Research Article

An Extension of Modular Sequence Spaces

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Received 1 April 2013; Accepted 2 June 2013

Academic Editor: Jinde Cao

The main aim of this paper is to present an extension of the modular sequence spaces by means of Cesàro mean of order one, to investigate several relevant algebraic and topological properties, and derive some other spaces in the sequel.

1. Introduction

Throughout the paper $w(X)$ will represent the spaces of all X valued sequences spaces, where (X, q) is a seminormed space, seminormed by q. For $X = C$, the space of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\bar{\theta} = (\theta, \theta, \theta, \ldots)$, where θ is the zero element of X.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

If convexity of Orlicz function M is replaced by $M(x + y) \le M(x) + M(y)$, then this function is called a modulus function introduced by Nakano [1].

Lindenstrauss and Tzafriri [2] used the idea of Orlicz function to construct sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}. \quad (1)$$

The space ℓ_M becomes a Banach space, with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) \le 1 \right\} \quad (2)$$

which is called an Orlicz space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \le p < \infty$.

Another generalization of Orlicz sequence spaces is due to Woo [3]. Let $\{M_k\}$ be a sequence of Orlicz functions. Define the vector space $\ell\{M_k\}$ by

$$\ell\{M_k\} = \left\{ x \in w : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\} \quad (3)$$

and equip this space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho} \right) \le 1 \right\}. \quad (4)$$

Then $\ell\{M_k\}$ becomes a Banach space and is called a modular sequence space. The space $\ell\{M_k\}$ also generalizes the concept of modular sequence space introduced earlier by Nakano [4], who considered the space $\ell\{M_k\}$ when $M_k(x) = x^{\alpha_k}$, where $1 \le \alpha_k < \infty$ for $k \ge 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \le kM(u)$ $(u \ge 0)$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu) \le kluM(u)$ for all values of u and for $l > 1$ (see [5]).

The previous Δ_2-condition implies $M(lu) \le Kl^{\log_2 K} M(u)$, for all $u > 0$, $l > 1$.

Bektaş and Altin [6], Parasar and Choudhary [7], Mursaleen et al. [8], Dutta and Başar [9], Dutta and Bilgin [10], Karakaya and Dutta [11], Tripathy and Dutta [12], Jebril [13], and many others have studied different summable spaces and other sequence spaces using Orlicz functions.

A *BK*-space (introduced by Zeller [14]) $(X, \| \cdot \|)$ is a Banach space of complex sequences $x = (x_k)$ in which the coordinate maps are continuous; that is, $|x_k^n - x_k| \to 0$, whenever $\|x^n - x\| \to 0$ as $n \to \infty$, where $x^n = (x_k^n)$, for all $n \in N$ and $x = (x_k)$.

Let A denote the set of all complex sequences which have only a finite number of nonzero coordinates, and let λ denote a *BK*-space of sequences $x = (x_k)$ which contains A. An element $x = (x_k)$ of λ will be called sectionally convergent if

$$x^{(n)} = \sum_{k=1}^{n} x_k e_k \longrightarrow x, \quad \text{as } n \longrightarrow \infty, \tag{5}$$

where $e_k = (\delta_{ki})$, where $\delta_{kk} = 1$, $\delta_{ki} = 0$ for $k \neq i$.

λ will be called *AK*-space if and only if each of its elements is sectionally convergent.

Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions, let X be a seminormed space with seminorm q, let $p = (p_k)$ be sequence of positive real numbers, and let $C = (c_{nk})_{n,k=0}^{\infty}$ be the Cesàro matrix of order one with $c_{nk} = 1/(n+1)$ if $0 \le k \le n$ and $c_{nk} = 0$, otherwise. Then for nonnegative real numbers s we define

$$\ell\{M_k, p, q, s, C\}$$
$$= \left\{ x \in w(X) : \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho(k+1)} \right) \right) \right]^{p_k} \right.$$
$$\left. < \infty, \text{ for some } \rho > 0 \right\}. \tag{6}$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \le \sup p_k = G$, $D = \max\{1, 2^{G-1}\}$. Then for all $a_k, b_k \in C$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \le D \left\{ |a_k|^{p_k} + |b_k|^{p_k} \right\}. \tag{7}$$

2. Main Results

In this section we give the theorems that characterize the structure of the class of sequences $\ell\{M_k, p, q, s, C\}$ and some other spaces which can be derived from this space.

