23. *Let $M : (0,1] \to \mathbb{R}$ be a solution of the functional equation*

$$M(xy) = M(x) M(y), \quad x, y \in (0,1] \tag{59}$$

and $M(x_0) > 1$ for some $x_0 \in (0,1]$. Assume also that a function $\psi : (0,1] \to \mathbb{R}$ satisfy the inequality

$$\left| \psi(xy) - M(x)\psi(y) - M(y)\psi(x) \right| \leq \varepsilon, \quad x, y \in (0,1] \tag{60}$$

for some $\varepsilon > 0$. Then

$$\psi(xy) = M(x)\psi(y) + M(y)\psi(x), \quad x, y \in (0,1]. \tag{61}$$

Theorem 24. *Let $(S, \cdot)$ be a semigroup and $\varphi_1, \ldots, \varphi_n : S \to S$ pairwise distinct automorphisms of S such that the set*

$\{\varphi_1, \dots, \varphi_n\}$ is a group with the operation of composition of mappings. Let, moreover, $\varepsilon : S \times S \to \mathbb{R}_+$ be a function for which there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of elements of S satisfying one of the following two conditions:

$$\lim_{k\to\infty} \varepsilon(u_k s, t) = 0, \quad s, t \in S;$$
$$\lim_{k\to\infty} \varepsilon(s, t\varphi_i(u_k)) = 0, \quad s, t \in S, \ i \in \{1, \dots, n\}. \tag{62}$$

If a function f, mapping S into a real normed space X, fulfils the inequality

$$\left\| f(s) + f(t) - \frac{1}{n}\sum_{i=1}^{n} f(s\varphi_i(t)) \right\| \le \varepsilon(s,t), \quad s, t \in S, \tag{63}$$

then f is a solution of the functional equation

$$f(s) + f(t) = \frac{1}{n}\sum_{i=1}^{n} f(s\varphi_i(t)), \quad s, t \in S. \tag{64}$$

The following result, concerning the parametric fundamental equation of information, has been obtained in [20].

Theorem 25. *Let $\alpha < 0$ and $f : (0,1) \to \mathbb{R}$ be a function such that*

$$\sup_{(x,y)\in D_2} \left| f(x) + (1-x)^\alpha f\left(\frac{y}{1-x}\right) - f(y) - (1-y)^\alpha f\left(\frac{x}{1-y}\right) \right| < \infty, \tag{65}$$

where

$$D_2 := \left\{ (p_1, p_2) \in \mathbb{R}^2 : p_1, p_2, p_1 + p_2 \in (0,1) \right\}. \tag{66}$$

Then

$$f(x) + (1-x)^\alpha f\left(\frac{y}{1-x}\right) = f(y) + (1-y)^\alpha f\left(\frac{x}{1-y}\right), \quad (x,y) \in D_2. \tag{67}$$

Let us recall (see [20]) that each solution $f : (0,1) \to \mathbb{R}$ of (67) is of the form

$$f(x) = cx^\alpha + d(1-x)^\alpha - d, \quad x \in (0,1) \tag{68}$$

with some $c, d \in \mathbb{R}$.

The next two theorems have been proved in [42, 43] and concern hyperstability of the polynomial and monomial equations (for details concerning those equations we refer the reader to [22]).

Theorem 26. *Let X and Y be real normed spaces. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=0}^{3} \frac{3!}{i!\,(3-i)!} (-1)^{(3-i)} f(ix + y) \right\| \le \varepsilon\left(\|x\|^p + \|y\|^p \right), \quad x, y \in X \setminus \{0\} \tag{69}$$

with some $\varepsilon > 0$ and $p < 0$, then

$$\sum_{i=0}^{3} \frac{3!}{i!\,(3-i)!} (-1)^{(3-i)} f(ix + y) = 0, \quad x, y \in X. \tag{70}$$

Theorem 27. *Let X and Y be real normed spaces and n a positive integer. If a function $f : X \to Y$ satisfies the inequality*

$$\left\| \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} (-1)^{(n-i)} f(ix + y) - n! f(x) \right\| \le \varepsilon\left(\|x\|^p + \|y\|^p \right), \quad x, y \in X \setminus \{0\} \tag{71}$$

with some $\varepsilon > 0$ and $p < 0$, then

$$\sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!} (-1)^{(n-i)} f(ix + y) = n! f(x), \quad x, y \in X. \tag{72}$$

The next theorem from [44] contains a hyperstability result for the Drygas equation.

