Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 383692, 8 pages
http://dx.doi.org/10.1155/2013/383692

Research Article

Univex Interval-Valued Mapping with Differentiability and Its Application in Nonlinear Programming

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Received 25 November 2012; Accepted 26 May 2013

Academic Editor: Lotfollah Najjar

Interval-valued univex functions are introduced for differentiable programming problems. Optimality and duality results are derived for a class of generalized convex optimization problems with interval-valued univex functions.

1. Introduction

Imposing the uncertainty upon the optimization problems is an interesting research topic. The uncertainty may be interpreted as randomness, fuzziness, or interval-valued fuzziness. The randomness occurring in the optimization problems is categorized as the stochastic optimization problems, and the imprecision (fuzziness) occurring in the optimization problems is categorized as the fuzzy optimization problems. In order to perfectly match the real situations, interval-valued optimization problems may provide an alternative choice for considering the uncertainty into the optimization problems. That is to say, the coefficients in the interval-valued optimization problems are assumed as closed intervals. Many approaches for interval-valued optimization problems have been explored in considerable details; see, for example, [1–3]. Recently, Wu has extended the concept of convexity for real-valued functions to LU-convexity for interval-valued functions, then he has established the Karush-Tucker conditions [4–6] for an optimization problem with interval-valued objective functions under the assumption of LU-convexity. Similar to the concept of nondominated solution in vector optimization problems, Wu has proposed a solution concept in optimization problems with interval-valued objective functions based on a partial ordering on the set of all closed intervals, then the interval-valued Wolfe duality theory [7] and Lagrangian duality theory [8] for interval-valued optimization problems have been proposed. Recently, Wu [9] has studied the duality theory for interval-valued linear programming problems.

In 1981, Hanson [10] introduced the concept of invexity and established Karush-Tucker type sufficient optimality conditions for a nonlinear programming problem. In [11], Kaul et al. considered a differentiable multiobjective programming problem involving generalized type I functions. They investigated Karush-Tucker type necessary and sufficient conditions and obtained duality results under generalized type I functions. The class of B-vex functions has been introduced by Bector and Singh [12] as a generalization of convex functions, and duality results are established for vector valued B-invex programming in [13]. Bector et al. [14] introduced the concept of univex functions as a generalization of B-vex functions introduced by Bector et al. [15]. Combining the concepts of type I and univex functions, Rueda et al. [16] gave optimality conditions and duality results for several mathematical programming problems. Aghezzaf and Hachimi [17] introduced classes of generalized type I functions for a differentiable multiobjective programming problem and derived some Mond-Weir type duality results under the above generalized type I assumptions. Gulati et al. [18] introduced the concept of (F, α, ρ, d)-V-type I functions and also studied sufficiency optimality conditions and duality multiobjective programming problems.

This paper aims at extending the Karush-Tucker optimality conditions to nonconvex optimization problem with interval-valued functions. First, we extend the concept of univexity for a real-valued function to an interval-valued function and present the concept of interval-valued univex functions. Then, the Karush-Tucker optimality conditions are proposed for an interval-valued function under the assumption of interval-valued univexity.

2. Preliminaries

Let one denotes by $\mathcal{I}$ the class of all closed intervals in R. $A = [a^L, a^U] \in \mathcal{I}$ denotes a closed interval, where a^L and a^U mean the lower and upper bounds of A, respectively. For every $a \in R$, we denote $a = [a, a]$.

Definition 1. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be in $\mathcal{I}$; one has

(i) $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$;

(ii) $-A = \{-a : a \in A\} = [-a^U, -a^L]$;

(iii) $A \times B = \{ab : a \in A \text{ and } b \in B\} = [\min_{ab}, \max_{ab}]$, where $\min_{ab} = \min\{a^Lb^L, a^Lb^U, a^Ub^L, a^Ub^U\}$ and $\max_{ab} = \max\{a^Lb^L, a^Lb^U, a^Ub^L, a^Ub^U\}$.

Then, we can see that

$$A - B = A + (-B) = \left[a^L - b^U, a^U - b^L\right],$$
$$kA = \{ka : a \in A\} = \begin{cases} \left[ka^L, ka^U\right] & \text{if } k \geq 0, \\ \left[ka^U, ka^L\right] & \text{if } k < 0, \end{cases} \tag{1}$$

where k is a real number.

By using *Hausdorff metric*, Neumaier [19] has proposed *Hausdorff metric* between the two closed intervals A and B as follows:

$$d_H(A, B) = \max\left\{\left|a^L - b^L\right|, \left|a^U - b^U\right|\right\}. \tag{2}$$

Definition 2. Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in R. One writes $A \preceq B$ if and only if $a^L \leq b^L$ and $a^U \leq b^U$, $A \prec B$ if and only if $A \preceq B$ and $A \neq B$, that is, the following (a1), (a2), or (a3) is satisfied:

(a1) $a^L < b^L$ and $a^U \leq b^U$;

(a2) $a^L \leq b^L$ and $a^U < b^U$;

(a3) $a^L < b^L$ and $a^U < b^U$.

