Research Article

Some Conclusion on Unique k-List Colorable Complete Multipartite Graphs

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If a graph G admits a k-list assignment L such that G has a unique L-coloring, then G is called uniquely k-list colorable graph, or UkLC graph for short. In the process of characterizing UkLC graphs, the complete multipartite graphs $K_{1*r,s}$ $(r, s \in N)$ are often researched. But it is usually not easy to construct the unique k-list assignment of $K_{1*r,s}$. In this paper, we give some propositions about the property of the graph $K_{1*r,s}$ when it is UkLC, which provide a very significant guide for constructing such list assignment. Then a special example of UkLC graphs $K_{1*r,s}$ as a application of these propositions is introduced. The conclusion will pave the way to characterize UkLC complete multipartite graphs.

1. Introduction

In this section, some definitions and results about list colorings which are referred to throughout the paper are introduced. For the necessary definitions and notation, we refer the reader to standard texts, such as [1]. Following the paper [2], we use the notation K_{s*r} $(r, s \in N$, N is the set of natural numbers) for a complete r-partite graph in which each part is of size s. Notation such as $K_{s*r,t}, (r, s, t \in N)$ is used similarly.

The idea of list colorings of graphs is due, independently, to Vizing [3] and Erdős et al. [4]. For a graph $G = (V, E)$ and each vertex $v \in V(G)$, let $L(v)$ denote a list of colors available for v. $L = \{L(v) \mid v \in V(G)\}$ is said to be *a list assignment* of G. If $|L(v)| = k$ for all $v \in V(G)$, then L is called k-list assignment of G. For example, the numbers nearby the vertices in Figure 1 are 2-list assignment of the graph. A *list coloring* from a given collection of lists is a proper coloring c such that $c(v)$ is chosen from $L(v)$. We will refer to such a coloring as an *L-coloring* [5]. In Figure 1, the set of circled numbers makes a 2-list coloring of the graph.

The list coloring model can be used in the channel assignment [6–8]. The fixed channel allocation scheme leads to low channel utilization across the whole channel. It requires a more effective channel assignment and management policy, which allows unused parts of channel to become available temporarily for other usages so that the scarcity of the channel can be largely mitigated [6]. It is a discrete optimization problem. A model for channel availability observed by the secondary users is introduced in [6]. We abstract each secondary network topology into a graph, where vertices represent wireless users such as wireless lines, WLANs, or cells, and edges represent interferences between vertices. In particular, if two vertices are connected by an edge in the graph, we assume that these two vertices cannot use the same spectrum simultaneously. In addition, we associate with each vertex a set, which represents the available spectra at this location. Due to the differences in the geographical location of each vertex, the sets of spectra of different nodes may be different. Then a list coloring model is constructed.

The research of list coloring consists of two parts: the choosability and the unique list colorability. Some relations
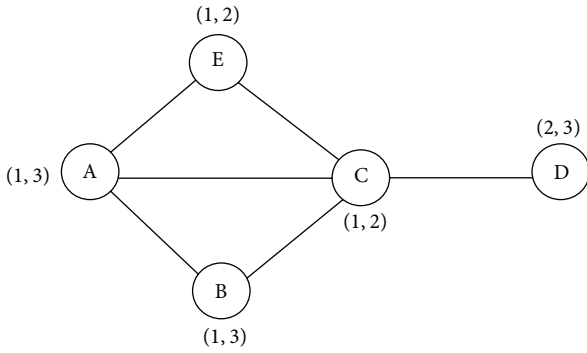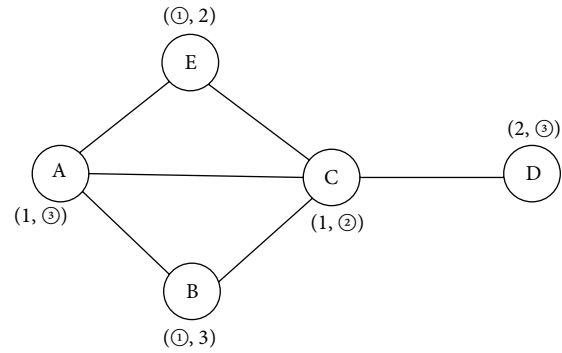
FIGURE 1: The list of graph.

FIGURE 2: The list coloring of graph.

between uniquely list colorability and choosability of a graph are presented in [9]. In this paper, we research the unique list colorability of graph.