Theorem 28. *Assume that D is a nonempty subset of a normed space X such that $0 \notin D$ and there exists an $n_0 \in \mathbb{N}$ with*

$$-x, nx \in D, \quad x \in D, \ n \in \mathbb{N}, \ n > n_0. \tag{73}$$

Let Y be a Banach space and $f : D \to Y$ fulfill the inequality

$$\| f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) \| \le c\left(\|x\|^p + \|y\|^p \right), \quad x, y \in D, \ x+y, x-y \in D \tag{74}$$

for some $c > 0$ and $p < 0$. Then f satisfies the conditional Drygas equation

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x, y \in D, \ x+y, \ x-y \in D. \tag{75}$$

Theorem 28 yields at once the following characterization of the inner product spaces.

Corollary 29. *Let X be a normed space and*

$$\sup_{x,y\in X\setminus\{0\}} \frac{\left| \|x+y\|^2 + \|x-y\|^2 - 2\|x\|^2 - 2\|y\|^2 \right|}{\|x\|^p + \|y\|^p} < \infty \tag{76}$$

for some $p < 0$. Then X is an inner product space.

Proof. Write $f(x) = \|x\|^2$ for $x \in X$. Then from Theorem 28 we easily derive that

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y), \quad x, y \in X \setminus \{0\}. \tag{77}$$

It is easy to see that this implies

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x, y \in X, \tag{78}$$

which yields the statement. □

The next hyperstability result has been proved in [45, Corollary 2.9] and is actually a particular consequence of two more general theorems proved there.

Theorem 30. *Let X be a normed space, Y be a Banach space, $p, q, \lambda \in \mathbb{R}_+$ and $0 < p + q \neq 4$. Assume also that a mapping $f : X \to Y$ satisfies the inequality*

$$\begin{aligned} \| f(2x+y) + f(2x-y) - 4f(x+y) \\ - 4f(x-y) - 24f(x) + 6f(y) \| \\ \leq \lambda \|x\|^p \|y\|^q, \quad x, y \in X. \end{aligned} \tag{79}$$

Then

$$\begin{aligned} & f(2x+y) + f(2x-y) \\ & = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y), \quad x, y \in X. \end{aligned} \tag{80}$$

A result on hyperstability of the equation of p-Wright affine functions has been obtained in [46] and it reads as follows.

Theorem 31. *Let X be a normed space over a field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, Y a Banach space, $p \in \mathbb{F}$, $A, k > 0$, $|p|^{2k} + |1-p|^{2k} < 1$ and $g : X \to Y$ satisfy*

$$\begin{aligned} \| g(px + (1-p)y) + g((1-p)x + py) - g(x) - g(y) \| \\ \leq A \|x\|^k \|y\|^k, \quad x, y \in X. \end{aligned} \tag{81}$$

Then g is a p-Wright affine function; that is,

$$\begin{aligned} g(px + (1-p)y) + g((1-p)x + py) \\ = g(x) + g(y), \quad x, y \in X. \end{aligned} \tag{82}$$

The next result has been proved in [47] and concerns the homogeneity equation.