Definition 3 (see [20]). Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals, the gH-difference of A and B is defined by

$$\left[a^L, a^U\right] \ominus_g \left[b^L, b^U\right] = \left[\min\left(a^L - b^L, a^U - b^U\right), \max\left(a^L - b^L, a^U - b^U\right)\right]. \tag{3}$$

For example, $[1, 3] \ominus_g [0, 3] = [0, 1]$, $[0, 3] \ominus_g [1, 3] = [-1, 0]$. And $a - b = [a, a] \ominus_g [b, b] = [a - b, a - b] = a - b$.

Proposition 4. (i) *For every $A, B \in \mathcal{I}$, $A \ominus_g B$ always exists and $A \ominus_g B \in \mathcal{I}$.*

(ii) *$A \ominus_g B \preceq 0$ if and only if $A \preceq B$.*

3. Interval-Valued Univex Functions

Definition 5 (interval-valued function). The function $f : \Omega \to \mathcal{I}$ is called an interval-valued function, where $\Omega \subseteq R^n$. Then, $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is a closed interval in R for each $\mathbf{x} \in R^n$, and $f(\mathbf{x})$ can be also written as $f(\mathbf{x}) = [f^L(\mathbf{x}), f^U(\mathbf{x})]$, where $f^L(\mathbf{x})$ and $f^U(\mathbf{x})$ are two real-valued functions defined on R^n and satisfy $f^L(\mathbf{x}) \leq f^U(\mathbf{x})$ for every $\mathbf{x} \in \Omega$.

Definition 6 (continuity of an interval-valued function). The function $f : \Omega \subseteq R^n \to \mathcal{I}$ is said to be continuous at $x \in \Omega$ if both $f^L(\mathbf{x})$ and $f^U(\mathbf{x})$ are continuous functions of $\mathbf{x}$.

The concept of gH-derivative of a function $f : (a, b) \to \mathcal{I}$ is defined in [19].

Definition 7. Let $x_0 \in (a, b)$ and h be such that $x_0 + h \in (a, b)$, then the gH-derivative of a function $f : (a, b) \to \mathcal{I}$ at x_0 is defined as

$$f'(x_0) = \lim_{x \to 0}\left[f(x_0 + h) \ominus_g f(x_0)\right]. \tag{4}$$

If $f'(x_0) \in \mathcal{I}$ exists, then we say that f is generalized Hukuhara differentiable (gH-differentiable, for short) at x_0.

Moreover, [21] also proved the following theorem.

Theorem 8. *Let $f : (a, b) \to \mathcal{I}$ be such that $f(x) = [f^L(x), f^U(x)]$. The function $f(x)$ is gH-differentiable if and only if $f^L(x)$ and $f^U(x)$ are differentiable real-valued functions. Furthermore,*

$$f'(x) = \left[\min\left\{\left(f^L\right)'(x), \left(f^U\right)'(x)\right\}, \max\left\{\left(f^L\right)'(x), \left(f^U\right)'(x)\right\}\right]. \tag{5}$$

Definition 9 (gradient of an interval-valued function). Let $f(\mathbf{x})$ be an interval-valued function defined on Ω, where Ω is an open subset of R^n. Let D_{x_i} $(i = 1, 2, \ldots, n)$ stand for the partial differentiation with respect to the ith variable x_i. Assume that $f^L(\mathbf{x})$ and $f^U(\mathbf{x})$ have continuous partial derivatives so that $D_{x_i}f^L(\mathbf{x})$ and $D_{x_i}f^U(\mathbf{x})$ are continuous. For $i = 1, 2, \ldots, n$, define

$$D_{x_i}f(\mathbf{x}) = \left[\min\left(D_{x_i}f^L(\mathbf{x}), D_{x_i}f^U(\mathbf{x})\right), \max\left(D_{x_i}f^L(\mathbf{x}), D_{x_i}f^U(\mathbf{x})\right)\right]. \tag{6}$$

We will say that $f(\mathbf{x})$ is differentiable at $\mathbf{x}$, and we write

$$\nabla f(\mathbf{x}) = \left(D_{x_1}f(\mathbf{x}), D_{x_2}f(\mathbf{x}), \ldots, D_{x_n}f(\mathbf{x})\right)^t. \tag{7}$$

We call $\nabla f(\mathbf{x})$ the gradient of the interval-valued univex function at $\mathbf{x}$.