The concept of *unique list coloring* was introduced by Dinitz and Martin [10] and independently by Mahmoodian and Mahdian [11], which can be used to study defining set of k-coloring [12] and critical sets in Latin squares [13]. Let G be a graph with n vertices, and suppose that for each vertex v in G, there exists a list of k colors $L(v)$, such that there exists a unique L-coloring for G; then G is called *uniquely k-list colorable graph* or a UkLC *graph* for short. It is obvious that the set of circled numbers makes a 2-list coloring of the graph in Figure 2. For a graph G, it is said to have *the property $M(k)$* if and only if it is not uniquely k-list colorable graph. So G has the property $M(k)$ if for any collection of lists assigned to its vertices, each of size k, either there is no list coloring for G or there exist two list colorings. Note that the m-number of a graph G, denoted by $m(G)$, is defined to be the least integer k such that G has the property $M(k)$.

It is clear from the definition of uniquely k-list colorable graphs that each UkLC graph is also a U$(k-1)$LC graph [14]. That is to say that, a graph which has the property $M(k-1)$ also has the property $M(k)$.

Mahdian and Mahmoodian [5] characterized uniquely 2-list colorable graphs. They showed the following.

Proposition 1 (see [5]). *A connected graph has the property $M(2)$ if and only if every block of G is either a cycle, a complete graph, or a complete bipartite graph.*

In paper [15], it is showed that recognizing uniquely k-list colorable graphs is Σ_2^p-complete for every $k \geq 3$; then uniquely 3-list colorable graphs are unlikely to have a nice characterization. But Ghebleh and Mahmoodian [14] and He et al. [16–18] have characterized the U3LC complete multipartite graphs, and one has the following.

Proposition 2 (see [14]). *The graphs $K_{3,3,3}$, $K_{2,4,4}$, $K_{2,3,5}$, $K_{2,2,9}$, $K_{1,2,2,2}$, $K_{1,1,2,3}$, $K_{1,1,1,2,2}$, $K_{1*4,6}$, $K_{1*5,5}$, and $K_{1*6,4}$ are U3LC.*

Proposition 3 (see [16]). *Let G be a complete multipartite graph; then G is U3LC if and only if it has one of the graphs in Proposition 2 as an induced subgraph.*

Wang et al. [19] have characterized U4LC complete multipartite graphs with at least 6 parts except for finitely many of them.

In the process of characterizing UkLC complete multipartite graphs, it is often researched that the property $M(k)$ of complete multipartite graphs has only one part whose size is more than one; that is, $K_{1*r,s}$ ($r, s \in N$). Paper [14] studied the property $M(3)$ of graphs $K_{1*r,3}$ and $K_{1,1,1,r}$. The following was concluded.

Proposition 4 (see [14]). *For every $r \geq 2$, $m(K_{1*r,3}) = m(K_{1,1,1,r}) = 3$.*

The property $M(4)$ of graphs $K_{1*r,5}$ and $K_{1*5,r}$ is researched in paper [19], and it is showed the following.

Proposition 5 (see [19]). *For every $r \geq 1$, $K_{1*5,r}$ and $K_{1*r,5}$ have the property $M(4)$, and if $r \geq 5$, then $m(K_{1*5,r}) = m(K_{1*r,5}) = 4$.*

Conclusions above are generalized by Wang et al. [20] recently.

Proposition 6 (see [20]). *For every $r \geq 1$, $k \geq 2$, $K_{1*r,(2k-3)}$ has the property $M(k)$.*

Proposition 7 (see [20]). *For every $r \geq 1$, $k \geq 2$, $K_{1*(2k-3),r}$ has the property $M(k)$.*

But there is no other conclusion about what are the maximal numbers r and s such that the graph $K_{1*r,s}$ is a UkLC graph for every k. Besides, the property of list assignment of UkLC graph $K_{1*r,s}$ is still unclear, and there is a lack of the necessary conditions for the UkLC graph $K_{1*r,s}$. It seems that the larger k is, the more difficult the characterizing UkLC graphs are.

In fact, if we want to proof that some graph $K_{1*r,s}$ is a UkLC graph, we must find a k-list assignment such that there exists a unique list coloring. In general it is not easy to construct such list assignment, and it usually requires a lot of skills. But if some properties of such graphs are known, the construction process perhaps will become easier. In addition, it is hoped that one can obtain some properties of UkLC graph $K_{1*r,s}$ for every k, not only for special k.

In this paper the property of the graph $K_{1*r,s}$ is researched when it is a UkLC graph. The paper is organized as follows. In Section 2, we give some propositions about the property of the graph $K_{1*r,s}$ when it is a UkLC graph. According to these propositions, a special example of UkLC graphs $K_{1*r,s}$ is introduced in Section 3. In Section 4, we discuss the results and give an open problem. The conclusion will pave the way to characterize UkLC complete multipartite graphs.

