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Research Article

About the Local Stability of the Four Cauchy Equations Restricted on a Bounded Domain and of Their Pexiderized Forms

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The Ulam-Hyers stability of functional equations is widely studied from various points of view by many authors. The present paper is concerned with local stability of the four Cauchy equations restricted on a bounded domain. These results can be easily adapted to the corresponding Pexiderized equations.

1. Introduction

After this introduction, in Section 2 the local stability of the additive equation

$$\varphi(x+y) = \varphi(x) + \varphi(y) \quad (1)$$

and, as a consequence, of the logarithmic equation

$$\varphi(xy) = \varphi(x) + \varphi(y) \quad (2)$$

both restricted on a bounded domain in R^2 is studied.

It is well known that the problem of stability was posed, for the additive equation, by S. Ulam and was solved by Hyers [1] in 1941, with reference to the equation valid on the whole space. Afterwards, stability was widely studied by many authors, from various points of view, considering further equations on the whole space or putting them in very general settings (see, for instance, [2, 3]).

As for the "local" stability of equations on a restricted domain, first results can be found in [4, 5] (see also [6]) and they concern substantially the set of functions $f : D_f \subset R \to (S, \|\cdot\|)$, satisfying the condition of δ-additivity

$$\|f(x+y) - f(x) - f(y)\| < \delta \quad (3)$$

restricted either on the triangular domain

$$E_0 = E(0,0;r) = \{(x,y) \in R^2 : x \geq 0, y \geq 0, x+y < r\} \quad (4)$$

for some given $r > 0$, or the unbounded domain

$$K_a = \{(x,y) \in R^2 : |x| + |y| > a\} \quad (5)$$

for some given $a > 0$.

In the present paper (Section 2) the bounded restricted domain of inequality (3) will be assumed to be the triangle

$$E = E(a,b;r) = \{(x,y) \in R^2 : x \geq a, y \geq b, x+y < a+b+r\} \quad (6)$$

for some given $(a,b) \in R^2$ and $r > 0$.

It has to be remarked that in the classical paper [1] by Hyers as well as in case of restricted domains studied in [4, 5], the solutions of the "equation" correlated to the inequality (3) on the given domain are either additive functions on the whole space (in cases of the general result by Hyers and of the domain K_a defined in (5)) or the restrictions to $D_f = [0, r)$ of functions F additive on R^2 (in case of a domain like E_0 defined in (4)).

On the contrary, when a restricted domain like that in (6) is assumed, the local solution of the corresponding additive equation, restricted to the same set, is different from the restriction on the domain of f of some function, which is additive in the whole space R^2 (see [7]).

Therefore, in order to adhere to the sense of Ulam's question in case of a restricted domain like $E(a,b;r)$ too, the locally δ-additive function f has to be compared with the local solution of the corresponding exact equation restricted to the same set $E(a,b;r)$.

In this frame, in Section 2, first the local stability of the additive Cauchy equation restricted to $E(a,b;r)$ will be proved (Theorem 1); then, as a consequence of this result, the local stability of the logarithmic Cauchy equation (2) will be proved (Theorem 6).

These results can be easily extended to the Pexiderized forms of the same equations.

Notice that the problem of the "local" stability for the remaining two Cauchy equations,

$$\varphi(x+y)=\varphi(x)\varphi(y), \tag{7}$$

$$\varphi(xy)=\varphi(x)\varphi(y), \tag{8}$$

restricted to bounded domains, requires a suitable slightly different approach because of the peculiar properties of the local solutions of such equations when they are restricted on bounded domains (see [8, 9]).

This problem will be the object of Section 3 of the present paper, where results of local stability of (7) and (8) will be proved (Theorems 15 and 16, resp.).

2. About the Additive and the Logarithmic Cauchy's Equations

2.1. A Result on Local Stability of the Additive Equation. In the set of functions f of a real variable with values in a normed space $S=(S,\|\cdot\|)$, let us consider the inequality

$$\|f(x+y)-f(x)-f(y)\|<\delta \tag{9}$$

for some $\delta>0$ and $(x,y)\in E(a,b;r)$, defined in (6), for given $(a,b)\in R^2$ and $r>0$.

As usual, the projections of E will be denoted by

$$\begin{aligned} E_x &:= \{x\in R : \exists y\in R \text{ such that } (x,y)\in E\},\\ E_y &:= \{y\in R : \exists x\in R \text{ such that } (x,y)\in E\},\\ E_{x+y} &:= \{x+y\in R : (x,y)\in E\}. \end{aligned} \tag{10}$$

For $E(a,b;r)$ as defined in (6), the projections are $E_x=[a,a+r)$, $E_y=[b,b+r)$, $E_{x+y}=[a+b,a+b+r)$, and the domain D_f of functions f satisfying (9) is $D_f=E_x\cup E_y\cup E_{x+y}$.

We purpose to check whether each $f: D_f\to(S,\|\cdot\|)$ satisfying (9) for $(x,y)\in E(a,b;r)$ is uniformly approached on D_f by some $\varphi: D_f\to S$ satisfying the additive equation restricted to $E(a,b;r)$, namely, by a function φ of the following form (see [7]):

$$\varphi(t)=\begin{cases} h_0(t)+\alpha, & t\in E_x,\\ h_0(t)+\beta, & t\in E_y,\\ h_0(t)+\alpha+\beta, & t\in E_{x+y}, \end{cases} \tag{11}$$

where $h_0: R\to S$ is additive on the whole space R^2 and $\alpha,\beta\in S$ are constant.

A positive answer is given by the following.

Theorem 1. *Let $f: D_f\to(S,\|\cdot\|)$, $(S,\|\cdot\|)$ being a Banach space, satisfy (9) for some $\delta>0$ and every $(x,y)\in E(a,b;r)$, defined in (6), for given $(a,b)\in R^2$ and $r>0$; $D_f=E_x\cup E_y\cup E_{x+y}$.*

Then there exists (at least) a function $H: R\to S$, additive on R^2, such that the function $F: D_f\to S$ defined by

$$F(t)=\begin{cases} H(t)+f(a)-H(a), & t\in E_x,\\ H(t)+f(b)-H(b), & t\in E_y,\\ H(t)+f(a)+f(b)-H(a)-H(b), & t\in E_{x+y} \end{cases} \tag{12}$$

has both the following properties:

(i) *F is a (local) solution of the additive equation restricted on $E(a,b;r)$;*

(ii) *F approaches uniformly f on D_f, and*

$$\|f(t)-F(t)\|<15\delta \tag{13}$$

holds for every $t\in D_f$.