Theorem 1. *Let $p = (p_k)$ be bounded sequence of positive reals; then $\ell\{M_k, p, q, s, C\}$ is a linear space over the field of complex numbers.*

Proof. Let $x, y \in \ell\{M_k, p, q, s, C\}$ and $\alpha, \beta \in C$. Then there exist some $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho_1(k+1)} \right) \right) \right]^{p_k} < \infty,$$
$$\sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} y_j}{\rho_2(k+1)} \right) \right) \right]^{p_k} < \infty. \tag{8}$$

We consider $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since each M_k is nondecreasing and convex, and since q is a seminorm,

$$\sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} (\alpha x_k + \beta y_k)}{\rho_3(k+1)} \right) \right) \right]^{p_k}$$
$$\le \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} (\alpha x_j)}{\rho_3(k+1)} \right) \right. \right.$$
$$\left. \left. + q \left(\frac{\sum_{j=0}^{k} (\beta y_j)}{\rho_3(k+1)} \right) \right) \right]^{p_k}$$
$$\le \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{2\rho_1(k+1)} \right) \right. \right.$$
$$\left. \left. + q \left(\frac{\sum_{j=0}^{k} y_j}{2\rho_2(k+1)} \right) \right) \right]^{p_k}$$
$$\le D \left\{ \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho_1(k+1)} \right) \right) \right]^{p_k} \right.$$
$$\left. + \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} y_j}{\rho_2(k+1)} \right) \right) \right]^{p_k} \right\} < \infty. \tag{9}$$

This completes the proof. $\square$

Theorem 2. *$\ell\{M_k, p, q, s, C\}$ is a paranormed space (need not total paranorm) space with paranorm g, defined as follows*

$$g(x) = \inf \left\{ \rho^{p_n/H} : \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho(k+1)} \right) \right) \right] \right.$$
$$\left. \le 1, n = 1, 2, \dots \right\}, \tag{10}$$

where $H = \max\{1, \sup_k p_k\}$.

Proof. Clearly $g(x) = g(-x)$. Since $M_k(0) = 0$, for all $k \in N$ we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = \theta$.

Now let $x, y \in \ell\{M_k, p, q, s, C\}$, and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho_1(k+1)} \right) \right) \right] \le 1,$$
$$\sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} y_j}{\rho_2(k+1)} \right) \right) \right] \le 1. \tag{11}$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} (x_j + y_j)}{\rho(k+1)} \right) \right) \right]$$

$$\leq \left(\frac{\rho_1}{\rho_1 + \rho_2} \right) \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho_1(k+1)} \right) \right) \right]$$

$$+ \left(\frac{\rho_2}{\rho_1 + \rho_2} \right) \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} y_j}{\rho_2(k+1)} \right) \right) \right] \leq 1. \tag{12}$$

Hence $g(x + y) \leq g(x) + g(y)$.

Finally let λ be a given non-zero scalar; then the continuity of the scalar multiplication follows from the following equality

$$g(\lambda x)$$

$$= \inf \left\{ \rho^{p_n/H} : \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} (\lambda x_j)}{\rho(k+1)} \right) \right) \right] \right.$$

$$\left. \leq 1, \, n = 1, 2, \dots \right\}$$

$$= \inf \left\{ (|\lambda| s)^{p_n/H} : \sum_{k=0}^{\infty} k^{-s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{s(k+1)} \right) \right) \right] \right.$$

$$\left. \leq 1, \, n = 1, 2, \dots \right\}, \quad \text{where } s = \frac{\rho}{|s|}. \tag{13}$$

This completes the proof. $\qquad \square$

The proof of the following theorem is easy, so omitted.