Theorem 32. *Let X and Y be normed spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $p, q \in \mathbb{R}$, $\varepsilon \in \mathbb{R}_+$, and $g : X \to Y$ satisfy*

$$\| g(\alpha x) - \alpha g(x) \| < \varepsilon (|\alpha|^p + \|x\|^q) \tag{83}$$

for any $\alpha \in \mathbb{K}$ and $x \in X$ such that $|\alpha|^p + \|x\|^q$ is defined. Assume also that $p < 1$ or $q < 0$. Then

$$g(\alpha x) = \alpha g(x), \quad x \in X \setminus \{0\}, \ \alpha \in \mathbb{K} \setminus \{0\}. \tag{84}$$

Moreover, if one of the following two conditions is valid:

(a) $p < 1$ *and* $q \geq 0$*;*
(b) $p > 0$ *and* $q < 0$*,*

then

$$g(\alpha x) = \alpha g(x), \quad x \in X, \alpha \in \mathbb{K}. \tag{85}$$

Some further hyperstability (but also superstability) results for the homogeneity equation can be found in [48, 49]. Unfortunately, they are too involved to be presented here. Therefore, we only give below the following simple corollary (see [48, Corollary 3]).

Theorem 33. *Let X be a real linear space, Y a Banach space, and $g : X \to Y$ satisfy*

$$\sup_{x \in X, \alpha \in (-\gamma, 0)} \| g(\alpha x) - \alpha g(x) \| < \infty \tag{86}$$

with some $\gamma > 0$. Then

$$g(\alpha x) = \alpha g(x), \quad x \in X, \ \alpha \in \mathbb{R}. \tag{87}$$

Pexiderized hyperstability of the functional equation of biadditivity, of the form

$$f(x+y, z+w) = f(x,z) + f(x,w) + f(y,z) + f(y,w), \tag{88}$$

has been considered in [50] (actually, for some reason, it has been called the bi-Jensen functional equation by the authors), where the following two theorems have been presented.

Theorem 34. *Let X be a normed space, Y a Banach space, $p < 0 \leq \varepsilon$, and a function $f : X \times X \to Y$ satisfy the inequality*

$$\begin{aligned} & \| f(x+y, z+w) - f_1(x,z) \\ & - f_2(x,w) - f_3(y,z) - f_4(y,w) \| \\ & \leq \varepsilon \left(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p \right), \quad x, y, z, w \in X \setminus \{0\} \end{aligned} \tag{89}$$

with some mappings $f_1, f_2, f_3, f_4 : X \times X \to Y$. Then f is biadditive; that is, (88) *holds for all $x, y, z, w \in X$.*

Theorem 35. *Let X be a normed space, Y a Banach space, $p < 0 \leq \varepsilon$, and a function $f : X \times X \to Y$ satisfy the inequality*

$$\begin{aligned} & \| f(x+y, z+w) - f_1(x,z) \\ & - f_2(x,w) - f_3(y,z) - f_4(y,w) \| \\ & \leq \varepsilon \left(\|x\|^p + \|y\|^p \right) \left(\|z\|^p + \|w\|^p \right), \quad x, y, z, w \in X \setminus \{0\} \end{aligned} \tag{90}$$

with some mappings $f_1, f_2, f_3, f_4 : X \times X \to Y$. Then f is biadditive.

For some further results, related somehow to the issue of hyperstability, we refer the reader to:

(i) [51, Theorem 8.3] (for a generalization of the quadratic equation);
(ii) [52] (for the equations of homomorphism and derivation in proper JCQ*-triples);
(iii) [53, Theorem 21.3] (for the equations of homomorphism for square symmetric groupoids, considered in a class of set-valued mappings);

(iv) [54, Theorem 1] (for a functional equation in one variable in a class of set-valued mappings);

(v) [55] (for functional equations of trigonometric forms in hypergroups).

5. Superstability

In this part of the paper we present several recent results on superstability of some functional equations. For numerous earlier results as well as the historical background of the subject we refer the reader to [6, 7, 9, 10].

The following definition explains how the notion of superstability for functional equations (in n variables) is understood nowadays.