2. Property of the UkLC Graph $K_{1*r,s}$

In this section, we list some theorems about the property of the graph $K_{1*r,s}$ when it is UkLC, as it is conducive to construct the list assignment of UkLC complete multipartite graphs and characterize the UkLC graphs.

Theorem 8. *For every $r \geq 1$, if $K_{1*r,s}$ is a UkLC graph, then $s \geq 2$ $(r, s \in N)$.*

Proof. If $s = 1$, $K_{1*r,s} = K_{1*(r+1)}$ has the property $M(2)$ by Proposition 1; so it is not a U2LC graph, nor a UkLC graph which is contradictory to the suppose. $\square$

In view of these facts, it is supposed that $s \geq 2$ for a UkLC graph $K_{1*r,s}$ in the following.

In the process of proving Theorem 9, for convenience, the $r + 1$ parts of $K_{1*r,s}$ are denoted by $V_i = \{v_i\}$ for $i = 1, 2, \ldots, r$ and $V_{r+1} = \{v_{r+1}, v_{r+2}, \ldots, v_{r+s}\}$. For the given k-list assignment L: $L(v_i) = \{c_{i,1}, c_{i,2}, \ldots, c_{i,k}\}$, $i = 1, 2, \ldots, r + s$, there is a unique k-list color c: $c(v_i) = c_{i,1}$, $i = 1, 2, \ldots, r + s$. Furthermore, one has $D_i = \{c_{i,1}\}$, $i = 1, 2, \ldots, r$; $\overline{D_i} = \{c_{i,2}, c_{i,3}, \ldots, c_{i,k}\}$, $i = 1, 2, \ldots, r$; $D_{r+1} = \{c_{r+1,1}, c_{r+2,1}, \ldots, c_{r+s,1}\}$; $\overline{D_{r+1}} = \{c_{r+1,2}, c_{r+1,3}, \ldots, c_{r+1,k}, c_{r+2,2}, c_{r+2,3}, \ldots, c_{r+2,k}, \ldots, c_{r+s,2}, c_{r+s,3}, \ldots, c_{r+s,k}\}$.

Theorem 9. *Suppose that $r \geq 1$, $k \geq 3$, and $s \geq 3$. Suppose that for some r, s, k, $K_{1*r,s}$ is a UkLC graph, and the unique k-list color c from the k-list assignment L is defined as above; then one has the following:*

(1) $c_{i,1} \neq c_{j,1}$, where $1 \leq i, j \leq r$ and $i \neq j$; $c_{i,1} \neq c_{j,1}$, where $1 \leq i \leq r$, $r + 1 \leq j \leq r + s$;

(2) $|D_{r+1}| \geq 2$;

(3) $|D_{r+1}| \leq s - 2$;

(4) $c_{i,1} \notin \{c_{j,2}, c_{j,3}, \ldots, c_{j,k}\}$, $i, j = r + 1, r + 2, \ldots, r + s$;

(5) $|\bigcup_{i=1}^{r+1} D_i| = |\bigcup_{i=1}^{r+1} \overline{D_i}|$ and $\overline{D_i} \subseteq \bigcup_{j \neq i} D_i$, $i, j = 1, 2, \ldots, r + 1$;

(6) there must be a i $(1 \leq i \leq r)$ such that $\overline{D_i} \subseteq D_{r+1}$.

Proof. Let $G = K_{1*r,s}$.

(1) From the definition of the UkLC graph, this conclusion is obvious.

(2) Suppose that $|D_{r+1}| = 1$, which means that $c_{r+1,1} = c_{r+2,1} = \cdots = c_{r+s,1}$.

Let $H = G - V_{r+1} = K_{1*r}$. We introduce a $(k-1)$-list assignment L' to H as follows. For every vertex v in H, if

$a \in L(v)$, then $L'(v) = L(v) \setminus \{c_{r+1,1}\}$; otherwise $L'(v) = L(v) \setminus \{b\}$ where $b \in L(v)$ and $b \neq c(v)$. Since L induces a list coloring c for G, H has exactly a L'-coloring, namely, the restriction of c on H. $H = K_{1*r}$ has the property $M(2)$ by Proposition 1; so it has the property $M(k)$, and we can obtain a new L'-coloring of H. From the construction of L', we know that the new L'-coloring can be extended to G. Thus, G has a new L-coloring which is different from c which is contradictory to the fact that c is the unique k-list color.

(3) We use the reduction to absurdity.

Case 1. One has $|D_{r+1}| = s$ which means that $c_{r+1,1}, c_{r+2,1}, \ldots, c_{r+s,1}$ are pairwise different.