Theorem 3. *Let* $\mathbf{M} = (M_k)$ *and* $\mathbf{T} = (T_k)$ *be sequences of Orlicz functions. For any two sequences* $p = (p_k)$ *and* $t = (t_k)$ *of bounded positive real numbers and for any two seminorms* q_1 *and* q_2 *one has*

(i) *if* q_1 *is stronger than* q_2, *then* $\ell\{M_k, p, q_1, s, C\} \subset \ell\{M_k, p, q_2, s, C\}$,

(ii) $\ell\{M_k, p, q_1, s, C\} \cap \ell\{M_k, p, q_2, s, C\} \subset \ell\{M_k, p, q_1 + q_2, s, C\}$,

(iii) $\ell\{M_k, p, q, s, C\} \cap \ell\{T_k, p, q, s, C\} \subset \ell\{M_k + T_k, p, q, s, C\}$,

(iv) $\ell\{M_k, p, q_1, s, C\} \cap \ell\{M_k, t, q_2, s, C\} \neq \varphi$,

(v) *If* $s_1 \leq s_2$, *then* $\ell\{M_k, p, q, s_1, C\} \subset \ell\{M_k, p, q, s_2, C\}$.

Theorem 4. *Let* $\mathbf{M} = (M_k)$ *and* $\mathbf{T} = (T_k)$ *be sequences of Orlicz functions which satisfy* Δ_2-*condition and* $s > 1$, *then*

$$\ell\{M_k, p, q, s, C\} \subseteq \ell\{T_k \circ M_k, p, q, s, C\}. \tag{14}$$

Proof. Let $x \in \ell\{M_k, p, q, s, C\}$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that each $M_k(u) < \varepsilon$ for $0 \leq u \leq \delta$. We write $y_k = M_k(q((\sum_{j=0}^{k} x_j)/\rho(k+1)))$ and consider

$$\sum_{k=0}^{\infty} k^{-s} [T_k(y_k)]^{p_k} = \sum_{1} k^{-s} [T_k(y_k)]^{p_k} + \sum_{2} k^{-s} [T_k(y_k)]^{p_k}, \tag{15}$$

where the first summation is over $y_k \leq \delta$ and the second is over $y_k > \delta$. Now we have

$$\sum_{1} k^{-s} [T_k(y_k)]^{p_k} < \max\left(1, \varepsilon^H\right) \sum_{k=1}^{\infty} k^{-s} < \infty, \quad \text{since } s > 1. \tag{16}$$

For $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \left(\frac{y_k}{\delta} \right). \tag{17}$$

Since each T_k is non-decreasing and convex, it follows that

$$T_k(y_k) < T_k \left(1 + \frac{y_k}{\delta} \right) < \frac{1}{2} T_k(2) + \frac{1}{2} T_k \left(2 \frac{y_k}{\delta} \right), \tag{18}$$

$$\text{for each } k \text{ in } N.$$

Since each T_k satisfies Δ_2-condition, we have

$$T_k(y_k) < \frac{1}{2} K \frac{y_k}{\delta} T_k(2) + \frac{1}{2} K \frac{y_k}{\delta} T_k(2) = K y_k \delta^{-1} T_k(2). \tag{19}$$

Hence

$$\sum_{2} k^{-s} [T_k(y_k)]^{p_k} \leq \max\left(1, \left(K \delta^{-1} M(2) \right)^H \right)$$

$$\times \sum_{k=1}^{\infty} k^{-s} (y_k)^{p_k} < \infty. \tag{20}$$

Thus

$$\sum_{k=0}^{\infty} k^{-s} [T_k(y_k)]^{p_k} = \sum_{1} k^{-s} [T_k(y_k)]^{p_k} + \sum_{2} k^{-s} [T_k(y_k)]^{p_k}$$

$$\leq \max\left(1, \varepsilon^H \right)$$

$$\times \sum_{k=1}^{\infty} k^{-s} + \max\left(1, \left(K \delta^{-1} M(2) \right)^H \right)$$

$$\times \sum_{k=1}^{\infty} k^{-s} (y_k)^{p_k} < \infty. \tag{21}$$

Hence $x \in \ell\{T_k \circ M_k, p, q, s, C\}$.

