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Research Article

The Intersection of Upper and Lower Semi-Browder Spectrum of Upper-Triangular Operator Matrices

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When $A \in B(H)$ and $B \in B(K)$ are given, we denote by M_C the operator acting on the infinite-dimensional separable Hilbert space $H \oplus K$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, it is proved that there exists some operator $C \in B(K, H)$ such that M_C is upper semi-Browder if and only if there exists some left invertible operator $C \in B(K, H)$such that M_C is upper semi-Browder. Moreover, a necessary and sufficient condition for M_C to be upper semi-Browder for some $C \in G(K, H)$ is given, where $G(K, H)$ denotes the subset of all of the invertible operators of $B(K, H)$.

1. Introduction

It is well known that if H is a Hilbert space, T is a bounded linear operator defined on H, and H_1 is an invariant closed subspace of T, then T can be represented in the following form:

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : H_1 \oplus H_1^{\perp} \longrightarrow H_1 \oplus H_1^{\perp}, \tag{1}$$

which motivated the interest in 2×2 upper-triangular operator matrices. For recent investigations on this subject, see references [1–23].

Throughout this paper, let H and K be separable infinite-dimensional complex Hilbert spaces, and let $B(H, K)$ be the set of all bounded linear operators from H into K; when $H = K$, we write $B(H, H)$ as $B(H)$. For $A \in B(H)$, $B \in B(K)$, and $C \in B(K, H)$, we have $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(H \oplus K)$. For $T \in B(H, K)$, let $R(T)$ and $N(T)$ denote the range and the kernel of T, respectively, and denote that $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim K/R(T)$. If $T \in B(H)$, the ascent $\operatorname{asc}(T)$ of T is defined to be the smallest nonnegative integer k which satisfies and $N(T^k) = N(T^{k+1})$. If such k does not exist, then the ascent of T is defined as infinity. Similarly, the descent $\operatorname{des}(T)$ of T is defined as the smallest nonnegative integer k for which $R(T^k) = R(T^{k+1})$ holds. If such k does not exist, then $\operatorname{des}(T)$ is defined as infinity, too. If the ascent and the descent of T are finite, then they are equal (see [6]). For $T \in B(H)$, if $R(T)$ is closed and $\alpha(T) < \infty$, then T is said to be an upper semi-Fredholm operator; if $\beta(T) < \infty$, which implies that $R(T)$ is closed, then T is said to be a lower semi-Fredholm operator. If $T \in B(H)$ is either upper or lower semi-Fredholm operator, then T is said to be a semi-Fredholm operator. If both $\alpha(T) < \infty$ and $\beta(T) < \infty$, then T is said to be a Fredholm operator. For a semi-Fredholm operator T, its index $\operatorname{ind}(T)$ is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

For a semi-Fredholm operator $T \in B(H)$, its shift Samuel multiplicity $s\_\mathrm{mul}(T)$ and backward shift Samuel multiplicity $b.s.\_\mathrm{mul}(T)$ are defined, respectively, by the following (see [24]):

$$s\_\mathrm{mul}(T) = \lim_{k \to \infty} \frac{\beta(T^k)}{k}, \qquad b.s.\_\mathrm{mul}(T) = \lim_{k \to \infty} \frac{\alpha(T^k)}{k}. \tag{2}$$

Moreover, it has been proved that $s\_\mathrm{mul}(T), b.s.\_\mathrm{mul}(T) \in \{0, 1, 2, \ldots, \infty\}$ and that $\operatorname{ind}(T) = b.s.\_\mathrm{mul}(T) - s\_\mathrm{mul}(T)$. These two invariants refine the Fredholm index and can be regarded as the stabilized dimensions of the kernel and the cokernel (see [24]).

In this paper, the sets of invertible operators and left invertible operators from H into K are denoted by $G(H,K)$ and $G_l(H,K)$, respectively; the sets of all Fredholm operators, upper semi-Fredholm operators, and lower semi-Fredholm operators from H into K are denoted by $\Phi(H,K)$, $\Phi_+(H,K)$, and $\Phi_-(H,K)$, respectively; the sets of all Browder operators, upper semi-Browder operators, and lower semi-Browder operators, on H are defined, respectively, by the following:

$$\Phi_b(H) := \{T \in \Phi(H) : \text{asc}(T) = \text{des}(T) < \infty\},$$
$$\Phi_{ab}(H) := \{T \in \Phi_+(H) : \text{asc}(T) < \infty\}, \quad (3)$$
$$\Phi_{sb}(H) := \{T \in \Phi_-(H) : \text{des}(T) < \infty\}.$$

Moreover, for $T \in B(H)$, we introduce its corresponding spectra as follows.

The spectrum is given as $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G(H)\}$.

The left spectrum is given as $\sigma_l(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin G_l(H)\}$.

The essential spectrum is defined as $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(H)\}$.

The upper semi-Fredholm spectrum is defined as $\sigma_{SF+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(X)\}$.