Definition 36. Let A be a nonempty set, (X,d) a metric space, and $\mathcal{F}_1, \mathcal{F}_2$ operators mapping a nonempty set $\mathcal{D} \subset X^A$ into X^{A^n}. We say that operator equation (1) is superstable if every $\varphi \in \mathcal{D}$ that is unbounded (i.e., $\sup_{x,y\in A} d(\varphi(x),\varphi(y)) = \infty$) and satisfies the inequality

$$\sup_{x_1,\dots,x_n\in A} d\left(\mathcal{F}_1\varphi\left(x_1,\dots,x_n\right),\mathcal{F}_2\varphi\left(x_1,\dots,x_n\right)\right) < \infty \tag{91}$$

is a solution of (1).

Let us start with the results that Moszner has proved in [56] (modificating the proofs from [57, 58]), and which concern the sine, homomorphism, Lobachevski and cosine equations.

Theorem 37. *Let G be a uniquely 2-divisible commutative group and A a finite-dimensional commutative normed algebra without the zero divisors. Then every unbounded function $f: G \to A$ such that*

$$\sup_{x,y\in G}\left\| f(x+y)\, f(x-y) - f(x)^2 + f(y)^2 \right\| < \infty \tag{92}$$

is a solution of the sine equation

$$f(x+y)\, f(x-y) = f(x)^2 - f(y)^2, \quad x,y\in G. \tag{93}$$

Theorem 38. *Let $(G,\cdot)$ be a commutative semigroup and $(A,\cdot)$ a groupoid equipped with*

(i) *an operation $\mathbb{R}_+ \times A \ni (\lambda, a) \mapsto \lambda a \in A$ such that*

$$\lambda(ab) = (\lambda a)\, b = a(\lambda b), \quad a,b\in A,\ \lambda\in\mathbb{R}_+; \tag{94}$$

(ii) *an element $0\in A$ such that $\lambda 0 = 0$ for $\lambda\in\mathbb{R}_+$ and $a^2 \neq 0$ for $a\in A\setminus\{0\}$;*

(iii) *a metric ρ satisfying the condition*

$$\rho(\lambda a,\lambda b) \le \lambda\rho(a,b), \quad a,b\in A,\ \lambda>0. \tag{95}$$

Moreover, assume that each nonzero element of A is cancellative on the left or on the right, the groupoid operation in A is continuous, and the unit sphere is compact in A. Then every unbounded function $f: G \to A$ such that

$$\sup_{x,y\in G}\rho\left(f(xy), f(x)\, f(y)\right) < \infty \tag{96}$$

is a homomorphism; that is,

$$f(xy) = f(x)\, f(y), \quad x,y\in G. \tag{97}$$

Theorem 39. *Let G be a uniquely 2-divisible commutative monoid and A a finite-dimensional commutative normed algebra without the zero divisors. Then every unbounded function $f: G\to A$ such that*

$$\sup_{x,y\in G}\left\| f\left(\frac{x+y}{2}\right)^2 - f(x)\, f(y)\right\| < \infty \tag{98}$$

is a solution of the Lobachevski equation

$$f\left(\frac{x+y}{2}\right)^2 = f(x)\, f(y), \quad x,y\in G. \tag{99}$$

Theorem 40. *Let G be a commutative group and A a finite-dimensional unital normed algebra without the zero divisors. Then every unbounded function $f: G\to A$ such that*

$$\sup_{x,y\in G}\left\| f(x+y) + f(x-y) - 2f(x)\, f(y)\right\| < \infty \tag{100}$$

is a solution of the cosine equation

$$f(x+y) + f(x-y) = 2f(x)\, f(y), \quad x,y\in G. \tag{101}$$

The next theorem, proved by Moszner in [56], generalizes Batko's result from [59].

Theorem 41. *Let G be a groupoid and A a finite-dimensional normed algebra without the zero divisors. Then every unbounded function $f: G\to A$ such that*

$$\sup_{x,y\in G}\left\| \left(f(x)+f(y)\right)\left(f(x+y) - f(x) - f(y)\right)\right\| < \infty \tag{102}$$

is a solution of the Dhombres equation

$$\left(f(x)+f(y)\right)\left(f(x+y) - f(x) - f(y)\right) = 0, \quad x,y\in G. \tag{103}$$

Using the method from the proof of Theorem 41, Moszner also got the superstability of the Mikusiński equation

$$f(x+y)\left(f(x+y) - f(x) - f(y)\right) = 0. \tag{104}$$

This result reads as follows.