In order to prove Theorem 1, let us premise two lemmas.

Lemma 2. *Let $f: D_f=E_x\cup E_y\cup E_{x+y}\to(S,\|\cdot\|)$ satisfy the condition (9) restricted to the set $E=E(a,b;r)$ defined in (6). Then the functions $\gamma_i:[0,r)\to S$, $i=1,2,3$, defined by*

$$\gamma_1(t):=f(a+t)-f(a),\qquad \gamma_2(t):=f(b+t)-f(b),$$
$$\gamma_3(t):=f(a+b+t)-f(a+b) \tag{14}$$

for $t\in[0,r)$ satisfy both the following inequalities:

$$\|\gamma_i(t)-\gamma_j(t)\|<2\delta \quad \text{for } i,j=1,2,3, \tag{15}$$

$$\|\gamma_i(\xi+\eta)-\gamma_i(\xi)-\gamma_i(\eta)\|<4\delta \quad \text{for } i=1,2,3,\quad (\xi,\eta)\in E_0=E(0,0;r). \tag{16}$$

Proof of Lemma 2. Let us prove (15) firstly for $i=1$, $j=2$. For $t\in[0,r)$ the points $(a+t,b)$ and $(a,b+t)$ belong to $E(a,b;r)$; hence, from (9) both the following inequalities hold

$$\begin{aligned} \|f(a+b+t)-\{f(a+t)+f(b)\}\|&<\delta,\\ \|f(a+b+t)-\{f(b+t)+f(a)\}\|&<\delta, \end{aligned} \tag{17}$$

whence,

$$\|f(a+t)+f(b)-f(b+t)-f(a)\|<2\delta, \tag{18}$$

namely, (15) for $i=1$, $j=2$.

Similarly we prove (15), for $i=1$, $j=3$, assuming the pairs (a,b) and $(a+t,b)$ with $t\in[0,r)$, from the formulas

$$\begin{gathered}\|\{f(a+b+t)-f(a+t)\}-f(b)\|<\delta,\\ \|\{f(a+b)-f(a)\}-f(b)\|<\delta,\end{gathered} \tag{19}$$

and for $i=2$, $j=3$, assuming the pairs (a,b), $(a,b+t)$, and $t\in[0,r)$.

In order to prove (16) let us assume $(\xi,\eta)\in E_0=E(0,0;r)$.

For $i=1$, from (15) and (9) we get

$$\begin{aligned}\Phi_1 &:= \gamma_1(\xi+\eta)-\gamma_1(\xi)-\gamma_1(\eta)\\ &= \{f(a+\xi+\eta)-f(a)\}-\{f(a+\xi)-f(a)\}\\ &\quad -\{f(a+\eta)-f(a)\}\\ &= f(a+\xi+\eta)-f(a+\xi)-\{f(b+\eta)-f(b)+\sigma\}\\ &= \{f(a+\xi+\eta)+f(b)\}-\{f(a+\xi)+f(b+\eta)\}-\sigma\\ &\qquad \|\sigma\|<2\delta;\end{aligned} \tag{20}$$

since $(a+\xi+\eta,b)\in E(a,b;r)$ and $(a+\xi,b+\eta)\in E(a,b;r)$, it follows from (9) that

$$\Phi_1=\{f(a+b+\xi+\eta)+\rho_1\}-\{f(a+b+\xi+\eta)+\rho_2\}-\sigma, \tag{21}$$

with $\|\sigma\|<2\delta$, $\|\rho_k\|<\delta$ for $k=1,2$, whence (16) for $i=1$.

Similarly, for $i=2$, a and b interchanged.

As for γ_3 with $(\xi,\eta)\in E_0=E(0,0;r)$, from (9)

$$\begin{aligned}\Phi_3 &:= \gamma_3(\xi+\eta)-\gamma_3(\xi)-\gamma_3(\eta)\\ &= \{f(a+b+\xi+\eta)-f(a+b)\}\\ &\quad -\{f(a+b+\xi)-f(a+b)\}\\ &\quad -\{f(a+b+\eta)-f(a+b)\}\\ &= \{f(a+\xi)+f(b+\eta)+\tau_1\}-\{f(a+\xi)+f(b)+\tau_2\}\\ &\quad -\{f(a)+f(b+\eta)+\tau_3\}+\{f(a)+f(b)+\tau_4\},\end{aligned} \tag{22}$$

where $\|\tau_k\|<\delta$, $k=1,2,3,4$, whence $\|\Phi_3\|<4\delta$.

Lemma 2 is proved. □

Lemma 3. *If $f: D_f=E_x\cup E_y\cup E_{x+y}\to(S,\|\cdot\|)$, S being a Banach space, satisfies (9) on $E=E(a,b;r)$, then each of the functions $\gamma_i(t)$, $i=1,2,3$, defined in (14) for $t\in[0,r)$, is uniformly approached on $[0,r)$ by the restriction of a function $H_i: R\to S$ additive on R^2.*

The following inequalities hold:

$$\|\gamma_i(t)-H_i(t)\|<12\delta,\quad t\in[0,r),\ i=1,2,3, \tag{23}$$

$$\|\gamma_i(t)-H_j(t)\|<14\delta,\quad t\in[0,r),\ i,j=1,2,3. \tag{24}$$

Proof of Lemma 3. Since each function γ_1, γ_2, γ_3 is 4δ-additive on $E(0,0;r)$ (from Lemma 2), formula (23) follows immediately from a known result [4, Lemma 2]:

Let $f:[0,d)\to(X,\|\cdot\|)$, X a Banach space, be δ-additive on $E_d=\{(x,y)\in R^2: 0\le x<d,\ 0\le y<d,\ x+y<d\}$.