The lower semi-Fredholm spectrum is presented as $\sigma_{SF-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-(X)\}$.

The Browder spectrum is presented as $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_b(H)\}$.

The upper semi-Browder spectrum is defined as $\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{ab}(X)\}$.

The lower semi-Browder spectrum is presented as $\sigma_{sb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{sb}(X)\}$.

Using the Samuel multiplicities, Zhang and Wu (see [20]) gave a necessary and sufficient condition for which $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K,H)$ and characterized the set of $\cap_{C\in B(K,H)}\sigma_{ab}(M_C)$. In this paper, our main goal is to characterize the intersection of $\cap_{C\in G_l(K,H)}\sigma_{ab}(M_C)$ and $\cap_{C\in G(K,H)}\sigma_{ab}(M_C)$. This paper is organized as follows. In Section 2, we give a necessary and sufficient condition for which $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in G_l(K,H)$ and get

$$\bigcap_{C\in G_l(K,H)} \sigma_{ab}(M_C) = \bigcap_{C\in B(K,H)} \sigma_{ab}(M_C). \quad (4)$$

In Section 3, we give a necessary and sufficient condition for which $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in G(K,H)$ and get

$$\begin{aligned}\bigcap_{C\in G(K,H)} \sigma_{ab}(M_C) &= \bigcap_{C\in \Phi(K,H)} \sigma_{ab}(M_C) \\ &= \left(\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)\right) \\ &\quad \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}.\end{aligned} \quad (5)$$

For the sake of convenience, we now present some lemmas which will be used in the sequel.

Lemma 1 (see [20, 24]). *An operator $T \in B(H)$ is semi-Fredholm if and only if T can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2 \oplus H_3$:*

$$T = \begin{pmatrix} T_1 & T_{12} & T_{13} \\ 0 & T_2 & T_{23} \\ 0 & 0 & T_3 \end{pmatrix} : H_1 \oplus H_2 \oplus H_3 \longrightarrow H_1 \oplus H_2 \oplus H_3, \quad (6)$$

where $\dim(H_3) < \infty$, T_1 is a right invertible operator, T_3 is a finite nilpotent operator, T_2 is a left invertible operator, and $\min\{\text{ind}(T_1), -\text{ind}(T_2)\} < \infty$. Moreover, $\text{ind}(T_1) = \alpha(T_1) = b.s._-\text{mul}(T)$, $\text{ind}(T_2) = -\beta(T_2) = -s_-\text{mul}(T)$, and $\text{ind}(T) = \alpha(T_1) - \beta(T_2)$.

Lemma 2 (see [18]). *Let $A \in B(H)$, $B \in B(K)$, and $C \in B(K,H)$.*

(1) *If $A \in \Phi_b(H)$, then $B \in \Phi_{ab}(K)$ if $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K,H)$.*

(2) *If $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K,H)$, then $A \in \Phi_{ab}(H)$.*

(3) *If $A \in \Phi_{ab}(H)$ and $B \in \Phi_{ab}(K)$, then $M_C \in \Phi_{ab}(H \oplus K)$ for any $C \in B(K,H)$.*

(4) *If $B \in \Phi_b(K)$, then $A \in \Phi_{ab}(H)$ if $M_C \in \Phi_{ab}(H\oplus K)$ for some $C \in B(K,H)$; $A \in \Phi_{sb}(H)$ if $M_C \in \Phi_{sb}(H \oplus K)$ for some $C \in B(K,H)$.*

(5) *If $M_C \in \Phi_b(H \oplus K)$ for some $C \in B(K,H)$, then $A \in \Phi_{ab}(H)$ and $B \in \Phi_{sb}(K)$.*

(6) *If two of A, B, and M_C are Browder, then so is the third.*

Lemma 3 (see [20]). *Let $T \in B(H)$. Then, T is upper semi-Browder if T can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2$:*

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix}, \quad (7)$$

where $\dim(H_1) < \infty$, T_1 is nilpotent, T_2 is left invertible, and $\beta(T_2) = s_-mul(T) = -ind(T)$.

Lemma 4 (see [20]). *Let $T \in B(H)$. Then, T is lower semi-Browder if T can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2$:*

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix}, \quad (8)$$

where $\dim(H_2) < \infty$, T_1 is right invertible, T_2 is nilpotent, and $\alpha(T_1) = b.s._-\text{mul}(T) = \text{ind}(T)$.