Adding new edges between any two vertices in $\{v_{r+1}, v_{r+2}, \ldots, v_{r+s}\}$, the resulting graph is $G' = K_{r+s}$. Note that c is also a proper L-coloring of G', and G' has the property $M(2)$ by Proposition 1; hence G' has the property $M(k)$. So we can obtain another coloring of G', which is also a legal L-coloring for G, which is contradictory to the fact that c is the unique k-list color.

Case 2. One has $|D_{r+1}| = s - 1$ which means that in $\{v_{r+1}, v_{r+2}, \ldots, v_{r+s}\}$ there are just two vertices assigned a common color, and the others are pairwise different.

Not loss of generality, we say that $c_{r+1,1} = c_{r+2,1}$ and $c_{k,1}$ are pairwise different for $k = r + 3, r + 4, \ldots, r + s$. Adding new edges between any two vertices in $\{v_1, v_2, \ldots, v_r, v_{r+3}, v_{r+4}, \ldots, v_{r+s}\}$, the resulting graph is $G'' = K_{r+s-2}$. A $(k-1)$-list assignment L' to G'' is introduced as follows. For every vertex v in G'', if $c_{r+1,1} \in L(v)$, then $L'(v) = L(v) \setminus \{c_{r+1,1}\}$; otherwise $L'(v) = L(v) \setminus \{b\}$ where $b \in L(v)$ and $b \neq c(v)$. It is obvious that $|L'(v)| = k - 1$ for every $v \in V(G'')$, and the restriction of c on G'' is an L'-coloring of G''. Obviously, $G'' = K_{1*(r+s-2)}$ has the property $M(2)$ by the Proposition 1 and the property $M(k-1)$. By the property $M(k-1)$ of G'', we can obtain a new L'-coloring c' of G'', which can be extended to G as follows. For every vertex v in G, if $v \in V(G'')$, then $c''(v) = c'(v)$; otherwise $c''(v) = c(v)$. From the construction of L', it is obvious that c'' is a new L-coloring of G which is contradictory to the fact that c is the unique k-list color.

In sum, $|D_{r+1}| \leq s - 2$.

(4) If $i = j$, then it is obvious that the conclusion is true. If $i \neq j$, then we suppose that the conclusion is wrong, which means that there are two numbers i_0 and j_0 such that $i_0 \neq j_0$ and $c_{i_0,1} \in \{c_{j_0,2}, c_{j_0,3}, \ldots, c_{j_0,k}\}$. It is clear that $c_{i_0,1} \neq c_{j_0,1}$. Let $c'(v_{j_0}) = c_{i_0,1}$ and let $c'(v_k) = c_{k,1}$ for $k = 1, 2, \ldots, r + s$ but $k \neq j_0$. Obviously, c' is a new L-coloring of G which is contradictory to the fact that c is the unique k-list color.

(5) Proof by contradiction. Suppose that $|\bigcup_{i=1}^{r+1} D_i| \neq |\bigcup_{i=1}^{r+1} \overline{D_i}|$, and there are i_0 and j_0 such that $1 \leq i_0 \leq r + s$, $2 \leq j_0 \leq k$ and $c_{i_0,j_0} \notin \bigcup_{i=1}^{r+1} D_i$. Let $c'(v_{i_0}) = c_{i_0,j_0}$ and let $c'(v_k) = c_{k,1}$ for $k = 1, 2, \ldots, r + s$ but $k \neq i_0$. Obviously c' is a new L-coloring of G which is contradictory to the fact that c is the unique k-list

color. Then from (3) we know that $\overline{D_i} \subseteq \bigcup_{j \neq i} D_i$, $i, j = 1, 2, \ldots, r + 1$.

(6) By contradiction. Suppose for every i $(1 \leq i \leq r)$ that $\overline{D_i} \nsubseteq D_{r+1}$; then it is obtained that $|\overline{D_i} \setminus D_{r+1}| \geq 1$. So we get that $|L(v_i) \setminus D_{r+1}| \geq 2$ for every $1 \leq i \leq r$. Let $H = G - V_{r+1} = K_{1*r}$. We introduce a 2-list assignment L' to H as follows. For every vertex v_i in H, we obtain $L'(v_i)$ by randomly getting rid of $k - 2$ elements from $L(v_i)$ such that $L'(v_i) \cap D_{r+1} = \Phi$ and $|L'(v_i)| = 2$, as can be done because $|L(v_i) \setminus D_{r+1}| \geq 2$. Since L induces a list coloring c for G, H has exactly one L'-coloring, namely, the restriction of c on H. $H = K_{1*r}$ has the property $M(2)$ by Proposition 1; so we can obtain a new L'-coloring of H. From the construction of L', we know that the new L'-coloring can be extended to G. Thus, G has a new L-coloring which is different from c which is contradictory to the fact that c is the unique k-list color. □

3. An Example of UkLC Graphs $K_{1*r,s}$

According the property of UkLC graph in Theorem 9, we construct a list assignment of graph $K_{1*r,s}$ for special r and s and prove that the graph is a UkLC graph in this section.