Then there exists at least one additive function $L: R\to X$ such that

$$\|f(x)-L(x)\|<3\delta\quad \text{for every } x\in[0,d). \tag{25}$$

Moreover, for every $i,j=1,2,3$ and $t\in[0,r)$, formula (24) follows from (15) and (23); in fact

$$\|\gamma_i(t)-H_j(t)\|=\|\{\gamma_i(t)-\gamma_j(t)\}+\{\gamma_j(t)-H_j(t)\}\|<14\delta. \tag{26}$$

This means that the restriction to $[0,r)$ of each additive H_j, $j=1,2,3$, approaches uniformly on $[0,r)$ each function γ_1, γ_2, γ_3.

Lemma 3 is proved. □

Proof of Theorem 1. According to (24) in Lemma 3, each function $\gamma_i(t)$, $i=1,2,3$, defined in (14) for $t\in[0,r)$, namely,

$$\begin{gathered}\gamma_1(t):=f(a+t)-f(a),\qquad \gamma_2(t):=f(b+t)-f(b),\\ \gamma_3(t):=f(a+b+t)-f(a+b),\end{gathered} \tag{27}$$

is uniformly approached on $[0,r)$ by each of the additive functions $H_j: R\to S$, $j=1,2,3$.

Let us define $F_i: D_f\to S$, $i=1,2,3$, as follows:

$$F_i(t)=\begin{cases}H_i(t)-H_i(a)+f(a), & t\in E_x,\\ H_i(t)-H_i(b)+f(b), & t\in E_y,\\ H_i(t)-H_i(a+b)+f(a)+f(b), & t\in E_{x+y}.\end{cases} \tag{28}$$

Such functions F_i, $i=1,2,3$, satisfy obviously the additive equation restricted to $E(a,b;r)$.

Moreover, thanks to Lemmas 2 and 3, each function F_i approaches uniformly f on D_f as in formula (13); in fact, for arbitrary $(x,y)\in E(a,b;r)$,

$$\begin{aligned}&x=a+t\in E_x,\qquad \gamma_1(t)=\gamma_1(x-a)=f(x)-f(a),\\ &\|F_i(x)-f(x)\|\\ &\quad=\|(H_i(x)-H_i(a)+f(a))-(\gamma_1(x-a)+f(a))\|\\ &\quad=\|H_i(x-a)-\gamma_1(x-a)\|<14\delta;\end{aligned} \tag{29}$$

similarly for $y\in E_y$, $\gamma_2(y-b)=f(y)-f(b)$.

On the projection E_{x+y}, where $x + y = a + b + t$ and $\gamma_3(x+y-a-b) = f(x+y) - f(a+b)$, we get from Lemma 3 and formula (9)

$$\begin{aligned} &\|F_i(x+y) - f(x+y)\| \\ &= \|H_i(x+y-a-b) + f(a) + f(b) \\ &\quad - (\gamma_3(x+y-a-b) + f(a+b))\| \\ &\leq \|H_i(x+y-a-b) - \gamma_3(x+y-a-b)\| \\ &\quad + \|f(a+b) - f(a) - f(b)\| < 15\delta. \end{aligned} \tag{30}$$

Therefore, each function $F_i : D_f \to S$, $i = 1, 2, 3$, satisfies (13), and Theorem 1 is proved. □

Remark 4. The foregoing study was developed as though the projections E_x, E_y, and E_{x+y} were pairwise disjoint.

If two of them overlap, for instance $E_x \cap E_y$ is nonempty, in every common point the values given by the different parts of formulas of approximating function F have to be the same.

More in particular, if the set $D_f = E_x \cup E_y \cup E_{x+y}$ is connected, in (28) the equations

$$f(a) - H_i(a) = f(b) - H_i(b) = f(a) + f(b) - H_i(a+b) \tag{31}$$

hold, whence $f(a)-H_i(a) = f(b)-H_i(b) = 0$. In this case, the locally δ-additive f is uniformly approached on the whole D_f by the restriction to D_f of a function $H : R \to S$, additive on R^2 ($H = H_i$, i either =1, or =2, or =3).

2.2. A Result on Local Stability of the Logarithmic Equation. On the ground of the results in Section 2.1 it is easy to prove the local stability of the logarithmic Cauchy equation (2) restricted to the bounded domain

$$\begin{aligned} J = J(a,b;r) := \{&(x,y) \in R^2 : e^a \leq x < e^{a+r}, \\ &e^b \leq y < e^{b+r}, e^{a+b} \leq xy < e^{a+b+r}\} \end{aligned} \tag{32}$$

for given $(a,b) \in R^2$ and $r > 0$.

The projections J_x, J_y, J_{xy} of J are given by

$$\begin{aligned} J_x &:= \{x \in R^+ : \exists y \in R^+ \text{ such that } (x,y) \in J\} = [e^a, e^{a+r}), \\ J_y &:= \{y \in R^+ : \exists x \in R^+ \text{ such that } (x,y) \in J\} = [e^b, e^{b+r}), \\ J_{xy} &:= \{xy \in R^+ : (x,y) \in J\} = [e^{a+b}, e^{a+b+r}). \end{aligned} \tag{33}$$

Since the local stability of (2) depends on the comparison of every f satisfying

$$\|f(xy) - f(x) - f(y)\| < \delta \tag{34}$$

for $(x,y) \in J(a,b;r)$ with some solution φ of the corresponding equation (2) restricted to $J(a,b;r)$, let us premise (Lemma 5) the local solution of (2).

Lemma 5. *Let S be a real linear space; if $\varphi : D_f = J_x \cup J_y \cup J_{xy} \to S$ satisfies (2) on the bounded domain $J = J(a,b;r)$ defined in (32), then there exists a function $h_0 : R \to S$, additive on R^2, such that*

$$\varphi(t) = \begin{cases} h_0(\ln t) - h_0(a) + \varphi(e^a), & t \in J_x, \\ h_0(\ln t) - h_0(b) + \varphi(e^b), & t \in J_y, \\ h_0(\ln t) - h_0(a) - h_0(b) + \varphi(e^{a+b}), & t \in J_{xy}. \end{cases} \tag{35}$$

Proof of Lemma 5. By the usual substitutions

$$\begin{gathered} x = e^u, \quad u = \ln x, \quad \varphi(x) = \varphi(e^u) =: g(u) = g(\ln x), \\ y = e^v, \quad v = \ln y, \quad \varphi(y) = \varphi(e^v) =: g(v) = g(\ln y), \\ u + v = \ln(xy), \\ \varphi(xy) = \varphi(e^{u+v}) =: g(u+v) = g(\ln(xy)), \end{gathered} \tag{36}$$

the domain $J(a,b;r)$ is transformed into a set $E(a,b;r)$ like the one defined in (6). Let us consider

$$\begin{aligned} E = E(a,b;r) = \{&(u,v) \in R^2 : u \geq a, v \geq b, \\ &u + v < a + b + r\}. \end{aligned} \tag{37}$$

Equation (2) is transformed into

$$g(u+v) = g(u) + g(v), \quad (u,v) \in E(a,b;r). \tag{38}$$

Therefore, using Theorem 1, we obtain (35) with additive h_0. □

The local stability of the logarithmic equation (2) is stated by the following.