Lemma 5 (see [20]). *For any given $A \in B(H)$ and $B \in B(K)$, $M_C \in \Phi_{ab}(H \oplus K)$ for some $C \in B(K,H)$ if $A \in \Phi_{ab}(H)$ and*

$$\begin{aligned} &s_-\text{mul}(A) = \infty \quad \text{if } B \notin \Phi_+(K), \\ &b.s._-\text{mul}(B) \le s_-\text{mul}(A) \quad \text{if } B \in \Phi_+(K). \end{aligned} \quad (9)$$

Lemma 6 (see [9]). *For any given $A \in B(H)$ and $B \in B(K)$, M_C is left invertible for some $C \in B(K,H)$ if A is left invertible and*

$$\begin{aligned} a(B) &\le \beta(A) \quad \text{if } R(B) \text{ is closed}, \\ \beta(A) &= \infty \quad \text{if } R(B) \text{ is not closed}. \end{aligned} \tag{10}$$

Lemma 7 (see [25]). *Let V be a linear subspace of H. Then, the following statements are equivalent.*

(1) *Any bounded operator $A \in B(H)$ with $R(A) \subseteq V$ is compact.*

(2) *V contains no closed infinite-dimensional subspace.*

2. $\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)$ and $\bigcap_{C\in G_l(K,H)} \sigma_{ab}(M_C)$

In [1, 20], the authors have proved that

$$\begin{aligned} &\bigcap_{C\in B(K,H)} \sigma_b(M_C) \\ &\quad = \sigma_{ab}(A) \cup \sigma_{sb}(B) \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) \\ &\qquad + \alpha(B-\lambda) \ne \beta(A-\lambda) \\ &\qquad + \beta(B-\lambda)\}. \end{aligned} \tag{11}$$

They, moreover, proved that

$$\begin{aligned} &\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C) \\ &\quad = \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B), \\ &\qquad s\_\mathrm{mul}(A-\lambda) < \infty\} \\ &\quad \cup \{\lambda \in \Phi(A) \cap \Phi_+(B) : b.s.\_\mathrm{mul}(B-\lambda) \\ &\qquad > s\_\mathrm{mul}(A-\lambda)\}. \end{aligned} \tag{12}$$

Comparing the above two kinds of spectra with the upper semi-Weyl spectrum and Weyl spectrum, one may expect that the following equality holds:

$$\begin{aligned} &\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C) \\ &\quad = \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B),\ \beta(A-\lambda) < \infty\} \\ &\quad \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) \\ &\qquad > \beta(A-\lambda) + \beta(B-\lambda)\}. \end{aligned} \tag{13}$$

However, it is not that case, as the following example shows.

Example 8. Let A be the unilateral shift on ℓ^2, that is,

$$V : \ell^2 \longrightarrow \ell^2,\ \{z_1, z_2, \ldots\} \longmapsto \{0, z_1, z_2, \ldots\}, \tag{14}$$

and let the operators A and B be defined by

$$A = V, \qquad B = \begin{pmatrix} (V^*)^2 & 0 \\ 0 & V^5 \end{pmatrix} : \ell^2 \oplus \ell^2 \longrightarrow \ell^2 \oplus \ell^2. \tag{15}$$

Then, we have $b.s.\_\mathrm{mul}(B) = 2 > s\_\mathrm{mul}(A) = 1$, while $\alpha(A) + \alpha(B) = 2 < \beta(A) + \beta(B) = 6$. Moreover, $0 \in \bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)$, while $0 \notin \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B)$ and $\beta(A-\lambda) < \infty\} \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) > \beta(A-\lambda) + \beta(B-\lambda)\}$. Thus, (13) does not hold.

In spite of the above counter example, we have the following.

Proposition 9. *For any given $A \in B(H)$ and $B \in B(K)$, one has*

$$\begin{aligned} &\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C) \supseteq \sigma_{ab}(A) \\ &\quad \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B), \beta(A-\lambda) < \infty\} \\ &\quad \cup \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) \\ &\qquad > \beta(A-\lambda) + \beta(B-\lambda)\}. \end{aligned} \tag{16}$$

Proof. From the proof of Theorem 2.3 in [20], we know that when $A \in \Phi_{ab}(H)$, $s\_\mathrm{mul}(A) < \infty$ if and only if $\beta(A) < \infty$. Combining this fact with Corollary 2.5 of [20], it is easy to see that

$$\begin{aligned} &\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C) \supseteq \sigma_{ab}(A) \cup \{\lambda \in \mathbb{C} : \lambda \in \sigma_{SF+}(B), \\ &\qquad \beta(A-\lambda) < \infty\}. \end{aligned} \tag{17}$$

Noting that $\beta(B-\lambda) < \infty$ implies that $R(B-\lambda)$ is closed, it follows from corollary 2.5 of [2] that

$$\begin{aligned} &\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C) \supseteq \bigcap_{C\in B(K,H)} \sigma_{aw}(M_C) \\ &\quad \supseteq \{\lambda \in \mathbb{C} : \alpha(A-\lambda) + \alpha(B-\lambda) \\ &\qquad > \beta(A-\lambda) + \beta(B-\lambda)\}, \end{aligned} \tag{18}$$

where $\sigma_{aw}(M_C) = \{\lambda \in \mathbb{C} : M_C - \lambda$ is not uppersemi-Fredholm operator with index less than or equal to $0\}$. □

Now, we are ready to present the main result of this section.