Theorem 10. *Let* $M = (k - 1)\binom{2k-2}{k-1}$ *where* $k \geq 2$*, the graph* $K_{1*(2k-2),M}$ *is a UkLC graph.*

Proof. For convenience, the $(2k - 1)$ parts of $K_{1*(2k-2),M}$ are denoted by $V_i = \{v_i\}$ for $i = 1, 2, \ldots, 2k - 2$ and $V_{2k-1} = \{u_1, u_2, \ldots, u_M\}$.

Let $A = \{1, 2, \ldots, 2k - 2\}$. We denote all $(k - 1)$-subsets of A by $\{A_1, A_2, \ldots, A_m\}$, where $m = \binom{2k-2}{k-1}$. Now consider $K_{1*(2k-2),M}$ with the following k-list of colors on vertices. For $i = 1, 2, \ldots, 2k - 2$, $L(v_i) = \{i, 2k - 1, 2k, \ldots, 3k - 3\}$. For every $t = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k - 1$, $L(u_{(j-1)*m+t}) = \{j + 2k - 2\} \cup A_t$.

For example, when $k = 3$, $K_{1*(2k-2),M} = K_{1*4,12}$, and the above mentioned k-list assignment L for $K_{1*4,12}$ is as follows:

$$\{\{1\underline{5}6\}, \{2\underline{5}6\}, \{3\underline{5}6\}, \{4\underline{5}6\},$$

$$\{5\underline{1}2, 5\underline{1}3, 5\underline{1}4, 5\underline{2}3, 5\underline{2}4, 5\underline{3}4, 6\underline{1}2, 6\underline{1}3, 6\underline{1}4, 6\underline{2}3, 6\underline{2}4, 6\underline{3}4\}\}. \tag{1}$$

Note that the list assignment L makes a total of $3k - 3$ colors. Since $K_{1*(2k-2),M}$ is a complete $(2k - 1)$-partite graph, the last part V_{2k-1} can take $k - 1$ colors at the most. From the construction of L, it is obtained obviously that $\{2k - 1, 2k, \ldots, 3k-3\}$ is the unique choice for a k-list coloring from L. Then the V_i $(1 \leq i \leq 2k - 2)$ must take the color i. In the example above, the colors in the list coloring are marked by underlines. So a unique k-list coloring from L is made and $K_{1*(2k-2),M}$ is a UkLC graph. □

Notice that for every $r \geq 1$, $k \geq 2$, $K_{1*(2k-3),r}$ has the property $M(k)$ by Proposition 7. Now the graph $K_{1*(2k-2),M}$ in Theorem 10 is a UkLC graph; so it is wrong with the

proposition "for every $r \geq 1$, $k \geq 2$, $K_{1*(2k-2),r}$ has the property $M(k)$". Therefore, $(2k - 3)$ is the maximal numbers in Proposition 7.

4. Discussion and Some Open Problems

It is not easy to characterize UkLC complete multipartite for any k. In fact, it is a very tricky job to construct a k-list assignment such that there exists a unique list coloring. Theorem 9 provides a direction for constructing such list assignment of $K_{1*r,s}$, and perhaps it makes construction easier for the researchers. Furthermore, Theorem 9 is true for every k ($k \geq 2$ and $k \in N$), and the conclusion is extensive.

It must be noted that the conditions in Theorem 9 are only necessary conditions of $K_{1*r,s}$ for UkLC graph, not sufficient conditions.

Theorem 10 can be regarded as a application of Theorem 9. And from Theorem 10, it is known that $(2k - 3)$ is exactly the maximal numbers in Proposition 7. But notice that the number $M = (k - 1)\binom{2k-2}{k-1}$ is not the minimal number for every k in Proposition 7. For example, when $k = 3$, $M = 12$ and $K_{1*4,12}$ is U3LC according to Theorem 10. In fact $K_{1*4,6}$ is U3LC by Propositions 2 and 3; so $M = 12$ is not the minimal number for $k = 3$ in Proposition 7. Moreover, it is very likely that for different k the minimal number is different in Proposition 7.

The following problem arises naturally from the work.

Problem. For every k, characterize all minimum number s such that the graph $K_{1*(2k-2),s}$ is a UkLC graph.

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