Example 10. Let $f : \mathbf{R}^2 \to \mathcal{I}$ defined by $f(\mathbf{x}) = [x_1^2 + x_2^2, 2x_1^2 + 2x_2^2 + 3]$. So $f^L(\mathbf{x}) = x_1^2 + x_2^2$ and $f^U(\mathbf{x}) = 2x_1^2 + 2x_2^2 + 3$. $D_{x_1} f^L(\mathbf{x}) = 2x_1$, $D_{x_2} f^L(\mathbf{x}) = 2x_2$, $D_{x_1} f^U(\mathbf{x}) = 4x_1$, $D_{x_2} f^U(\mathbf{x}) = 4x_2$. Thus,

$$D_{x_1} f(\mathbf{x}) = \begin{cases} [2x_1, 4x_1] & \text{if } x_1 \geq 0, \\ [4x_1, 2x_1] & \text{if } x_1 < 0, \end{cases}$$
$$D_{x_2} f(\mathbf{x}) = \begin{cases} [2x_2, 4x_2] & \text{if } x_2 \geq 0, \\ [4x_2, 2x_2] & \text{if } x_2 < 0. \end{cases} \tag{8}$$

Thus,

$$\nabla f(\mathbf{x}) = \begin{cases} ([2x_1, 4x_1], [2x_2, 4x_2])^t & \text{if } x_1 \geq 0, x_2 \geq 0, \\ ([2x_1, 4x_1], [4x_2, 2x_2])^t & \text{if } x_1 \geq 0, x_2 < 0, \\ ([4x_1, 2x_1], [2x_2, 4x_2])^t & \text{if } x_1 < 0, x_2 \geq 0, \\ ([4x_1, 2x_1], [4x_2, 4x_2])^t & \text{if } x_1 < 0, x_2 < 0. \end{cases} \tag{9}$$

Further,

$$\nabla^L f(\mathbf{x}) = \begin{cases} (2x_1, 2x_2)^t & \text{if } x_1 \geq 0, x_2 \geq 0, \\ (2x_1, 4x_2)^t & \text{if } x_1 \geq 0, x_2 < 0, \\ (4x_1, 2x_2)^t & \text{if } x_1 < 0, x_2 \geq 0, \\ (4x_1, 4x_2)^t & \text{if } x_1 < 0, x_2 < 0. \end{cases}$$
$$\nabla^U f(\mathbf{x}) = \begin{cases} (4x_1, 4x_2)^t & \text{if } x_1 \geq 0, x_2 \geq 0, \\ (4x_1, 2x_2)^t & \text{if } x_1 \geq 0, x_2 < 0, \\ (2x_1, 4x_2)^t & \text{if } x_1 < 0, x_2 \geq 0, \\ (2x_1, 4x_2)^t & \text{if } x_1 < 0, x_2 < 0. \end{cases} \tag{10}$$

Remark 11. If $f^L = f^U$, then $\nabla f(\mathbf{x})$ of interval-valued functions is the extension of $\nabla f(\mathbf{x})$, where $f : \Omega \to R$.

The concept of convexity plays an important role in the optimization theory. In recent years, the concept of convexity has been generalized in several directions by using novel and innovative techniques. An important generalization of convex functions is the introduction of univex functions, which was introduced by Bector et al. [15].

Let K be a nonempty open set in R^n, and let $f : K \to R$, $\eta : K \times K \to R^n$, $\Phi : R \to R$, and $b : K \times K \times [0, 1] \to R^+$, $b = b(\mathbf{x}, \mathbf{y}, \lambda)$. If the function f is differentiable, then b does not depend on λ; see [12] or [15].

Definition 12. A differentiable real-valued function f is said to be univex at $\mathbf{y} \in K$ with respect to η, Φ, b if for all $\mathbf{x} \in K$

$$b(\mathbf{x}, \mathbf{y}) \Phi [f(\mathbf{x}) - f(\mathbf{y})] \geq \eta^t(\mathbf{x}, \mathbf{y}) \nabla f(\mathbf{y}). \tag{11}$$

Let K be a nonempty open set in R^n, and let $f : K \to \mathcal{I}$ be an interval-valued function, $\eta : K \times K \to R^n$, $\Phi : \mathcal{I} \to \mathcal{I}$, and $b : K \times K \times [0, 1] \to R^+$, $b = b(\mathbf{x}, \mathbf{y}, \lambda)$.

Definition 13 (interval-valued univex function). A differentiable interval-valued function f is said to be univex at $\mathbf{y} \in K$ with respect to η, Φ, b if for all $\mathbf{x} \in K$

$$b(\mathbf{x}, \mathbf{y}) \Phi [f(\mathbf{x}) \ominus_g f(\mathbf{y})] \geq \eta^t(\mathbf{x}, \mathbf{y}) \nabla f(\mathbf{y}). \tag{12}$$

Remark 14. (i) An interval-valued univex function is the extension of a univex function by $f^L = f^U$.

(ii) $\Phi : \mathcal{I} \to \mathcal{I}$ could be deduced from $\phi : R \to R$ by $\Phi(A) := \{y : \exists x \in A, \phi(x) = y, y \in R\}$.

Example 15. Consider the real-valued function ϕ_1 given by $\phi_1(x) = x + 1$, $x \in R$, then we can obtain $\Phi_1([a^L, a^U]) = [a^L + 1, a^U + 1]$. If $\phi_2(x) = |x|$, $x \in R$. Then

$$\Phi_2([a^L, a^U]) = \begin{cases} [a^L, a^U] & \text{if } a^L \geq 0, \\ [-a^U, -a^L] & \text{if } a^U \leq 0, \\ [0, \max(-a^L, a^U)] & \text{if } a^L < 0,\ a^U \geq 0. \end{cases} \tag{13}$$

Example 16. Let $f(x) = [x^2, 2x^2 + 3]$, $x \in R$, $b = 1$, $\eta(x, y) = x - y$, $\Phi = \Phi_2$, then $f(x)$ is univex with respect to b, η, and Φ.