Theorem 6. *Let $(S, \|\cdot\|)$ be a Banach space; if $f : D_f = J_x \cup J_y \cup J_{xy} \to (S, \|\cdot\|)$ satisfies*

$$\|f(xy) - f(x) - f(y)\| < \delta \tag{39}$$

for some $\delta > 0$ and every $(x,y) \in J(a,b;r)$, defined in (32), for given $(a,b) \in R^2$ and $r > 0$, then there exists (at least) a function $H : R \to S$, additive on R^2, such that the function $L : D_f \to S$ defined by

$$L(t) = \begin{cases} H(\ln t) - H(a) + f(e^a), & t \in J_x = [e^a, e^{a+r}), \\ H(\ln t) - H(b) + f(e^b), & t \in J_y = [e^b, e^{b+r}), \\ H(\ln t) - H(a) - H(b) \\ \quad + f(e^a) + f(e^b), & t \in J_{xy} = [e^{a+b}, e^{a+b+r}), \end{cases} \tag{40}$$

satisfies both the following properties:

(i) *L is a local solution of the logarithmic equation on the restricted domain $J(a,b;r)$;*

(ii) *L approaches uniformly f on D_f, and*

$$\|f(t) - L(t)\| < 15\delta \tag{41}$$

holds for every $t \in D_f$.

Proof of Theorem 6. The usual substitutions $x = e^u$, $y = e^v$ (like in proof of Lemma 5) transform the inequality (39) restricted to the set $J(a,b;r)$ into

$$\|g(u+v) - g(u) - g(v)\| < \delta \tag{42}$$

restricted to the set $E = E(a,b;r)$, defined in (6).

Now, we can follow the same line of proof as in Section 2.1 by defining the functions $\gamma_i : [0,r) \to S$, $i = 1,2,3$, related to g; namely,

$$\gamma_1(t) := g(a+t) - g(a), \qquad \gamma_2(t) := g(b+t) - g(b),$$
$$\gamma_3(t) := g(a+b+t) - g(a+b); \tag{43}$$

then there exist functions $H_i : R \to S$, $i = 1,2,3$, additive on R^2, such that each of the functions G_i, $i = 1,2,3$,

$$G_i(t) = \begin{cases} H_i(t) - H_i(a) + g(a), & t \in E_u = [a, a+r), \\ H_i(t) - H_i(b) + g(b), & t \in E_v = [b, b+r), \\ H_i(t) - H_i(a) - H_i(b) & \\ \quad +g(a) + g(b), & t \in E_{u+v} = [a+b, a+b+r), \end{cases} \tag{44}$$

is a local solution of the equation $g(u+v) = g(u) + g(v)$ restricted to $E(a,b;r)$, and

$$\|g(t) - G_i(t)\| < 15\delta, \quad t \in D_g = E_u \cup E_v \cup E_{u+v} \tag{45}$$

holds for $i = 1,2,3$.

Now let us come back to f, by the substitutions which transformed (39) into (42), beginning by the transformation of functions G_i defined in (44); on J_x (from E_u)

$$G_i(\ln x) = H_i(\ln x) - H_i(a) + f(e^a), \quad x = e^u \in [e^a, e^{a+r}), \tag{46}$$

similarly, for $G_i(\ln y)$ on J_y (from E_v) and for $G_i(\ln xy)$ on J_{xy} (from E_{u+v}).

By the definition

$$L_i(\tau) := G_i(\ln \tau), \quad i = 1,2,3, \tag{47}$$

formula (44) changes into

$$L_i(\tau) = \begin{cases} H_i(\ln \tau) - H_i(a) + f(e^a), & \tau \in J_x = [e^a, e^{a+r}), \\ H_i(\ln \tau) - H_i(b) + f(e^b), & \tau \in J_y = [e^b, e^{b+r}), \\ H_i(\ln \tau) - H_i(a) - H_i(b) & \\ \quad +f(e^a) + f(e^b), & \tau \in J_{xy} = [e^{a+b}, e^{a+b+r}). \end{cases} \tag{48}$$

Obviously each $L_i(\tau)$, $i = 1,2,3$, satisfies the logarithmic equation restricted to $J(a,b;r)$.

In order to prove the approximation stated in (41), let us begin by the projection J_x: for $x \in J_x$; then $u = \ln x$, $g(u) = f(x)$, and $u \in [a, a+r)$; hence, from (44)

$$G_i(u) = G_i(\ln x) = L_i(x) = H_i(\ln x) - H_i(a) + f(e^a) \tag{49}$$

and from (45)

$$\|f(x) - L_i(x)\| < 15\delta \quad \text{for } x \in J_x. \tag{50}$$

Similarly for f on J_y and on J_{xy}.

Therefore, (41) is true with $L = L_1$ or $L = L_2$ or $L = L_3$, and Theorem 6 is proved. □

Remark 7. Remarks about the consequence of a possible overlapping of the projections of the given restricted domain, like those in Remark 4, could be repeated here.

2.3. About the Pexiderized Forms of the Foregoing Equations.

Stability results for the Pexiderized forms of the additive and the logarithmic equations, namely,

$$\begin{aligned} A(x+y) &= B(x) + C(y), \quad (x,y) \in E(a,b;r), \\ A(xy) &= B(x) + C(y), \quad (x,y) \in J(a,b;r), \end{aligned} \tag{51}$$

can be easily stated on the ground of the foregoing Theorems 1 and 6.