Theorem 10. *For any given $A \in B(H)$ and $B \in B(K)$, one has*

$$\bigcap_{C\in G_l(K,H)} \sigma_{ab}(M_C) = \bigcap_{C\in B(K,H)} \sigma_{ab}(M_C). \tag{19}$$

Proof. Since $\bigcap_{C\in G_l(K,H)} \sigma_{ab}(M_C) \supseteq \bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)$ is obvious, it is sufficient to prove that if $M_C \in \Phi_{ab}(H \oplus K)$, then there exists some left invertible operator $Q \in B(K,H)$ such that $M_Q \in \Phi_{ab}(H \oplus K)$.

Suppose that $M_C \in \Phi_{ab}(H\oplus K)$. It follows from Lemma 5 that $A \in \Phi_{ab}(H)$ and

$$\begin{aligned} s\_\ \mathrm{mul}\ (A) &= \infty \quad \text{if } B \notin \Phi_+(K), \\ b.s.\_\ \mathrm{mul}\ (B) &\le s\_\ \mathrm{mul}\ (A) \quad \text{if } B \in \Phi_+(K). \end{aligned} \tag{20}$$

There are two cases to consider.

Case 1. Assume that $A \in \Phi_{ab}(H)$, $s\_\mathrm{mul}(A) = \infty$, and $B \notin \Phi_+(K)$. Then, it follows from Lemma 3 that A can be decomposed into the following form:

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2, \quad (21)$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator, and $\beta(A_2) = s\_\mathrm{mul}(A) = \infty$. So, we can let

$$Q = \begin{pmatrix} 0 \\ 0 \\ V \end{pmatrix} : K \longrightarrow H_1 \oplus R(A_2) \oplus (H_2 \ominus R(A_2)), \quad (22)$$

where $V \in B(K, (H_2 \ominus R(A_2)))$ is unitary. Obviously, Q is left invertible. Now, M_Q can be rewritten as

$$M_Q = \begin{pmatrix} A_1 & A_{12} & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & V \\ 0 & 0 & B \end{pmatrix} : H_1 \oplus H_2 \oplus K \longrightarrow H_1 \oplus R(A_2) \oplus (H_2 \ominus R(A_2)) \oplus K. \quad (23)$$

Since A_2 is left invertible and V is invertible, then there exist unique A_2' and V' such that $A_2'A_2 = I_{H_2}$ and $V'V = I_K$, and

$$\begin{pmatrix} A_2' & 0 & 0 \\ 0 & V' & 0 \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ 0 & V \\ 0 & B \end{pmatrix} = I_{H_2} \oplus I_K. \quad (24)$$

This implies that $\begin{pmatrix} A_2 & 0 \\ 0 & V \\ 0 & B \end{pmatrix}$ is left invertible. And, hence, Lemma 2 leads to $M_Q \in \Phi_{ab}(H \oplus K)$.

Case 2. Assume that $A \in \Phi_{ab}(H)$, $b.s.\_\mathrm{mul}(B) \le s\_\mathrm{mul}(A)$, and $B \in \Phi_+(K)$. Then, it follows from Lemma 3 that A can be decomposed into the following form:

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2, \quad (25)$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator, and $\beta(A_2) = s\_\mathrm{mul}(A)$. By the assumption that $B \in \Phi_+(K)$ and Lemma 1, we know that B can be decomposed into the following form with respect to some orthogonal decomposition $K = K_1 \oplus K_2 \oplus K_3$:

$$B = \begin{pmatrix} B_1 & * & * \\ 0 & B_2 & * \\ 0 & 0 & B_3 \end{pmatrix}, \quad (26)$$

where $\dim(K_3) < \infty$, B_1 is a right invertible operator, B_2 is a left invertible operator, B_3 is a finite nilpotent operator, and the parts marked by $*$ can be any operators. Moreover, $\infty > \alpha(B_1) = b.s.\_\mathrm{mul}(B)$. Thus, $\beta(A_2) \ge \alpha(B_1)$, and then there exists some left invertible $C_1 \in B(N(B_1),\ H_2 \ominus R(A_2))$. Noting that $\dim((K_1 \ominus N(B_1)) \oplus K_2 \oplus K_3) = \dim(H_1 \oplus R(A_2)) = \infty$, we can let $C_2 \in G((K_1 \ominus N(B_1)) \oplus K_2 \oplus K_3, H_1 \oplus R(A))$. Consider

$$Q = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} : N(B_1) \oplus [K_1 \ominus N(B_1) \oplus K_2 \oplus K_3] \longrightarrow (H_2 \ominus R(A_2)) \oplus [H_1 \oplus R(A_2)]. \quad (27)$$