Example 17. Let $f(x) = [x^3, x^3 + 1]$, $x \in R$,

$$b(x, y) = \begin{cases} \dfrac{y^2}{x - y} & \text{if } x \geq y, \\ 0 & \text{if } x \leq y, \end{cases}$$
$$\eta(x, y) = \begin{cases} x^2 + y^2 + xy & \text{if } x \geq y, \\ x - y & \text{if } x \leq y. \end{cases} \tag{14}$$

Let $\Phi : \mathcal{I} \to \mathcal{I}$ be defined by $\Phi([a^L, a^U]) = 3[a^L, a^U]$, then $f(x)$ is univex with respect to b, η and Φ.

4. Optimality Criteria

Let $f(\mathbf{x}), g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})$ be differentiable interval-valued functions defined on a nonempty open set $X \subseteq R^n$. Throughout this paper we consider the following primal problem (P):

$$\begin{aligned} &\min \quad f(\mathbf{x}) \\ &\text{s.t.} \quad g(\mathbf{x}) \preceq 0, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{P}$$

Let $P := \{\mathbf{x} \in X : g(\mathbf{x}) \preceq 0, i = 1, 2, \ldots, m\}$. We say $\mathbf{x}^*$ is an optimal solution of (P) if $f(\mathbf{x}) \succeq f(\mathbf{x}^*)$ for all P-feasible $\mathbf{x}$. In this section, we obtain sufficient optimality conditions for a feasible solution $\mathbf{x}^*$ to be efficient or properly efficient for (P) in the form of the following theorems.

Theorem 18. *Let* $\mathbf{x}^*$ *be* P*-feasible. Suppose that*

(i) *there exist* η, Φ_0, b_0, Φ_i, b_i, $i = 1, 2, \ldots, m$ *such that*

$$b_0(\mathbf{x}, \mathbf{x}^*)\Phi_0\left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*)\right] \succeq \eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*), \quad (15)$$

$$-b_i(\mathbf{x}, \mathbf{x}^*)\Phi_i\left[g_i(\mathbf{x}^*)\right] \succeq \eta^t(\mathbf{x}, \mathbf{x}^*)\nabla g_i(\mathbf{x}^*) \quad (16)$$

for all feasible $\mathbf{x}$*;*

(ii) *there exist* $\mathbf{y}^* = (y_1, y_2, \ldots, y_m)^t \in R^m$ *such that*

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*) = 0, \quad (17)$$

$$\mathbf{y}^* \geq 0. \quad (18)$$

Further, suppose that

$$\Phi_0(A) \succeq 0 \Longrightarrow A \succeq 0, \quad (19)$$

$$A \preceq 0 \Longrightarrow \Phi_i(A) \succeq 0, \quad (20)$$

$$b_0(\mathbf{x}, \mathbf{x}^*) > 0, \qquad b_i(\mathbf{x}, \mathbf{x}^*) > 0 \quad (21)$$

for all feasible $\mathbf{x}$*. Then,* $\mathbf{x}^*$ *is an optimal solution of* (P)*.*

Proof. Let $\mathbf{x}$ be P-feasible. Then,

$$g_i(\mathbf{x}) \preceq 0. \quad (22)$$

From (20), we conclude that

$$\Phi_i\left[g_i(\mathbf{x})\right] \succeq 0. \quad (23)$$

Thus,

$$\begin{aligned} &\Phi_i^L\left[g_i(\mathbf{x})\right] \geq 0, \\ &\Phi_i^U\left[g_i(\mathbf{x})\right] \geq 0. \end{aligned} \quad (24)$$

By (15) and Definition 2, we have

$$\begin{aligned} &\left\{b_0(\mathbf{x}, \mathbf{x}^*)\Phi_0\left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*)\right]\right\}^L \geq \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*)\right\}^L, \\ &\left\{b_0(\mathbf{x}, \mathbf{x}^*)\Phi_0\left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*)\right]\right\}^U \geq \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*)\right\}^U. \end{aligned} \quad (25)$$

From (17),

$$\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*) + \eta^t(\mathbf{x}, \mathbf{x}^*)\sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*) = 0. \quad (26)$$

It follows from Definition 2 that

$$\begin{aligned} &\left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*)\right\}^L + \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^L = 0, \\ &\left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*)\right\}^U + \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^U = 0. \end{aligned} \quad (27)$$

It is equivalent to

$$\begin{aligned} &\left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*)\right\}^L = -\left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^L, \\ &\left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*)\right\}^U = -\left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^U. \end{aligned} \quad (28)$$