In fact, when the inequality

$$\|\psi(x+y) - \lambda(x) - \mu(y)\| < \delta \tag{52}$$

is satisfied for every $(x,y) \in E(a,b;r)$, the statement of Theorem 1 can be easily adapted to the condition (52), because the functions $\lambda(x)$, $\mu(y)$ and $\psi(x+y)$ play a role like that of the restrictions of f to the projections E_x, E_y, E_{x+y} in case of a unique function f.

Similarly for

$$\|\rho(xy) - \tau(x) - \sigma(y)\| < \delta \tag{53}$$

restricted to $J(a,b;r)$, by use of Theorem 6.

Remark 8. In case of a Pexiderized equation on restricted domain, overlapping of the projections of the given bounded domain obviously produces no changes in the result.

3. About the Remaining Two Cauchy Equations (7) $\varphi(x+y) = \varphi(x)\varphi(y)$ and (8) $\varphi(xy) = \varphi(x)\varphi(y)$ on a Bounded Restricted Domain

3.1. Preliminaries. As for (7) $\varphi(x+y) = \varphi(x)\varphi(y)$ the restricted domain is assumed to be $E(a,b;r)$ defined in (6); the domain of (8) $\varphi(xy) = \varphi(x)\varphi(y)$ is $J(a,b;r)$ defined in (32) for fixed real a, b and $r > 0$.

Let us premise the local solutions of the above equations (see papers [8, 9] and [10], resp.).

Lemma 9. *Let $\varphi : D_\varphi = E_x \cup E_y \cup E_{x+y} \to R$ satisfy (7) restricted to $E(a,b;r)$ defined in (6).*

If and only if there exists some $(x', y') \in E(a,b;r)$, $(x' \in E_x, y' \in E_y)$ such that $\varphi(x') \neq 0$ and $\varphi(y') \neq 0$, the following properties (P_1), (P_2), (P_3) hold:

(P_1) *$\varphi(t) \neq 0$ for every $t \in E_{x+y}$ (hence for every $t \in D_\varphi$);*

(P_2) *sgn $\varphi(t)$ is constant on each projection E_x, E_y, E_{x+y} (not necessarily the same in different projections);*

(P_3) *$\varphi(t)$ is given on D_φ by the following formulas:*

(i) *if $t \in E_x$, $\varphi(t) = Ae^{G(t)-G(a)}$,*

(ii) *if $t \in E_y$, $\varphi(t) = Be^{G(t)-G(b)}$,*

(iii) *if $t \in E_{x+y}$, $\varphi(t) = ABe^{G(t)-G(a+b)}$,*

where $G : R \to R$ is additive on R^2; $A \neq 0$, $B \neq 0$ are constant.

Remark 10. Notice that φ restricted to each of the projections E_x, E_y, E_{x+y} is the restriction of a solution $\Phi : R \to R$ of the equation

$$\Phi(x+y) = K\Phi(x)\Phi(y) \tag{54}$$

valid on the whole R^2, for suitable $K \neq 0$.

Since this equation can be written as

$$K\Phi(x+y) = K\Phi(x)K\Phi(y), \tag{55}$$

we get $K\Phi(t) = e^{h_0(t)}$, for some additive $G(t)$, whence formulas in (P_3) of Lemma 9 for

$$K = \frac{e^{G(a)}}{A}, \qquad K = \frac{e^{G(b)}}{B}, \qquad K = \frac{e^{G(a+b)}}{AB}, \tag{56}$$

respectively, in E_x, E_y, E_{x+y}.

Lemma 11 (see [10]). *The general nowhere vanishing solution $\varphi : D_\varphi = J_x \cup J_y \cup J_{xy} \to R$ of (8) restricted to the set $J(a,b;r)$ defined in (32) is given by the following formulas:*

$$\begin{aligned} \varphi(x) &= e^{h(\ln x)} \cdot e^{-h(a)} \cdot \varphi(e^a), \quad x \in J_x = [e^a, e^{a+r}), \\ \varphi(y) &= e^{h(\ln y)} \cdot e^{-h(b)} \cdot \varphi(e^b), \quad y \in J_y = [e^b, e^{b+r}), \\ \varphi(xy) &= e^{h(\ln xy)} \cdot e^{-h(a+b)} \cdot \varphi(e^{a+b}), \\ & xy \in J_{xy} = [e^{a+b}, e^{a+b+r}), \end{aligned} \tag{57}$$

where $h : R \to R$ is additive on R^2.

Remark 12. As in Remark 10, we can see that the local solution φ of (8), restricted to each of the projections J_x, J_y, J_{xy}, is the restriction of a solution $\Psi : R^+ \to R$ of a more general equation

$$\Psi(xy) = K\Psi(x)\Psi(y), \quad (x,y) \in R^+ \times R^+ \tag{58}$$

for suitable $K \neq 0$.

From $K\Psi(xy) = K\Psi(x)K\Psi(y)$, it follows that $\Psi(t) = (1/K)e^{h(\ln t)}$, with

$$\begin{aligned} K &= \frac{e^{h(a)}}{\varphi(e^a)} \quad \text{on the projection } J_x, \\ K &= \frac{e^{h(b)}}{\varphi(e^b)} \quad \text{on } J_y, \\ K &= \frac{e^{h(a+b)}}{\varphi(e^{a+b})} \quad \text{on } J_{xy}. \end{aligned} \tag{59}$$

3.2. How the Question of Local Stability of (7) *or* (8) *Has to Be Properly Formulated?* The foregoing Remarks 10 and 12, which point out a connection of the restricted equation under consideration with more general equations, namely,

$$\Psi(x+y) = K\Psi(x)\Psi(y), \qquad \Psi(xy) = K\Psi(x)\Psi(y), \tag{60}$$

suggest the following forms of perturbation of such equations:

$$f(x+y) = e^{\delta\cdot\theta(x,y)} f(x) f(y), \tag*{$(7)_\delta$}$$

$$f(xy) = e^{\delta\cdot\theta(x,y)} f(x) f(y), \tag*{$(8)_\delta$}$$

for $-1 < \theta(x,y) < 1$ and some fixed $\delta > 0$.