Theorem 42. *Let G be a group and A a finite-dimensional normed algebra without the zero divisors. Then every unbounded function $f: G\to A$ such that*

$$\sup_{x,y\in G}\left\| f(x+y)\left(f(x+y) - f(x) - f(y)\right)\right\| < \infty \tag{105}$$

is a solution of (104).

The above theorem generalizes (to some extent) the following result, which has been obtained in [60], by another method of proof and under stronger assumptions.

Theorem 43. *Let $\varepsilon \geq 0$ and G be a commutative group. If a function $f : G \to \mathbb{C}$ satisfies*

$$\left|f(x+y)\left(f(x+y)-f(x)-f(y)\right)\right| \leq \varepsilon, \quad x, y \in G, \tag{106}$$

then f is additive or

$$\left|f(x)\right| \leq 2\sqrt{6\varepsilon}, \quad x \in G. \tag{107}$$

Chahbi in [61] has dealt with the equation

$$f\left(x+f(x)^k y+f(x)^k f(y)^k z\right)=f(x) f(y) f(z), \tag{108}$$

where $k \in \mathbb{N}$, and showed the following result on its superstability.

Theorem 44. *Let X be a linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $f : X \to \mathbb{K}$, $\varphi : X \times X \to \mathbb{R}_+$ hemicontinuous (see [61] for the definition) at the origin functions such that*

$$\begin{aligned}&\left|f\left(x+f(x)^k y+f(x)^k f(y)^k z\right)-f(x) f(y) f(z)\right| \\ &\quad \leq \varphi(y, z), \quad x, y, z \in X.\end{aligned} \tag{109}$$

Then f is bounded or satisfies (108) *for every $x, y, z \in X$.*

The form of (108) has been motivated by the Gołąb-Schinzel equation

$$f(x+f(x) y)=f(x) f(y) \tag{110}$$

and some other equations related to it. A survey on superstability results for such equations can be found in [3, pages 29–32] (for more information and further references on those equations see also [62]).

The following result comes from [63].

Theorem 45. *Assume that $\varepsilon > 0$ and $a \geq 1$. If a function $f : \mathbb{R} \to \mathbb{R}$ fulfills the inequality*

$$\left|f(x+y)-a^{x^2 y+x y^2} f(x) f(y)\right|<\varepsilon, \quad x, y \in \mathbb{R}, \tag{111}$$

then either f is bounded or it is a solution of the equation

$$f(x+y)=a^{x^2 y+x y^2} f(x) f(y), \quad x, y \in \mathbb{R}. \tag{112}$$

Let R be a ring (not necessarily commutative) uniquely divisible by 2 and $H := R^3$. Then $(H, \cdot)$, where

$$\begin{aligned}&(x, y, z) \cdot\left(x', y', z'\right) \\ &\quad:=\left(x+x', y+y', z+z'+\frac{1}{2}\left(x y'-y x'\right)\right), \\ &\qquad (x, y, z),\left(x', y', z'\right) \in H,\end{aligned} \tag{113}$$

is a noncommutative group, which in the case when $R = \mathbb{R}$ is isomorphic to the Heisenberg group. Denote by i a selfmap of H given by

$$i(x, y, z):=(y, x,-z), \quad(x, y, z) \in H. \tag{114}$$

With this notations, we have the following theorem (proved in [64]) on the superstability of a functional equation connected with the d'Alembert and Stetkær equations.

Theorem 46. *Assume that $\varepsilon > 0$. If a function $f : H \to \mathbb{C}$ fulfills the inequality*

$$\left|f(ab)+f(ai(b))-2 f(a) f(b)\right| \leq \varepsilon, \quad a, b \in H, \tag{115}$$

then either

$$\left|f(a)\right| \leq \frac{1+\sqrt{1+2\varepsilon}}{2}, \quad a \in H \tag{116}$$

or

$$f(ab)+f(ai(b))=2 f(a) f(b), \quad a, b \in H. \tag{117}$$

The next result has been proved in [65].