From (16), we have

$$\begin{aligned} &\left\{-b_i(\mathbf{x}, \mathbf{x}^*)\Phi_i\left[g_i(\mathbf{x}^*)\right]\right\}^L \geq \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla g_i(\mathbf{x}^*)\right\}^L, \\ &\left\{-b_i(\mathbf{x}, \mathbf{x}^*)\Phi_i\left[g_i(\mathbf{x}^*)\right]\right\}^U \geq \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla g_i(\mathbf{x}^*)\right\}^U. \end{aligned} \quad (29)$$

From Definition 1, we have

$$\begin{aligned} &-\left\{b_i(\mathbf{x}, \mathbf{x}^*)\Phi_i\left[g_i(\mathbf{x}^*)\right]\right\}^U \geq \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla g_i(\mathbf{x}^*)\right\}^L, \\ &-\left\{b_i(\mathbf{x}, \mathbf{x}^*)\Phi_i\left[g_i(\mathbf{x}^*)\right]\right\}^L \geq \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla g_i(\mathbf{x}^*)\right\}^U. \end{aligned} \quad (30)$$

Thus,

$$\begin{aligned} &\left\{b_0(\mathbf{x}, \mathbf{x}^*)\Phi_0\left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*)\right]\right\}^L \\ &\quad \geq \left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\nabla f(\mathbf{x}^*)\right\}^L \\ &\quad = -\left\{\eta^t(\mathbf{x}, \mathbf{x}^*)\sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^L \\ &\quad \geq \sum_{i=1}^{m}\left\{b_i(\mathbf{x}, \mathbf{x}^*)\Phi_i\left[g_i(\mathbf{x}^*)\right]\right\}^U \\ &\quad \geq \sum_{i=1}^{m}\left\{b_i(\mathbf{x}, \mathbf{x}^*)\Phi_i\left[g_i(\mathbf{x}^*)\right]\right\}^L \\ &\quad \geq 0, \\ &\left\{b_0(\mathbf{x}, \mathbf{x}^*)\Phi_0\left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*)\right]\right\}^U \\ &\quad \geq \left\{b_0(\mathbf{x}, \mathbf{x}^*)\Phi_0\left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*)\right]\right\}^L \\ &\quad \geq 0. \end{aligned} \quad (31)$$

So,

$$b_0(\mathbf{x}, \mathbf{x}^*)\Phi_0\left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*)\right] \succeq 0. \quad (32)$$

From (21), it follows that

$$\Phi_0\left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*)\right] \succeq 0. \quad (33)$$

By (19),

$$f(\mathbf{x}) \ominus_g f(\mathbf{x}^*) \succeq 0. \quad (34)$$

From Proposition 4, it follows that

$$f(\mathbf{x}) \succeq f(\mathbf{x}^*). \quad (35)$$

Therefore, $\mathbf{x}^*$ is an optimal solution of (P). □

Theorem 19. *Let* $\mathbf{x}^*$ *be* P*-feasible. Suppose that*

(i) *there exist* η, Φ_0, b_0, Φ_i, b_i, $i = 1, 2, \ldots, m$ *such that*

$$\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla f(\mathbf{x}^*) \succeq 0 \Longrightarrow b_0(\mathbf{x}, \mathbf{x}^*) \Phi_0 \left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*) \right] \succeq 0, \tag{36}$$

$$-b_i(\mathbf{x}, \mathbf{x}^*) \Phi_i [g_i(\mathbf{x}^*)] \preceq 0 \Longrightarrow \eta^t(\mathbf{x}, \mathbf{x}^*) \nabla g_i(\mathbf{x}^*) \preceq 0 \tag{37}$$

for all feasible $\mathbf{x}$*;*

(ii) *there exist* $\mathbf{y}^* = (y_1, y_2, \ldots, y_m)^t \in R^m$ *such that*

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*) = 0, \tag{38}$$

$$\mathbf{y}^* \geq 0. \tag{39}$$

Further, suppose that

$$\Phi_0(A) \succeq 0 \Longrightarrow A \succeq 0, \tag{40}$$

$$A \preceq 0 \Longrightarrow \Phi_i(A) \succeq 0, \tag{41}$$

$$b_0(\mathbf{x}, \mathbf{x}^*) > 0, \qquad b_i(\mathbf{x}, \mathbf{x}^*) > 0 \tag{42}$$

for all feasible $\mathbf{x}$*. Then,* $\mathbf{x}^*$ *is an optimal solution of* (P)*.*

Proof. Let $\mathbf{x}$ be P-feasible. Then, $g_i(\mathbf{x}^*) \preceq 0$, from (41), we obtain that

$$\Phi_i[g_i(\mathbf{x}^*)] \succeq 0. \tag{43}$$

So,

$$-b_i(\mathbf{x}, \mathbf{x}^*) \Phi_i [g_i(\mathbf{x}^*)] \preceq 0. \tag{44}$$

By (37),

$$\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla g_i(\mathbf{x}^*) \preceq 0. \tag{45}$$

Thus,

$$-\sum_{i=1}^{m} y_i \eta^t(\mathbf{x}, \mathbf{x}^*) \nabla g_i(\mathbf{x}^*) \succeq 0. \tag{46}$$