Moreover, it is known (see [8, 9]) that the local solutions of the restricted equations (7) or (8), which vanish somewhere, are expressed by formulas containing arbitrary functions; therefore, the problem of the local stability seems to be significant in the set of nowhere vanishing functions f.

In this frame, the perturbed forms $(7)_\delta$ and $(8)_\delta$ can be written equivalently as

$$e^{-\delta} < \frac{f(x+y)}{f(x) f(y)} < e^{\delta}, \tag*{$(7)'_\delta$}$$

$$e^{-\delta} < \frac{f(xy)}{f(x) f(y)} < e^{\delta}. \tag*{$(8)'_\delta$}$$

The stability results which follow are framed in this context.

3.3. A Sign Property concerning the Perturbed Forms of the Exponential Equation and the Power Equation. Here, we will be concerned with the condition $(7)'_\delta$, $(x,y) \in E(a,b;r)$ defined in (6) for some fixed $\delta > 0$, in the set of functions $f : D_f = E_x \cup E_y \cup E_{x+y} \to R$, such that $f(t) \neq 0$ for every $t \in D_f$.

Let us premise a remark about signs of nowhere vanishing functions f satisfying $(7)'_\delta$ on $E(a,b;r)$. From Lemma 9, Property (P_2), it is known that every nowhere vanishing solution of the exponential Cauchy equation restricted to $E(a,b;r)$ keeps a constant sign in each of the projections E_x, E_y, E_{x+y} of $E(a,b;r)$.

We will see that a similar property is true also for every solution of the restricted condition $(7)'_\delta$, which is rewritten here as follows:

$$f(x+y) = e^{\delta\cdot\theta(x,y)} f(x) f(y), \quad -1 < \theta(x,y) < 1. \tag*{$(7)'_{\delta\theta}$}$$

From $(7)'_{\delta\theta}$, assuming $(x, y) = (a, b+\xi)$ and $(x, y) = (a+\xi, b)$ with $0 \leq \xi < r$, we get

$$f(a+b+\xi) = e^{\theta'\delta} f(a) f(b+\xi), \quad \theta' = \theta(a, b+\xi), \tag{61}$$

$$f(a+b+\xi) = e^{\theta''\delta} f(a+\xi) f(b), \quad \theta'' = \theta(a+\xi, b), \tag{62}$$

whence

$$f(a) f(b+\xi) = e^{(\theta''-\theta')\delta} f(a+\xi) f(b), \quad \xi \in [0, r), \tag{63}$$

$$f(b+\xi) = e^{(\theta''-\theta')\delta} \frac{f(b)}{f(a)} f(a+\xi). \tag{64}$$

Moreover, from $(7)'_{\delta\theta}$ for $(x, y) = (a+\xi, b+\xi)$, $0 \leq \xi < r/2$, $\theta''' = \theta(a+\xi, b+\xi)$, $f(a+b+2\xi) = e^{\theta'''\delta} f(a+\xi) f(b+\xi) = e^{\theta'''\delta} f(a+\xi)^2 e^{(\theta''-\theta')\delta} f(b)/f(a)$; hence, $f(t)$ has constant sign in E_{x+y}.

As a consequence, from (62), (64) it follows that f has constant signs also in E_x and in E_y (the signs of $f(a)$ and of $f(b)$, resp.).

This proves the following.

Lemma 13. *Every nowhere vanishing function $f : D_f = E_x \cup E_y \cup E_{x+y} \rightarrow R$ satisfying $(7)_\delta$ in $E(a, b; r)$ keeps constant sign in each of the projections E_x, E_y, E_{x+y} of $E(a, b; r)$.*

Similarly, we can prove a sign property concerning the perturbed form of the power equation.

Let us consider now the condition $(8)'_\delta$; namely,

$$e^{-\delta} < \frac{f(xy)}{f(x) f(y)} < e^{\delta}, \quad (x, y) \in J(a, b; r) \tag{65}$$

for some fixed $\delta > 0$, assuming $f(t) \neq 0$ for every $t \in D_f = J_x \cup J_y \cup J_{xy}$.

The usual substitutions of variables x, y allow us to use the foregoing results about the exponential equation. Put

$$\begin{aligned} &x = e^u, \quad u = \ln x, \quad f(x) = f(e^u) =: g(u), \quad a \leq u < a+r \\ &y = e^v, \quad v = \ln y, \quad f(y) = f(e^v) =: g(v), \quad b \leq v < b+r, \end{aligned} \tag{66}$$

whence $f(xy) =: g(u+v)$, $a+b \leq u+v < a+b+r$.

Then (65) is transformed into

$$e^{-\delta} < \frac{g(u+v)}{g(u) g(v)} < e^{\delta}, \quad (u, v) \in E(a, b; r); \tag{67}$$

namely

$$g(u+v) = e^{\delta\cdot\theta(u,v)} g(u) g(v), \quad -1 < \theta(u, v) < 1. \tag{65_θ}$$

Therefore, from Lemma 13, it follows that $g(u)$ has constant sign ($=\operatorname{sgn} g(a)$) in $[a, a+r)$, whence $f(x)$ has constant sign ($=\operatorname{sgn} f(e^a)$) in $[e^a, e^{a+r})$; similarly for $g(v)$ in $[b, b+r)$, namely for $f(y)$ in $[e^b, e^{b+r})$ and for $g(u+v)$ in $[a+b, a+b+r)$, namely $f(xy)$ in $[e^{a+b}, e^{a+b+r})$.

Hence, the following result is proved.

Lemma 14. *Every nowhere vanishing function $f : D_f = J_x \cup J_y \cup J_{xy} \rightarrow R$ satisfying (65) restricted to $J(a, b; r)$ has constant signs in each of the projections J_x, J_y, J_{xy} of $J(a, b; r)$.*

3.4. A Result of Local Stability for the Exponential Cauchy Equation. In the set of functions $f : D_f = E_x \cup E_y \cup E_{x+y} \rightarrow R$ such that $f(t) \neq 0$ for every $t \in D_f$, let us consider the inequality $(7)'_\delta$, with $(x, y) \in E(a, b; r)$ for some fixed $\delta > 0$.