Obviously, Q is left invertible, and M_Q can be rewritten as

$$M_Q = \begin{pmatrix} A_1 & A_{12} & C_{11} & 0 & C_{12} & C_{13} \\ 0 & A_{21} & C_{21} & 0 & C_{22} & C_{23} \\ 0 & 0 & 0 & C_1 & 0 & 0 \\ 0 & 0 & B_{11} & 0 & * & * \\ 0 & 0 & 0 & 0 & B_2 & * \\ 0 & 0 & 0 & 0 & 0 & B_3 \end{pmatrix} : H_1 \oplus H_2 \oplus (K_1 \ominus N(B_1) \oplus N(B_1)) \oplus K_2 \oplus K_3 \longrightarrow H_1 \oplus R(A_2) \oplus (H_2 \ominus dR(A_2)) \oplus K_1 \oplus K_2 \oplus K_3, \quad (28)$$

where A_{21} and B_{11} are invertible and C_1 and B_2 are left invertible. Similar to the proof of Case 1, through direct calculation we can show that $\begin{pmatrix} A_{21} & C_{21} & 0 \\ 0 & 0 & C_1 \\ 0 & B_{11} & 0 \end{pmatrix}$ is left invertible. Also since $\dim(H_1) < \infty$ and $\dim(K_3) < \infty$, we have $A_1 \in \Phi_b(H_1)$ and $B_3 \in \Phi_b(K_3)$. Thus, it follows from Lemma 2 that $M_C \in \Phi_{ab}(H \oplus K)$. □

By duality, we have the following.

Theorem 11. *For any given $A \in B(H)$ and $B \in B(K)$, one has*

$$\bigcap_{C \in G_r(K,H)} \sigma_{sb}(M_C) = \bigcap_{C \in B(K,H)} \sigma_{sb}(M_C). \quad (29)$$

3. $\bigcap_{C \in G(K,H)} \sigma_{ab}(M_C)$ and $\bigcap_{C \in \Phi(K,H)} \sigma_{ab}(M_C)$

In this section, we give the characterization of invertible and Fredholm perturbations of upper semi-Browder spectra of 2×2 upper-triangular matrices. We begin with some lemmas.

Lemma 12 (see [19]). *For a given pair $(A, B) \in B(H) \times B(K)$, if either A or B is a compact operator, then, for each $C \in \Phi(K, H)$, M_C is not a semi-Fredholm operator.*

In particular, if B is not compact, then M_C is not semi-Browder for any invertible operator C.

Lemma 13. *The following statements are equivalent.*

(i) *B is not compact.*

(ii) *For each given $A \in \Phi_{ab}(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in G(K, H)$ such that M_C is an upper semi-Browder operator.*

(iii) *For each given $A \in \Phi_{ab}(H)$, if $\beta(A) = \infty$, then there exists an operator $C \in \Phi(K, H)$ such that M_C is an upper semi-Browder operator.*

Proof. Obviously, we only need to prove the implications (i) ⇒ (ii) and (iii) ⇒ (i).

(iii) ⇒ (i). If B is compact, then it follows from Lemma 12 that M_C is not a semi-Fredholm operator for each $C \in \Phi(K, H)$, which contradicts with (iii). Thus, B is not compact.

(i) $\Rightarrow$ (ii). Suppose that B is not compact. Then, we consider the following two cases.

Case 1. Assume that $R(B)$ is closed. It follows from Lemma 3 that A can be decomposed into the following form with respect to some orthogonal decomposition $H = H_1 \oplus H_2$:

$$A = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \longrightarrow H_1 \oplus H_2, \tag{30}$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, and A_2 is a left invertible operator. Noting that $\beta(A) = \infty$, we have $\beta(A_2) = \infty$. Since the assumption that B is not compact, we have that $\dim N(B)^{\perp} = \infty$. Also since $\beta(A_2) = \infty$, let $R(A_2)^{\perp} = H_3 \oplus H_4$ with $\dim(H_3) = \dim N(B)$ and $\dim(H_4) = \infty$. Define an operator $C : K \to H$ by

$$\begin{aligned} C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} &: N(B) \oplus N(B)^{\perp} \\ &\longrightarrow H_3 \oplus \left(H_1 \oplus R\left(A_2\right) \oplus H_4\right), \end{aligned} \tag{31}$$

where $C_1 \in B(N(B), H_3)$ and $C_2 \in B(N(B)^{\perp}, H_1 \oplus R(A_2) \oplus H_4)$ are invertible operators. Obviously, $C \in B(K, H)$ is invertible. Next, we claim that M_C is an upper semi-Browder operator. To see this, M_C can be rewritten as

$$\begin{aligned} M_C = \begin{pmatrix} A_1 & A_{12} & C_{11} & 0 \\ 0 & A_{22} & C_{21} & 0 \\ 0 & 0 & 0 & C_1 \\ 0 & 0 & C_{41} & 0 \\ 0 & 0 & B_1 & 0 \end{pmatrix} &: H_1 \oplus H_2 \\ &\oplus N(B)^{\perp} \oplus N(B) \\ &\longrightarrow H_1 \oplus R\left(A_2\right) \oplus H_3 \oplus H_4 \oplus K, \end{aligned} \tag{32}$$