Then, we have

$$\begin{aligned} &-\left\{\sum_{i=1}^{m} y_i \eta^t(\mathbf{x}, \mathbf{x}^*) \nabla g_i(\mathbf{x}^*)\right\}^U \\ &\quad = \left\{-\sum_{i=1}^{m} y_i \eta^t(\mathbf{x}, \mathbf{x}^*) \nabla g_i(\mathbf{x}^*)\right\}^L \geq 0, \\ &-\left\{\sum_{i=1}^{m} y_i \eta^t(\mathbf{x}, \mathbf{x}^*) \nabla g_i(\mathbf{x}^*)\right\}^L \\ &\quad = \left\{-\sum_{i=1}^{m} y_i \eta^t(\mathbf{x}, \mathbf{x}^*) \nabla g_i(\mathbf{x}^*)\right\}^U \geq 0. \end{aligned} \tag{47}$$

From (38) and Definition 2, it follows that

$$\begin{aligned} \{\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla f(\mathbf{x}^*)\}^L + \left\{\eta^t(\mathbf{x}, \mathbf{x}^*) \sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^L = 0, \\ \{\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla f(\mathbf{x}^*)\}^U + \left\{\eta^t(\mathbf{x}, \mathbf{x}^*) \sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^U = 0. \end{aligned} \tag{48}$$

It is equivalent to

$$\begin{aligned} \{\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla f(\mathbf{x}^*)\}^L = -\left\{\eta^t(\mathbf{x}, \mathbf{x}^*) \sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^L, \\ \{\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla f(\mathbf{x}^*)\}^U = -\left\{\eta^t(\mathbf{x}, \mathbf{x}^*) \sum_{i=1}^{m} y_i \nabla g(\mathbf{x}^*)\right\}^U. \end{aligned} \tag{49}$$

Therefore,

$$\begin{aligned} \{\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla f(\mathbf{x}^*)\}^L \geq 0, \\ \{\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla f(\mathbf{x}^*)\}^U \geq 0. \end{aligned} \tag{50}$$

From Definition 2, we obtain that

$$\eta^t(\mathbf{x}, \mathbf{x}^*) \nabla f(\mathbf{x}^*) \succeq 0. \tag{51}$$

By (36),

$$b_0(\mathbf{x}, \mathbf{x}^*) \Phi_0 \left[f(\mathbf{x}) \ominus_g f(\mathbf{x}^*) \right] \succeq 0. \tag{52}$$

Then, from (40) and (42), we have

$$f(\mathbf{x}) \ominus_g f(\mathbf{x}^*) \succeq 0. \tag{53}$$

From Proposition 4, it follows that

$$f(\mathbf{x}) \succeq f(\mathbf{x}^*). \tag{54}$$

Therefore, $\mathbf{x}^*$ is an optimal solution of (P). □

5. Duality

Consider the following:

$$\begin{aligned} \max \quad & f(\mathbf{u}) \\ \text{s.t.} \quad & \nabla f(\mathbf{u}) + \sum_{i=1}^{m} y_i \nabla g_i(\mathbf{u}) = 0, \\ & y_i \nabla g_i(\mathbf{u}) \succeq 0 \\ & y_i \geq 0. \end{aligned} \tag{D}$$

Theorem 20 (weak duality). *Let* $\mathbf{x}$ *be* P*-feasible, and let* $(\mathbf{u}, \mathbf{y})$ *be* D*-feasible. Assume that there exist* η, Φ_0, b_0, Φ_i, b_i, $i = 1, 2, \ldots, m$ *such that*

$$\begin{aligned} b_0(\mathbf{x}, \mathbf{u}) \Phi_0 \left[f(\mathbf{x}) \ominus_g f(\mathbf{u}) \right] \succeq \eta^t(\mathbf{x}, \mathbf{u}) \nabla f(\mathbf{u}), \\ -b_i(\mathbf{x}, \mathbf{u}) \Phi_i [g_i(\mathbf{u})] \succeq \eta^t(\mathbf{x}, \mathbf{u}) \nabla g_i(\mathbf{u}) \end{aligned} \tag{55}$$

at $\mathbf{u}$*;*

$$\Phi_0(A) \geq 0 \Longrightarrow A \geq 0, \qquad b_0(\mathbf{x},\mathbf{u}) > 0, \quad b_i(\mathbf{x},\mathbf{u}) \geq 0 \tag{56}$$

and $\sum_{i=1}^m b_i(\mathbf{x},\mathbf{u}) y_i \Phi_i(g_i(\mathbf{u})) \geq 0$. *Then,* $f(\mathbf{x}) \succeq f(\mathbf{u})$.