This completes the proof. $\qquad \square$

Taking $M_k(x) = x$, for all $x \in [0, \infty)$ and k in N, in Theorem 4, we get the next corollary.

Corollary 5. *Let* $\mathbf{M} = (M_k)$ *be any sequence of Orlicz functions which satisfy* Δ_2*-condition and* $s > 1$*, then*

$$\ell\{p, q, s, C\} \subseteq \ell\{M_k, p, q, s, C\}. \tag{22}$$

We will write $f \approx g$ for non-negative functions f and g whenever $C_1 f \leq g \leq C_2 f$ for some $C_j > 0$, $j = 1, 2$.

Theorem 6. *Let* $\mathbf{M} = (M_k)$ *and* $\mathbf{T} = (T_k)$ *be sequences of Orlicz functions. If* $M_k \approx T_k$ *for each* $k \in N$*, then* $\ell\{M_k, p, q, s, C\} = \ell\{T_k, p, q, s, C\}$.

Proof. The proof is obvious. $\square$

Theorem 7. *Let* $\mathbf{M} = (M_k)$ *be a sequence of Orlicz functions. If* $\lim_{t \to 0}(M_k(t)/t) > 0$ *and* $\lim_{t \to 0}(M_k(t)/t) < \infty$*, for each* $k \in N$*, then* $\ell\{M_k, p, q, s, C\} = \ell\{p, q, s, C\}$.

Proof. If the given conditions are satisfied, we have $M_k(t) \approx t$ for each k, and the proof follows from Theorem 6. $\square$

If we take $s = 0$, the sequence space $\ell\{M_k, p, q, s, C\}$ reduces to the following sequence space:

$$\ell\{M_k, p, q, C\}$$
$$= \left\{ x \in w(X) : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho(k+1)} \right) \right) \right]^{p_k} \right. \tag{23}$$
$$\left. < \infty, \text{ for some } \rho > 0 \right\}.$$

Theorem 8. *Let* $p = (p_k)$ *be bounded sequence of positive reals, and let* (X, q) *be a complete seminormed space, then* $\ell\{M_k, p, q, C\}$ *is a complete paranormed space paranormed by* h*, defined by*

$$h(x) = \inf \left\{ \rho^{p_n/H} : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho(k+1)} \right) \right) \right] \right. \tag{24}$$
$$\left. \leq 1, n = 1, 2, \dots \right\},$$

where $H = \max\{1, \sup_k p_k\}$.

Proof. Let (x^i) be a Cauchy sequence in $\ell\{M_k, p, q, C\}$. Let $\delta > 0$ be fixed, and let $r > 0$ be such that for a given $0 < \varepsilon <$

1, $\varepsilon/r\delta > 0$, and $r\delta \geq 1$. Then there exists a positive integer n_0 such that

$$h(x^s - x^t) < \frac{\varepsilon}{r\delta}, \quad \forall s, t \geq n_0.$$

$$\Longrightarrow \inf \left\{ \rho^{p_n/H} : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k}(x_j^s - x_j^t)}{\rho(k+1)} \right) \right) \right] \right.$$
$$\left. \leq 1, n = 1, 2, \dots \right\} < \frac{\varepsilon}{r\delta}, \quad \forall s, t \geq n_0.$$
$$\tag{25}$$

Hence we have

$$\sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k}(x_j^s - x_j^t)}{h(x^s - x^t)(k+1)} \right) \right) \right] \leq 1, \quad \forall s, t \geq n_0.$$
$$\tag{26}$$