where $A_{22} \in B(H_2, R(A_2))$ is invertible and $B_1 \in B(N(B)^{\perp}, R(B))$ is left invertible. By Lemma 2 and the fact that $A_1 \in \Phi_b(H_1)$, it is sufficient to prove that

$$M_1 =: \begin{pmatrix} A_{22} & C_{21} & 0 \\ 0 & 0 & C_1 \\ 0 & C_{41} & 0 \\ 0 & B_1 & 0 \end{pmatrix} \tag{33}$$

is semi-Browder. For this, we only need to show that M_1 is left invertible. In fact, since A_{22} is invertible and B_1 and C_1 are left invertible, we can set A'_{22}, B'_1, and C'_1 such that

$$A'_{22} A_{22} = I_{H_2}, \qquad B'_1 B_1 = I_{N(B)^{\perp}}, \qquad C'_1 C_1 = I_{N(B)}. \tag{34}$$

Direct calculation shows that

$$\begin{aligned} &\begin{pmatrix} A'_{22} & 0 & 0 & -A'_{22} C_{21} B'_1 \\ 0 & 0 & 0 & B'_1 \\ 0 & C'_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{22} & C_{21} & 0 \\ 0 & 0 & C_1 \\ 0 & C_{41} & 0 \\ 0 & B_1 & 0 \end{pmatrix} \\ &\quad = \begin{pmatrix} I_{H_2} & 0 & 0 \\ 0 & I_{N(B)^{\perp}} & 0 \\ 0 & 0 & I_{N(B)} \end{pmatrix}, \end{aligned} \tag{35}$$

which implies that M_1 is left invertible. Noting that $A_1 \in \Phi(H_1)$, by Lemma 2 we have that M_C is upper semi-Browder.

Case 2. Assume that $R(B)$ is not closed. If B is not compact, then by Lemma 7, $R(B)$ contains a closed infinite-dimensional subspace. Without loss of generality, suppose that $\widetilde{K_1}$ is a closed subspace of $R(B)$ with $\dim \widetilde{K_1} = \infty$ and $\dim \widetilde{K_1}^{\perp} = \infty$. Let $K_1 = \{x \in N(B)^{\perp} : Bx \in \widetilde{K_1}\}$. Thus, K_1 is a closed subspace of $N(B)^{\perp}$, and $\dim(K_1) = \infty$. Denote $K_2 = N(B)^{\perp} \ominus K_1$. Without loss of generality, we may assume that $\dim(K_2) = \infty$ (otherwise, suppose that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of K_1. Denote $K'_1 = \operatorname{span}\{e_n : n = 2, 4, 6, \ldots\}$ and $\widetilde{K'_1} = \{Bx : x \in K'_1\}$, then K_1 and $\widetilde{K_1}$ can be replaced by K'_1 and $\widetilde{K'_1}$, resp.). Since $\beta(A_2) = \infty$, let $R(A_2)^{\perp} = H_3 \oplus H_4$ with $\dim(H_3) = \dim N(B)$ and $\dim H_4 = \infty$. Define an operator $C : K \to H$ by

$$\begin{aligned} C = \begin{pmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{pmatrix} &: K_1 \oplus K_2 \oplus N(B) \\ &\longrightarrow \left(H_1 \oplus R\left(A_2\right)\right) \oplus H_4 \oplus H_3, \end{aligned} \tag{36}$$

where C_1, C_2, and C_3 are unitary operators. Obviously, C is invertible. M_C can be rewritten as

$$\begin{aligned} M_C = \begin{pmatrix} A_1 & A_{12} & C_{11} & 0 & 0 \\ 0 & A_{22} & C_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_3 \\ 0 & 0 & 0 & C_2 & 0 \\ 0 & 0 & B_{11} & B_{12} & 0 \\ 0 & 0 & 0 & B_{22} & 0 \end{pmatrix} &: H_1 \oplus H_2 \\ &\oplus K_1 \oplus K_2 \oplus N(B) \\ &\longrightarrow H_1 \oplus R\left(A_2\right) \oplus H_3 \oplus H_4 \oplus \widetilde{K_1} \oplus \widetilde{K_1}^{\perp}, \end{aligned} \tag{37}$$

where A_{22} and B_{11} are invertible and $C_1 = \begin{pmatrix} C_{11} \\ C_{21} \end{pmatrix}$.