From $(7)'_\delta$

$$-\delta < \ln \frac{f(x+y)}{f(x) f(y)} < \delta; \tag{68}$$

since

$$0 < \frac{f(x+y)}{f(x) f(y)} = \frac{|f(x+y)|}{|f(x) f(y)|}, \tag{69}$$

f satisfies

$$\left|\ln|f(x+y)| - \ln|f(x)| - \ln|f(y)|\right| < \delta; \tag{70}$$

namely for $\lambda(t) := \ln|f(t)|$,

$$|\lambda(x+y) - \lambda(x) - \lambda(y)| < \delta, \quad (x, y) \in E(a, b; r). \tag{71}$$

On the ground of Theorem 1, there exists (at least) one additive function $H : R \rightarrow R$, such that the function $L : D_f \rightarrow S$ defined by

$$\begin{aligned} L(x) &= H(x) + \lambda(a) - H(a), \quad x \in E_x, \\ L(y) &= H(y) + \lambda(b) - H(b), \quad y \in E_y, \\ L(x+y) &= H(x+y) + \lambda(a) + \lambda(b) - H(a) - H(b), \\ &\qquad x+y \in E_{x+y}, \end{aligned} \tag{72}$$

is a local solution of the additive Cauchy equation restricted to $E(a, b; r)$, such that

$$|\lambda(t) - L(t)| < 15\delta \quad \text{for every } t \in D_f. \tag{73}$$

Since $|f(t)| = e^{\lambda(t)}$, whence,

$$f(t) = \operatorname{sgn} f(t) \cdot e^{\lambda(t)}, \quad t \in D_f, \tag{74}$$

we get

$$f(t) = \operatorname{sgn} f(t) \cdot e^{L(t)+15\delta\cdot\theta(t)}, \quad -1 < \theta(t) < 1,\ t \in D_f, \tag{75}$$

substitution of $L(t)$ by its explicit formulas gives the following:

$$\begin{aligned} \text{for } t \in E_x: \quad e^{-15\delta\cdot\theta(t)} f(t) &= \operatorname{sgn} f(a) \cdot e^{H(t)+\ln|f(a)|-H(a)} \\ &= \operatorname{sgn} f(a) \cdot |f(a)| \cdot e^{-H(a)} \cdot e^{H(t)} \\ &= f(a) \cdot e^{-H(a)} \cdot e^{H(t)}, \end{aligned} \tag{76}$$

and similarly

$$\text{for } t \in E_y, \quad e^{-15\delta\cdot\theta(t)} f(t) = f(b) \cdot e^{-H(b)} \cdot e^{H(t)},$$

$$\text{for } t \in E_{x+y}, \quad e^{-15\delta\cdot\theta(t)} f(t) = f(a) \cdot f(b) \cdot e^{-H(a)-H(b)} \cdot e^{H(t)}. \tag{77}$$

By defining

$$F(t) := e^{-15\delta\cdot\theta(t)} f(t), \quad t \in E_x \cup E_y \cup E_{x+y}, \ -1 < \theta(t) < 1, \tag{78}$$

it is easily proved that

$$F(x+y) = F(x)F(y) \quad \text{for } (x,y) \in E(a,b;r). \tag{79}$$

Moreover, from (78),

$$e^{-15\delta} \cdot F(t) < f(t) < e^{15\delta} \cdot F(t), \quad t \in D_f. \tag{80}$$

This means that the values of $f(t)$ in D_f are "near" (in dependence on δ) the values of a local solution $F(t)$ of the corresponding equation restricted to the same domain $E(a,b;r)$ and give the following theorem of local stability.

Theorem 15. *If the function $f : D_f = E_x \cup E_y \cup E_{x+y} \to R$, is nowhere vanishing in its domain D_f and satisfies $(7)'_\delta$, for some given $\delta > 0$ and every $(x,y) \in E(a,b;r)$ defined in* (6) *for given $(a,b) \in R^2$ and $r > 0$, then there exists (at least) an additive function $H : R \to R$ such that the function $F : D_f \to R$*

$$F(t) = \begin{cases} f(a)e^{-H(a)} \cdot e^{H(t)}, & t \in E_x, \\ f(b)e^{-H(b)} \cdot e^{H(t)}, & t \in E_y, \\ f(a)f(b)e^{-H(a)-H(b)} \cdot e^{H(t)}, & t \in E_{x+y} \end{cases} \tag{81}$$

has both the properties:

(i) *F is a nowhere vanishing local solution of the exponential Cauchy equation $F(x+y) = F(x)F(y)$ restricted to $E(a,b;r)$;*

(ii) *the values $F(t)$ are near the values $f(t)$ on D_f; more exactly*

$$e^{-15\delta} \cdot F(t) < f(t) < e^{15\delta} \cdot F(t), \quad t \in D_f. \tag{82}$$

3.5. A Result on Local Stability of the Power Cauchy Equation. In the set of nowhere vanishing functions $f : D_f = J_x \cup J_y \cup J_{xy} \to R$, let us consider the inequality (65), $(x,y) \in J(a,b;r) \subset R^2$ defined in (32), for some given $\delta > 0$.

The usual substitutions

$$x = e^u, \quad u = \ln x, \quad f(x) = f(e^u) =: g(u),$$
$$y = e^v, \quad v = \ln y, \quad f(y) = f(e^v) =: g(v), \tag{83}$$
$$xy = e^{u+v}, \quad f(xy) =: g(u+v),$$

transform the condition (65) into

$$e^{-\delta} < \frac{g(u+v)}{g(u)g(v)} < e^{\delta}, \quad (u,v) \in E(a,b;r). \tag{65$_g$}$$

Hence, thanks to Theorem 15 (referred to $g(t)$ instead of $f(t)$), there exists (at least) one additive function $H : R \to R$ such that the function $G : D_g \to R$ defined by

$$G(u) = g(a) \cdot e^{-H(a)} \cdot e^{H(u)}, \quad u \in E_u,$$
$$G(v) = g(b) \cdot e^{-H(b)} \cdot e^{H(v)}, \quad v \in E_v,$$
$$G(u+v) = g(a)g(b) \cdot e^{-H(a)-H(b)} \cdot e^{H(u+v)}, \quad u+v \in E_{u+v}, \tag{84}$$

satisfies the exponential equation $G(u+v) = G(u)G(v)$ restricted to $E(a,b;r)$ and approaches g on $D_g = E_u \cup E_v \cup E_{u+v}$ as follows:

$$e^{-15\delta} G(t) < g(t) < e^{15\delta} G(t), \quad t \in D_g. \tag{82$_g$}$$

Formula $(82)_g$ can be rewritten as

$$e^{15\delta\cdot\theta(t)} g(t) = G(t), \quad -1 < \theta(t) < 1, \tag{82$'_g$}$$

for $t \in E_u \cup E_v \cup E_{u+v}$.