It follows that

$$M_k \left(q \left(\frac{\sum_{j=0}^{k}(x_j^s - x_j^t)}{h(x^s - x^t)(k+1)} \right) \right) \leq 1, \tag{27}$$
$$\forall s, t \geq n_0, \quad k \in N.$$

For $r > 0$ with $M_k(r\delta/2) \geq 1$, we have

$$M_k \left(q \left(\frac{\sum_{j=0}^{k}(x_j^s - x_j^t)}{h(x^s - x^t)(k+1)} \right) \right) \leq M_k \left(\frac{r\delta}{2} \right), \tag{28}$$
$$\forall s, t \geq n_0, \quad k \in N.$$

Since M_k is non-decreasing for each $k \in N$, we have

$$q \left(\frac{\sum_{j=0}^{k}(x_j^s - x_j^t)}{k+1} \right) \leq \frac{r\delta}{2} \cdot \frac{\varepsilon}{r\delta} = \frac{\varepsilon}{2}. \tag{29}$$

Hence it follows that (x_k^s) is a Cauchy sequence in (X, q) for each $k \in N$. But (X, q) is complete, and so (x_k^s) is convergent in (X, q) for each $k \in N$.

Let $\lim_{s \to \infty} x_k^s = x_k$ exists for each $k \in N$.

Now we have for all $s, t \geq n_0$,

$$\inf \left\{ \rho^{p_n/H} : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k}(x_j^s - x_j^t)}{\rho(k+1)} \right) \right) \right] \right.$$
$$\tag{30}$$
$$\left. \leq 1, n = 1, 2, \dots \right\} < \varepsilon.$$

Then we have

$$\lim_{t \to \infty} \left\{ \inf \left\{ \rho^{p_n/H} : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k}(x_j^s - x_j^t)}{\rho(k+1)} \right) \right) \right] \right. \right.$$
$$\left. \left. \leq 1, n = 1, 2, \dots \right\} \right\} < \varepsilon, \quad \forall s \geq n_0.$$
$$\tag{31}$$

Using the continuity of Orlicz functions, we have

$$\inf \left\{ \rho^{p_n/H} : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} \left(x_j^s - \lim_{t \to \infty} x_j^t \right)}{\rho (k+1)} \right) \right) \right] \right.$$

$$\left. \leq 1, \, n = 1, 2, \ldots \right\} < \varepsilon, \quad \forall s \geq n_0. \tag{32}$$

This implies

$$\inf \left\{ \rho^{p_n/H} : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} \left(x_j^s - x_j \right)}{\rho} \right) \right) \right] \right.$$

$$\left. \leq 1, \, n = 1, 2, \ldots \right\} < \varepsilon, \quad \forall s \geq n_0. \tag{33}$$

It follows that $(x^s - x) \in \ell\{M_k, p, q, C\}$.

Since $(x^s) \in \ell\{M_k, p, q, C\}$ and $\ell\{M_k, p, q, C\}$ is a linear space, so we have $x = x^s - (x^s - x) \in \ell\{M_k, p, q, C\}$.

This completes the proof. □

If we take $s = 0$ and $p_k = l$, a constant the sequence space $\ell\{M_k, p, q, s, C\}$ reduces to the following sequence space:

$$\ell\{M_k, q, C\}$$

$$= \left\{ x \in w(X) : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] \right. \tag{34}$$

$$\left. < \infty, \text{ for some } \rho > 0 \right\}.$$

Theorem 9. *Let (X, q) be a complete normed space; then $\ell\{M_k, q, C\}$ is a Banach space normed by $\|\cdot\|$, defined by*

$$\|x\| = \inf \left\{ \rho : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] \leq 1 \right\}. \tag{35}$$

Proof. We prove that $\|\cdot\|$ is a norm on $\ell\{M_k, q, C\}$. The completeness part can be proved using similar arguments as applied to prove the previous theorem.