Proof. It is similar to the proof of Theorem 18. □

Theorem 21 (weak duality). *Let* $\mathbf{x}$ *be* P*-feasible, and let* $(\mathbf{u},\mathbf{y})$ *be* D*-feasible. Assume that there exist* η, Φ_0, b_0, Φ_1, b_1 *such that*

$$\eta^t(\mathbf{x},\mathbf{u}) \nabla f(\mathbf{u}) \succeq 0 \Longrightarrow b_0(\mathbf{x},\mathbf{u}) \Phi_0 \left[f(\mathbf{x}) \ominus_g f(\mathbf{u}) \right] \succeq 0, \tag{57}$$

$$-b_1(\mathbf{x},\mathbf{u}) \Phi_1 \left[\sum_{i=1}^m y_i g_i(\mathbf{u}) \right] \preceq 0 \Longrightarrow \sum_{i=1}^m y_i \eta^t(\mathbf{x},\mathbf{u}) \nabla g_i(\mathbf{u}) \preceq 0 \tag{58}$$

at $\mathbf{u}$*;*

$$\Phi_0(A) \geq 0 \Longrightarrow A \geq 0, \tag{59}$$

$$A \geq 0 \Longrightarrow \Phi_1(A) \geq 0, \tag{60}$$

$$b_0(\mathbf{x},\mathbf{u}) > 0, \qquad b_1(\mathbf{x},\mathbf{u}) \geq 0. \tag{61}$$

Then, $f(\mathbf{x}) \succeq f(\mathbf{u})$.

Proof. Since $(\mathbf{u},\mathbf{y})$ is D-feasible, then $y_i^t \nabla g_i(\mathbf{u}) \succeq 0$, from (60) and (61),

$$-b_1(\mathbf{x},\mathbf{u}) \Phi_1 \left[\sum_{i=1}^m y_i g_i(\mathbf{u}) \right] \preceq 0. \tag{62}$$

Then, we have

$$\sum_{i=1}^m y_i \eta^t(\mathbf{x},\mathbf{u}) \nabla g_i(\mathbf{u}) \preceq 0. \tag{63}$$

Thus,

$$\begin{gathered} -\sum_{i=1}^m y_i \eta^t(\mathbf{x},\mathbf{u}) \nabla g_i(\mathbf{u}) \succeq 0, \\ -\left\{ \sum_{i=1}^m y_i \eta^t(\mathbf{x},\mathbf{u}) \nabla g_i(\mathbf{u}) \right\}^U = \left\{ -\sum_{i=1}^m y_i \eta^t(\mathbf{x},\mathbf{u}) \nabla g_i(\mathbf{u}) \right\}^L \geq 0, \\ -\left\{ \sum_{i=1}^m y_i \eta^t(\mathbf{x},\mathbf{u}) \nabla g_i(\mathbf{u}) \right\}^L = \left\{ -\sum_{i=1}^m y_i \eta^t(\mathbf{x},\mathbf{u}) \nabla g_i(\mathbf{u}) \right\}^U \geq 0. \end{gathered} \tag{64}$$

Since $(\mathbf{u},\mathbf{y})$ is D-feasible we can obtain that,

$$\nabla f(\mathbf{u}) + \sum_{i=1}^m y_i \nabla g_i(\mathbf{u}) = 0. \tag{65}$$

So,

$$\eta^t(\mathbf{x},\mathbf{u}) \nabla f(\mathbf{u}) + \eta^t(\mathbf{x},\mathbf{u}) \sum_{i=1}^m y_i \nabla g_i(\mathbf{u}) = 0. \tag{66}$$

By Definition 2, it follows that

$$\begin{gathered} \left\{ \eta^t(\mathbf{x},\mathbf{u}) \nabla f(\mathbf{u}) \right\}^L + \left\{ \eta^t(\mathbf{x},\mathbf{u}) \sum_{i=1}^m y_i \nabla g_i(\mathbf{u}) \right\}^L = 0, \\ \left\{ \eta^t(\mathbf{x},\mathbf{u}) \nabla f(\mathbf{u}) \right\}^U + \left\{ \eta^t(\mathbf{x},\mathbf{u}) \sum_{i=1}^m y_i \nabla g_i(\mathbf{u}) \right\}^U = 0. \end{gathered} \tag{67}$$

Therefore,

$$\begin{gathered} \left\{ \eta^t(\mathbf{x},\mathbf{u}) \nabla f(\mathbf{u}) \right\}^L = -\left\{ \eta^t(\mathbf{x},\mathbf{u}) \sum_{i=1}^m y_i \nabla g_i(\mathbf{u}) \right\}^L \geq 0, \\ \left\{ \eta^t(\mathbf{x},\mathbf{u}) \nabla f(\mathbf{u}) \right\}^U = -\left\{ \eta^t(\mathbf{x},\mathbf{u}) \sum_{i=1}^m y_i \nabla g_i(\mathbf{u}) \right\}^U \geq 0. \end{gathered} \tag{68}$$

Then,

$$\eta^t(\mathbf{x},\mathbf{u}) \nabla f(\mathbf{u}) \succeq 0. \tag{69}$$

By (57),

$$b_0(\mathbf{x},\mathbf{u}) \Phi_0 \left[f(\mathbf{x}) \ominus_g f(\mathbf{u}) \right] \succeq 0. \tag{70}$$