From the definition of G,

for $u \in E_u = [a, a+r)$ then $g(u) = f(x)$, $G(u) = G(\ln x) = f(e^a) \cdot e^{-H(a)} \cdot e^{H(\ln x)}$,

similarly for $v \in E_v = [b, b+r)$ then $g(v) = f(y)$, $G(\ln y) = f(e^b) \cdot e^{-H(b)} \cdot e^{H(\ln y)}$,

and for $u+v \in E_{u+v} = [a+b, a+b+r)$: $G(\ln(xy)) = f(e^a)f(e^b) \cdot e^{-H(a)-H(b)} \cdot e^{H(\ln xy)}$.

Hence, by defining $\Phi : D_f \to R$ as follows:

$$\Phi(x) := e^{15\delta\cdot\theta(u)} f(x) = G(\ln x) = f(e^a)e^{-H(a)}e^{H(\ln x)},$$
$$\Phi(y) := e^{15\delta\cdot\theta(v)} f(y) = G(\ln y) = f(e^b)e^{-H(b)}e^{H(\ln y)},$$
$$\Phi(xy) := e^{15\delta\cdot\theta(u+v)} f(xy) = G(\ln(xy)) = f(e^a)f(e^b)e^{-H(a)-H(b)}e^{H(\ln(xy))}, \tag{85}$$

we get

$$e^{15\delta\cdot\theta(t)} f(t) = \Phi(t), \quad -1 < \theta(t) < 1, \ t \in J_x \cup J_y \cup J_{xy}; \tag{86}$$

namely,

$$e^{-15\delta}\Phi(t) < f(t) < e^{15\delta}\Phi(t), \quad t \in D_f = J_x \cup J_y \cup J_{xy}. \tag{87}$$

This proves the following property of local stability of the "power" Cauchy equation.

Theorem 16. *Let the nowhere vanishing function $f : D_f \subset R^+ \to R$ satisfy the condition* (65) *for some given $\delta > 0$ and every $(x,y) \in J(a,b;r)$ defined in* (32), *for given $(a,b) \in R^2$ and $r > 0$; $D_f = J_x \cup J_y \cup J_{xy}$; then there exists (at least)* *an additive function $H : R \to R$ such that the function $\Phi : D_f \to R$ defined by*

$$\Phi(t) = \begin{cases} f(e^a)e^{-H(a)} \cdot e^{H(\ln t)}, & t \in J_x, \\ f(e^b)e^{-H(b)} \cdot e^{H(\ln t)}, & t \in J_y, \\ f(e^a) f(e^b) e^{-H(a)-H(b)} \cdot e^{H(\ln t)}, & t \in J_{xy} \end{cases} \quad (88)$$

has both the following properties:

(i) *Φ is a local solution of the Cauchy equation $\Phi(xy) = \Phi(x)\Phi(y)$ restricted to $J(a,b;r)$;*

(ii) *the values of $\Phi(t)$ are near the values $f(t)$ in D_f; more exactly*

$$e^{-15\delta} \cdot \Phi(t) < f(t) < e^{15\delta} \cdot \Phi(t), \quad t \in D_f = J_x \cup J_y \cup J_{xy}. \quad (89)$$

3.6. Remark about the Pexiderized Forms of the Foregoing Equations. According to the remarks at the end of Section 2, the stability results given by Theorems 15 and 16 can be easily adapted to the Pexiderized forms of the corresponding equations, namely, to

$$\begin{gathered} \alpha(x+y) = \beta(x)\gamma(y) \quad \text{restricted to } E(a,b;r), \\ \rho(xy) = \sigma(x)\tau(y) \quad \text{restricted to } J(a,b;r), \end{gathered} \quad (90)$$

for nowhere vanishing functions

$$\begin{gathered} \beta : E_x \longrightarrow R, \qquad \gamma : E_y \longrightarrow R, \qquad \alpha : E_{x+y} \longrightarrow R, \\ \sigma : J_x \longrightarrow R, \qquad \tau : J_y \longrightarrow R, \qquad \rho : J_{xy} \longrightarrow R. \end{gathered} \quad (91)$$

References

[1] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[2] D. H. Hyers, "The stability of homomorphisms and related topics," in *Global Analysis-Analysis on Manifolds*, M. Th. Rassias, Ed., pp. 140–153, Teubner, Leipzig, Germany, 1983.

[3] D. H. Hyers and T. M. Rassias, "Approximate homomorphisms," *Aequationes Mathematicae*, vol. 44, no. 2-3, pp. 125–153, 1992.

[4] F. Skof, "Sull'approssimazione delle applicazioni localmente δ-additive," *Atti dell' Accademia delle Scienze di Torino*, vol. 117, pp. 337–389, 1983.

[5] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.

[6] N. Brillouët-Belluot, J. Brzdęk, and K. Ciepliński, "On some recent developments in Ulam's type stability," *Abstract and Applied Analysis*, vol. 2012, Article ID 716936, 41 pages, 2012.

[7] Z. Daróczy and L. Losonczi, "Über die Erweiterung der auf einer Punktmenge additiven Funktionen," *Publicationes Mathematicae Debrecen*, vol. 14, pp. 239–245, 1967.

[8] J. Aczél and F. Skof, "Local Pexider and Cauchy Equations," *Aequationes Mathematicae*, vol. 73, no. 3, pp. 311–320, 2007.

[9] F. Skof, "The general solution of the exponential Cauchy equation on a bounded restricted domain," *Aequationes Mathematicae*, vol. 73, no. 1-2, pp. 144–155, 2007.

[10] F. Skof, "Extension results for restricted Cauchy equations ," *Atti dell' Accademia delle Scienze di Torino*, vol. 141, pp. 33–42; 51–60, 2007.