Next, we prove that $M_C \in \Phi_{ab}(H \oplus K)$. Noting that $\dim(H_1) < \infty$, then, by Lemma 2, it is sufficient to prove that

$$M_1 =: \begin{pmatrix} A_{22} & C_{21} & 0 & 0 \\ 0 & 0 & 0 & C_3 \\ 0 & 0 & C_2 & 0 \\ 0 & B_{11} & B_{12} & 0 \\ 0 & 0 & B_{22} & 0 \end{pmatrix} \tag{38}$$

is left invertible. For this, let A'_{22}, B'_{11}, C'_1, and C'_2 be operators satisfying

$$\begin{aligned} A'_{22} A_{22} &= I_{H_2}, \qquad B'_{11} B_{11} = I_{K_1}, \\ C'_2 C_2 &= I_{K_2}, \qquad C'_3 C_3 = I_{N(B)}. \end{aligned} \tag{39}$$

Direct calculation shows that

$$\begin{aligned}
&\begin{pmatrix} A_{22}' & 0 & A_{22}'C_{21}B_{11}'B_{12}C_2' & -A_{22}'C_{21}B_{11}' & 0 \\ 0 & 0 & B_{11}'B_{12}C_2' & B_{11}' & 0 \\ 0 & 0 & C_2' & 0 & 0 \\ 0 & C_3' & 0 & 0 & \end{pmatrix} \\
&\quad\times \begin{pmatrix} A_{22} & C_{21} & 0 & 0 \\ 0 & 0 & 0 & C_3 \\ 0 & 0 & C_2 & 0 \\ 0 & B_{11} & B_{12} & 0 \\ 0 & 0 & B_{22} & 0 \end{pmatrix} \\
&\quad= \begin{pmatrix} I_{H_2} & 0 & 0 & 0 \\ 0 & I_{K_1} & 0 & 0 \\ 0 & 0 & I_{K_2} & 0 \\ 0 & 0 & 0 & I_{N(B)} \end{pmatrix},
\end{aligned} \tag{40}$$

which implies that M_1 is left invertible.

Combining Case 1 with Case 2, the lemma is proved. □

Similarly, we have the following.

Lemma 14. *The following statements are equivalent:*

(i) *A is not compact.*

(ii) *For each given $B \in \Phi_{sb}(H)$, if $\alpha(B) = \infty$, then there exists an operator $C \in G(K, H)$ such that M_C is a lower semi-Browder operator.*

(iii) *For each given $B \in \Phi_{sb}(H)$, if $\alpha(B) = \infty$, then there exists an operator $C \in \Phi(K, H)$ such that M_C is a lower semi-Browder operator.*

One is now ready to prove the main result of this section.

Theorem 15. *For a given pair $(A, B) \in B(H) \times B(K)$, one has*

$$\begin{aligned}
\bigcap_{C\in G(K,H)} \sigma_{ab}(M_C) &= \bigcap_{C\in \Phi(K,H)} \sigma_{ab}(M_C) \\
&= \left(\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)\right) \\
&\quad \cup \{\lambda \in \mathbb{C} : B - \lambda \ \textit{is compact}\}.
\end{aligned} \tag{41}$$

Proof. According to Lemma 12, it is clear that

$$\begin{aligned}
\bigcap_{C\in G(K,H)} \sigma_{ab}(M_C) &\supseteq \bigcap_{C\in \Phi(K,H)} \sigma_{ab}(M_C) \\
&\supseteq \left(\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)\right) \\
&\quad \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}.
\end{aligned} \tag{42}$$

For the conversion, without loss of generality, suppose that

$$0 \notin \left(\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)\right) \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}. \tag{43}$$

Then, B is not compact, and there exists some $C \in B(K, H)$ such that $M_C \in \Phi_{ab}(H\oplus K)$, and, hence, $A \in \Phi_{ab}(H)$.

Case 1. $\beta(A) = \infty$. It follows from Lemma 13 that there exists some $C \in G(K, H)$ such that M_C is an upper semi-Browder operator. This implies that $\lambda \notin \bigcap_{C\in G(K,H)} \sigma_{ab}(M_C)$. In this case, we have proved. Consider that

$$\begin{aligned}
\bigcap_{C\in G(K,H)} \sigma_{ab}(M_C) &\subseteq \left(\bigcap_{C\in B(K,H)} \sigma_{ab}(M_C)\right) \\
&\quad \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}.
\end{aligned} \tag{44}$$

Case 2. Consider $\beta(A) < \infty$. This implies that $A \in \Phi(H)$, and, thus, $B \in \Phi_+(K)$ since $M_C \in \Phi_{ab}(H \oplus K)$. It follows from Lemma 5 that $b.s.\_\mathrm{mul}(B) \leq s\_\mathrm{mul}(A)$. Moreover, using Lemmas 1 and 3, we have

$$\begin{aligned}
A &= \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} : H_1 \oplus H_2 \longmapsto H_1 \oplus H_2, \\
B &= \begin{pmatrix} B_1 & * & * \\ 0 & B_2 & * \\ 0 & 0 & B_3 \end{pmatrix} : K_1 \oplus K_2 \oplus K_3 \longmapsto K_1 \oplus K_2 \oplus K_3,
\end{aligned} \tag{45}$$