From (59) and (61),

$$f(\mathbf{x}) \ominus_g f(\mathbf{u}) \succeq 0. \tag{71}$$

thus,

$$f(\mathbf{x}) \succeq f(\mathbf{u}). \tag{72}$$

□

Theorem 22 (strong duality). *If* $\mathbf{x}^*$ *is* P*-optimal and a constraint qualification is satisfied at* $\mathbf{x}^*$*, then there exists* $\mathbf{y}^* = (y_1, y_2, \ldots, y_m)^t \in R^m$ *such that* $(\mathbf{x}^*, \mathbf{y}^*)$ *is* D*-feasible and the values of the objective functions for* (P) *and* (D) *are equal at* $\mathbf{x}^*$ *and* $(\mathbf{x}^*, \mathbf{y}^*)$*, respectively. Furthermore, if for all* P*-feasible* $\mathbf{x}$ *and* D*-feasible* $(\mathbf{u},\mathbf{y})$*, the hypotheses of Theorem 19 are satisfied, then* $(\mathbf{x}^*, \mathbf{y}^*)$ *is* D*-optimal.*

Proof. Since a constraint qualification is satisfied at $\mathbf{x}^*$, there exists $\mathbf{y}^* \in R^m$ such that the following Kuhn-Tucker conditions are satisfied:

$$\begin{gathered} \nabla f(\mathbf{x}^*) + \sum_{i=1}^m y_i \nabla g_i(\mathbf{x}^*) = 0, \\ \sum_{i=1}^m y_i \nabla g_i(\mathbf{x}^*) = 0, \\ y_i \geq 0. \end{gathered} \tag{73}$$

Therefore, $(\mathbf{x}^*, \mathbf{y}^*)$ is D-feasible.

Suppose that $(\mathbf{x}^*, \mathbf{y}^*)$ is not D-optimal. Then, there exists a D-feasible $(\mathbf{u}, \mathbf{y})$ such that $f(\mathbf{u}) \succ f(\mathbf{x}^*)$. This contradicts Theorem 20. Therefore, $(\mathbf{x}^*, \mathbf{y}^*)$ is D-optimal. □

6. Numerical Example

Consider the following example:

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) = \left[x_1 - \sin(x_2) + 1, x_1 - \sin(x_2) + 3\right] \\ \text{subject to} \quad & g_1(\mathbf{x}) \\ & \quad = \left[\sin(x_1) - 4\sin(x_2) - 2, \right. \\ & \qquad \left. \sin(x_1) - 4\sin(x_2)\right] \preceq 0, \\ & g_2(\mathbf{x}) = \left[2\sin(x_1) + 7\sin(x_2) + x_1 - 7, \right. \\ & \qquad \left. 2\sin(x_1) + 7\sin(x_2) + x_1 - 6\right] \\ & \quad \preceq 0, \\ & g_3(\mathbf{x}) = \left[2x_1 + 2x_2 - 5, 2x_1 + 2x_2 - 3\right] \preceq 0, \\ & g_4(\mathbf{x}) = \left[4x_1^2 + 4x_2^2 - 12, 4x_1^2 + 4x_2^2 - 9\right] \preceq 0, \\ & g_5(\mathbf{x}) = \left[-\sin(x_1) - 1, -\sin(x_1)\right] \preceq 0, \\ & g_6(\mathbf{x}) = \left[-\sin(x_2) - 1, -\sin(x_2)\right] \preceq 0. \end{aligned} \tag{74}$$

Note that the interval-valued objective function is univex with respect to $b = 1, \eta(\mathbf{x}, \mathbf{u}) = \mathbf{x} - \mathbf{u}$, $\Phi([a^L, a^U]) = [a^L, a^U]$, and every g_i $(i = 1, 2, \ldots, 6)$ is univex with respect to $b = 1$, $\Phi([a^L, a^U]) = [a^L, a^U]$

$$\eta(\mathbf{x}, \mathbf{u}) = \left(\frac{\sin x_1 - \sin u_1}{\cos u_1}, \frac{\sin x_2 - \sin u_2}{\cos u_2}\right)^T, \tag{75}$$

where $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{u} = (u_1, u_2)^T$.

It is easy to see that the problem satisfies the assumptions of Theorem 18. Then,

$$\begin{aligned} & (1, -\cos x_2)^T + \mu_1(\cos x_1, -4\cos x_2)^T \\ & \quad + \mu_2(2\cos x_1 + 1, 7\cos x_2)^T + \mu_3(2, 2)^T \\ & \quad + \mu_4(8x_1, 8x_2)^T + \mu_5(-\cos x_1, 0)^T \\ & \quad + \mu_6(0, -\cos x_2)^T = (0, 0)^T. \end{aligned} \tag{76}$$

After some algebraic calculations, we obtain that $\mathbf{x}^* = (0, \sin^{-1}(6/7))^T$ and $\mathbf{u}^* = (0, 1/7, 0, 0, 10/7, 0)^T$. Therefore, $\mathbf{x}^*$ is a solution.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant no. 60974082), and the Science Plan Foundation of the Education Bureau of Shaanxi Province (nos. 11JK1051, 2013JK1098, 2013JK1130, and 2013JK1182).

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