If $x = \theta$, then it is obvious that $\|x\| = 0$. Conversely assume $\|x\| = 0$. Then using the definition of norm, we have

$$\inf \left\{ \rho : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] \leq 1 \right\} = 0. \tag{36}$$

This implies that for a given $\varepsilon > 0$, there exists some ρ_ε ($0 < \rho_\varepsilon < \varepsilon$) such that

$$\sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho_\varepsilon (k+1)} \right) \right) \right] \leq 1. \tag{37}$$

It follows that

$$M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho_\varepsilon (k+1)} \right) \right) \leq 1, \quad \forall k \in N. \tag{38}$$

Thus

$$M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\varepsilon (k+1)} \right) \right) \leq M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho_\varepsilon (k+1)} \right) \right)$$

$$\leq 1, \quad \forall k \in N. \tag{39}$$

Suppose $(\sum_{j=0}^{n_i} x_j)/(n_i + 1) \neq 0$, for some i. Let $\varepsilon \to 0$ then $(\sum_{j=0}^{n_i} x_j)/(\varepsilon(n_i + 1)) \to \infty$.

It follows that $M_k(q((\sum_{j=0}^{k} x_j)/(\varepsilon(k+1)))) \to \infty$ as $\varepsilon \to 0$ for some $n_i \in N$. This is a contradiction.

Therefore $(\sum_{j=0}^{k} x_j)/(k+1) = 0$, for all $k \in N$.

It follows that $x_k = 0$ for all $k \geq 1$. Hence $x = \theta$.

Again proof of the properties $\|x + y\| \leq \|x\| + \|y\|$ and for any scalar α, $\|\alpha x\| = |\alpha| \|x\|$ are similar to that of Theorem 2. □

It is easy to see that $\|x^i\| \to 0$ implies that $x_k^i \to 0$ for each $i \geq 1$. Hence we have the following proposition.

Proposition 10. *The space $\ell\{M_k, q, C\}$ is a BK-space.*

Now we study the *AK*-characteristic of the space $\ell\{M_k, q, s, C\}$. Before that we give a new definition and prove some results which will be required.

Definition 11. For any sequence of Orlicz functions $\mathbf{M} = (M_k)$, we define

$$h\{M_k, q, C\} = \left\{ x \in w(X) : \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] \right.$$

$$\left. < \infty, \text{ for every } \rho > 0 \right\}. \tag{40}$$

Clearly $h\{M_k, q, C\}$ is a subspace of $\ell\{M_k, q, C\}$. The topology of $h\{M_k, q, C\}$ is the one it inherits from $\|\cdot\|$.

Proposition 12. *Let $\mathbf{M} = (M_k)$ be a sequence of Orlicz functions which satisfy Δ_2-condition. Then*

$$\ell\{M_k, q, C\} = h\{M_k, q, C\}. \tag{41}$$

Proof. It is enough to prove that $\ell\{M_k, q, C\} \subseteq h\{M_k, q, C\}$.

Let $x \in \ell\{M_k, q, C\}$, then for some $\rho > 0$,

$$\sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] < \infty. \tag{42}$$

Choose an arbitrary $\eta > 0$. If $\rho \leq \eta$ then

$$\sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\eta (k+1)} \right) \right) \right]$$
$$< \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] < \infty. \tag{43}$$

Let now $\eta < \rho$ and put $l = \rho/\eta > 1$.