Theorem 47. *Assume that X is a normed space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and Y is a Banach algebra over $\mathbb{F}$ in which the norm is multiplicative, that is,*

$$\|ab\|=\|a\|\|b\|, \quad a, b \in Y. \tag{118}$$

If a mapping $f : X \to Y$ fulfills

$$\delta:=\sup_{x, y \in X}\|f(x+y)-f(x) f(y)\|<\infty, \tag{119}$$

then either

$$\|f(x)\| \leq \frac{1+\sqrt{1+4\delta}}{2}, \quad x \in X \tag{120}$$

or

$$f(x+y)=f(x) f(y), \quad x, y \in X. \tag{121}$$

In [66], Kim dealt with the pexiderized Lobachevski equation

$$f\left(\frac{x+y}{2}\right)^2=g(x) h(y) \tag{122}$$

and proved the following theorem.

Theorem 48. *Let $\varepsilon \geq 0$ and G be a uniquely 2-divisible commutative semigroup. If nonzero and nonconstant functions $f, g, h : G \to \mathbb{R}$ satisfy the inequality*

$$\left|f\left(\frac{x+y}{2}\right)^2-g(x) h(y)\right| \leq \varepsilon, \quad x, y \in G, \tag{123}$$

then either there exist $C_1, C_2, C_3 > 0$ such that

$$\left|g(x)\right| \leq C_1, \quad\left|h(x)\right| \leq C_2, \quad\left|f(x)\right| \leq C_3, \quad x \in G \tag{124}$$

or both g and h satisfy (99).

An immediate consequence of Theorem 48 is the following corollary.

Corollary 49. *Let $\varepsilon \geq 0$ and G be a uniquely 2-divisible commutative semigroup. If nonzero and nonconstant functions $f, g : G \to \mathbb{R}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g(x) f(y) \right| \leq \varepsilon, \quad x, y \in G, \tag{125}$$

then either there exist $C_1, C_2 > 0$ such that

$$|g(x)| \leq C_1, \quad |f(x)| \leq C_2, \quad x \in G \tag{126}$$

or both g and f satisfy (99).

The below superstability outcomes for the functional equations

$$f(x+y) = f(x) g(y) + f(y), \tag{127}$$

$$f(xy) = f(x) g(y) + f(y) \tag{128}$$

have been obtained in [67] ((127) and (128) with $g(x) \equiv 1$ become the classical Cauchy equations and therefore we exclude this case here).

Theorem 50. *Let V be a linear space and let functions $f, g : V \to \mathbb{C}$ be such that*

$$\sup_{x,y \in V} |f(x+y) - f(x) g(y) - f(y)| < \infty. \tag{129}$$

Then the following three statements hold:

(i) *if $f(x) \equiv 0$, then g is arbitrary;*

(ii) *if f is nonzero and bounded or $f(0) \neq 0$, then g is also bounded;*

(iii) *if f is unbounded, then $f(0) = 0$, g is unbounded, and* (127) *holds for all $x, y \in V$.*

Theorem 51. *Let V be a linear space. If functions $f, g : V \to \mathbb{C}$ are such that*

$$\sup_{x,y \in V} |f(xy) - f(x) g(y) - f(y)| < \infty, \tag{130}$$

then the following three statements hold:

(i) *if $f(x) \equiv 0$, then g is arbitrary;*

(ii) *if f is nonzero and bounded or $f(1) \neq 0$, then g is also bounded;*

(iii) *if f is unbounded, then $f(1) = 0$, g is unbounded, and* (128) *holds for all $x, y \in V$.*

The next two theorems have been obtained in [68]. It is assumed in them that X is a commutative group, Λ is a finite subgroup of the group of automorphisms of X (the action of $\lambda \in \Lambda$ on $x \in X$ is denoted by λx), and N is the cardinality of Λ.