where $\dim(H_1) < \infty$, A_1 is nilpotent, A_2 is a left invertible operator, $\dim(K_3) < \infty$, B_1 is a right invertible operator, B_2 is a left invertible operator, B_3 is a finite nilpotent operator, and the parts marked by $*$ can be any operators. Moreover, $\beta(A_2) = s\_\mathrm{mul}(A)$, and $\alpha(B_1) = b.s.\_\mathrm{mul}(B)$. Hence, $\alpha(B_1) \leq \beta(A_2)$. Now, put $H_2 = R(A_2) \oplus H_3 \oplus H_4$, where $\dim(H_3) = \alpha(B_1) < \infty$. Noting that $\dim H_1 \oplus R(A_2) \oplus H_4 = N(B_1)^\perp \oplus K_2 \oplus K_3 = \infty$, there exist unitaries $C_{33} \in B(N(B_1), H_3)$ and $C' \in B(H_1 \oplus R(A_2) \oplus H_4, N(B_1)^\perp \oplus K_2 \oplus K_3)$. Let $C = \begin{pmatrix} C_{33} & 0 \\ 0 & C' \end{pmatrix}$. Obviously, $C = \begin{pmatrix} C_{33} & 0 \\ 0 & C' \end{pmatrix} \in G(H \oplus K)$.

Consider that operator

$$\begin{aligned}
&M_C \\
&= \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : H \oplus K \rightarrow H \oplus K \\
&= \begin{pmatrix} A_1 & A_{12} & C_{11} & 0 & C_{13} & C_{14} \\ 0 & A_2 & C_{21} & 0 & C_{23} & C_{24} \\ 0 & 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & C_{41} & 0 & C_{43} & C_{44} \\ 0 & 0 & B_{11} & 0 & * & * \\ 0 & 0 & 0 & 0 & B_2 & * \\ 0 & 0 & 0 & 0 & 0 & B_3 \end{pmatrix} : H_1 \oplus H_2 \\
&\quad \oplus N(B_1)^\perp \oplus N(B_1) \oplus K_2 \oplus K_3 \\
&\quad \longrightarrow H_1 \oplus R(A_2) \oplus H_3 \oplus H_4 \oplus K_1 \oplus K_2 \oplus K_3,
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
C' &= \begin{pmatrix} C_{11} & C_{13} & C_{14} \\ C_{21} & C_{23} & C_{24} \\ C_{41} & C_{43} & C_{44} \end{pmatrix} : H_1 \oplus H_2 \oplus N(B_1)^\perp \\
&\quad \longrightarrow H_1 \oplus R(A_2) \oplus H_4, \quad B_1 = \begin{pmatrix} B_{11} \\ 0 \end{pmatrix}.
\end{aligned} \tag{47}$$

We claim that $M_C \in \Phi_{ab}(H \oplus K)$. In fact, since A_1 and B_3 are Browder operators, then, by Lemma 2, it is sufficient to show that

$$M =: \begin{pmatrix} A_2 & C_{21} & 0 & C_{23} \\ 0 & 0 & C_{33} & 0 \\ 0 & C_{41} & 0 & C_{43} \\ 0 & B_{11} & 0 & * \\ 0 & 0 & 0 & B_2 \end{pmatrix} : H_2 \oplus N(B_1)^{\perp} \oplus N(B_1) \oplus K_2 \longrightarrow R(A_2) \oplus H_3 \oplus H_4 \oplus K_1 \oplus K_2 \quad (48)$$

is upper semi-Browder. Observe that A_2 and B_2 are left invertible; C_{33} and B_{11} are invertible. Direct calculation shows that M is injective. Since $A \in \Phi(H)$ and $B \in \Phi_+(K)$, we have $M_C \in \Phi_{ab}(H \oplus K)$, and, hence, M is an upper semi-Fredholm operator. Thus, M is left invertible. Combining this with Lemma 2 yields $M_C \in \Phi_{ab}(H \oplus K)$, which means that $\lambda \notin \bigcap_{C \in G(K,H)} \sigma_{ab}(M_C)$. Thus,

$$\bigcap_{C \in G(K,H)} \sigma_{ab}(M_C) \subseteq \bigcap_{C \in B(K,H)} \sigma_{ab}(M_C) \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}. \quad (49)$$

Combining Case 1 with Case 2 leads to

$$\bigcap_{C \in G(K,H)} \sigma_{ab}(M_C) = \bigcap_{C \in \phi(K,H)} \sigma_{ab}(M_C) = \left(\bigcap_{C \in B(K,H)} \sigma_{ab}(M_C) \right) \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\}. \quad (50)$$

This completes the proof. □

By duality, we have

Theorem 16. *For a given pair $(A, B) \in B(H) \times B(K)$, one has*

$$\bigcap_{C \in G(K,H)} \sigma_{sb}(M_C) = \bigcap_{C \in \Phi(K,H)} \sigma_{sb}(M_C) = \left(\bigcap_{C \in B(K,H)} \sigma_{sb}(M_C) \right) \cup \{\lambda \in \mathbb{C} : A - \lambda \textit{ is compact}\}. \quad (51)$$

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