Since each M_k satisfies the Δ_2-condition, there exist constants K_k such that

$$\sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\eta (k+1)} \right) \right) \right]$$
$$\leq \sum_{k=0}^{\infty} K_k \left(\frac{\rho}{\eta} \right)^{\log_2 K_k} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right]. \tag{44}$$

Let $S = \sup_k K_k (\rho/\eta)^{\log_2 K_k}$. Then for every $\eta > 0$

$$\sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\eta (k+1)} \right) \right) \right]$$
$$\leq S \sum_{k=0}^{\infty} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] < \infty. \tag{45}$$

This completes the proof. □

Proposition 13. *Let* (X, q) *be a complete normed space, then* $h\{M_k, q, C\}$ *is an AK-space.*

Proof. Let $x \in h\{M_k, q, C\}$. Then for each $\varepsilon, 0 < \varepsilon < 1$, we can find an s_0 such that

$$\sum_{k \geq s_0} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\varepsilon (k+1)} \right) \right) \right] \leq 1. \tag{46}$$

Hence for $s \geq s_0$,

$$\left\| x - x^{(s)} \right\|$$
$$= \inf \left\{ \rho > 0 : \sum_{k \geq s+1} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] \leq 1 \right\}$$
$$\leq \inf \left\{ \rho > 0 : \sum_{k \geq s} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] \leq 1 \right\} < \varepsilon. \tag{47}$$

Thus we can conclude that $h\{M_k, q, C\}$ is an *AK* space. □

Combining Proposition 10 and Proposition 12, we have the following theorem.

Theorem 14. *Let* $\mathbf{M} = (M_k)$ *be a sequence of Orlicz functions which satisfy* Δ_2-*condition, then* $\ell\{M_k, q, C\}$ *is an AK-space.*

Proposition 15. *The space* $h\{M_k, q, C\}$ *is a closed subspace of* $\ell\{M_k, q, C\}$.

Proof. Let $\{x^s\}$ be a sequence in $h\{M_k, q, C\}$ such that $\|x^s - x\| \to 0$, where $x \in \ell\{M_k, q, C\}$.

To complete the proof we need to show that $x \in h\{M_k, q, C\}$; that is,

$$\sum_{k \geq 0} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] < \infty \quad \text{for every } \rho > 0. \tag{48}$$

To $\rho > 0$ there corresponds an l such that $\|x^l - x\| \leq \rho/2$. Then using convexity of each M_k,

$$\sum_{k \geq 0} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right]$$
$$= \sum_{k \geq 0} \left[M_k \left(q \left(\frac{2 \sum_{j=0}^{k} x_j^l - 2 \left(\sum_{j=0}^{k} x_j^l - \sum_{j=0}^{k} x_j \right)}{2\rho (k+1)} \right) \right) \right]$$
$$\leq \frac{1}{2} \sum_{k \geq 0} \left[M_k \left(q \left(\frac{2 \sum_{j=0}^{k} x_j^l}{\rho (k+1)} \right) \right) \right]$$
$$+ \frac{1}{2} \sum_{k \geq 0} \left[M_k \left(q \left(\frac{2 \sum_{j=0}^{k} \left(x_j^l - x_j \right)}{\rho (k+1)} \right) \right) \right]$$
$$\leq \frac{1}{2} \sum_{k \geq 0} \left[M_k \left(q \left(\frac{2 \sum_{j=0}^{k} x_j^l}{\rho (k+1)} \right) \right) \right]$$
$$+ \frac{1}{2} \sum_{k \geq 0} \left[M_k \left(q \left(\frac{2 \sum_{j=0}^{k} \left(x_j^l - x_j \right)}{\|x^l - x\| (k+1)} \right) \right) \right]. \tag{49}$$

Now from Theorem 9, using the definition of norm $\| \cdot \|$, we have

$$\sum_{k \geq 0} \left[M_k \left(q \left(\frac{2 \sum_{j=0}^{k} \left(x_j^l - x_j \right)}{\|x^l - x\| (k+1)} \right) \right) \right] \leq 1. \tag{50}$$

It follows that

$$\sum_{k \geq 0} \left[M_k \left(q \left(\frac{\sum_{j=0}^{k} x_j}{\rho (k+1)} \right) \right) \right] < \infty \quad \text{for every } \rho > 0. \tag{51}$$

Thus $x \in h\{M_k, q, C\}$. □

Hence we have the following corollary.

Corollary 16. *The space* $h\{M_k, q, C\}$ *is a BK-space.*

Acknowledgment

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