Theorem 52. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $f, g, h : X \to \mathbb{K}$, the function*

$$\begin{aligned} & X \times X \ni (x, y) \\ & \longmapsto \frac{1}{N} \sum_{\lambda \in \Lambda} f(x + \lambda y) - f(x) g(y) - h(y) \in \mathbb{K} \end{aligned} \tag{131}$$

is bounded, and the function f is unbounded, then the function g satisfies the functional equation

$$\frac{1}{N} \sum_{\lambda \in \Lambda} g(x + \lambda y) = g(x) g(y), \quad x, y \in X. \tag{132}$$

Theorem 53. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If $f, g, h : X \to \mathbb{K}$, the function*

$$\begin{aligned} & X \times X \ni (x, y) \\ & \longmapsto \frac{1}{N} \sum_{\lambda \in \Lambda} f(x + \lambda y) - f(y) g(x) - h(x) \in \mathbb{K} \end{aligned} \tag{133}$$

is bounded, and the function f is unbounded, then the function g satisfies (132).

The next three theorems do not actually provide superstability results in the sense of Definition 36. However, we present them here, because they seem to be of some interest and are attempts to extend the notion of superstability analogously to the notion of φ-hyperstability.

The subsequent theorem, proved in [69], gives a partial affirmative answer to a problem posed by Th. M. Rassias during the 31st ISFE ($\Re z$ and $\Im z$ denote the real and imaginary parts of a complex number z, resp.).

Theorem 54. *Let $(S, \cdot)$ be a commutative semigroup, $\varphi : S^2 \to \mathbb{R}_+$, $\psi : S \to \mathbb{R}_+$, and $f : S \to \{z \in \mathbb{C} : -\pi < \Im z \leq \pi\}$ functions such that*

$$\begin{aligned} |f(x \cdot y) - f(x) - f(y)| \leq \varphi(x, y), \quad x, y \in S, \\ |f(x)| \leq \psi(x), \quad x \in S. \end{aligned} \tag{134}$$

Assume also that there exists a $p \in \{s \in S : \Re f(s) > 0\}$ with

$$\begin{aligned} \sum_{m=0}^{\infty} \varphi\left(p, p^{m+1}\right) < \infty, \\ \psi(x \cdot p) \leq \psi(x), \quad x \in S. \end{aligned} \tag{135}$$

Then f satisfies the Cauchy equation

$$f(x \cdot y) = f(x) + f(y), \quad x, y \in S. \tag{136}$$

The next result has been proved in [70].

Theorem 55. *Let G be a commutative group and $\varphi : G \to \mathbb{R}_+$. If $f : G \to \mathbb{C}$ is an unbounded function such that*

$$\begin{aligned} & |f(x+y+z) + f(x+y-z) + f(y+z-x) \\ & \quad + f(z+x-y) - 4 f(x) f(y) f(z)| \\ & \quad \leq \varphi(x), \quad x, y, z \in G, \end{aligned} \tag{137}$$

then

$$f(x+y+z)+f(x+y-z)+f(y+z-x)+f(z+x-y) = 4f(x)f(y)f(z), \quad x,y,z \in G. \tag{138}$$

The last presented theorem is the main result of [71] and includes a few outcomes from [72–74].

Theorem 56. *Assume that G is a commutative group and $\varphi : G \to \mathbb{R}_+$. If nonzero functions $f, g, h, k : G \to \mathbb{C}$ fulfill*

$$|f(x+y) - g(x-y) - 2h(x)k(y)| \le \varphi(x), \quad x \in G, \tag{139}$$

then either k is bounded or there is a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of G such that the limit

$$\lim_{n\to\infty} \frac{k(y_n+y)+k(y_n-y)}{k(y_n)} =: l(y) \tag{140}$$

exists for every $y \in G$ and h, l satisfy the functional equation

$$h(x+y)+h(x-y) = h(x)l(y), \quad x,y \in G. \tag{141}$$

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