

## Research Article

# Weighted Composition Operators from Hardy to Zygmund Type Spaces

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This paper aims at studying the boundedness and compactness of weighted composition operator between spaces of analytic functions. We characterize boundedness and compactness of the weighted composition operator  $uC_\phi$  from the Hardy spaces  $H^p$  to the Zygmund type spaces  $\mathcal{X}_\alpha = \{f \in H(D) : \sup_{z \in D} (1 - |z|^2)^\alpha |f''(z)| < \infty\}$  and the little Zygmund type spaces  $\mathcal{X}_{\alpha,0}$  in terms of function theoretic properties of the symbols  $u$  and  $\phi$ .

## 1. Introduction

Let  $D = \{z : |z| < 1\}$  be the open unit disk in the complex plane  $\mathbb{C}$  and  $T = \{z : |z| = 1\}$  its boundary, and  $H(D)$  denotes the set of all analytic functions on  $D$ . An analytic self-map  $\phi : D \rightarrow D$  induces the composition operator  $C_\phi$  on  $H(D)$ , defined by  $C_\phi(f) = f(\phi(z))$  for  $f$  analytic on  $D$ . It is a well-known consequence of Littlewood's subordination principle that the composition operator  $C_\phi$  is bounded on the classical Hardy  $H^p$  ( $0 < p \leq \infty$ ) spaces, Bergman  $A^p$  ( $0 < p \leq \infty$ ) spaces, and Bloch spaces (see, e.g., [1–4]).

Let  $u$  be a fixed analytic function on the open unit disk. Define a linear operator  $uC_\phi$  on the space of analytic functions on  $D$ , called a weighted composition operator, by  $uC_\phi f = u \cdot (f \circ \phi)$ , where  $f$  is an analytic function on  $D$ . We can regard this operator as a generalization of a multiplication operator and a composition operator. In recent years the weighted composition operator has received much attention and appears in various settings in the literature. For example, it is known that isometries of many analytic function spaces are weighted composition operators (see [5], for instance). Their boundedness and compactness have been studied on various Banach spaces of analytic functions, such as Hardy, Bergman, BMOA, Bloch-type, and Zygmund spaces; see, for example, [6–11]. Also, it has been studied from one Banach space of analytic functions to another; one may see [12–23].

The purpose of this paper is to consider the weighted composition operators from the Hardy space  $H^p$  ( $0 < p < \infty$ ) to the Zygmund type spaces  $\mathcal{X}_\alpha$ . Our main goal is to characterize boundedness and compactness of the operators  $uC_\phi$  from  $H^p$  to  $\mathcal{X}_\alpha$  in terms of function theoretic properties of the symbols  $u$  and  $\phi$ .

Now we give a detailed definition of these spaces. For  $0 \leq r < 1$ ,  $f(z) \in H(D)$ , we set

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty, \quad (1)$$

$$M_\infty(r, f) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

For  $0 < p \leq \infty$ , the Hardy space  $H^p$  consists of those functions  $f \in H(D)$ , for which

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f) < \infty. \quad (2)$$

It is well known that with norm (2) the  $H^p$  space is a Banach space if  $1 \leq p \leq \infty$ , for  $0 < p < 1$ ,  $H^p$  space is a nonlocally convex topological vector space, and  $d(f, g) = \|f - g\|_p^p$  is a complete metric for it. For more information about the  $H^p$  space, one may see these books, for example, [24, 25].

For  $\alpha > 0$  the  $\alpha$ -Bloch space  $\beta_\alpha$  consists of all analytic functions  $f$  defined on  $D$  such that

$$\|f\|_{\beta_\alpha} = \sup\{(1 - |z|^2)^\alpha |f'(z)| : z \in D\}, \quad 0 < \alpha < +\infty. \tag{3}$$

The space  $\mathcal{X}_\alpha$  consists of all analytic functions  $f$  defined on  $D$  such that

$$\|f\|_{\mathcal{X}_\alpha} = \sup\{(1 - |z|^2)^\alpha |f''(z)| : z \in D\}, \quad 0 < \alpha < +\infty. \tag{4}$$

When  $\alpha = 1$ , it is called the Zygmund space. From a theorem by Zygmund (see [26, vol. I, p. 263] or [24, Theorem 5.3]), we see that  $f \in \mathcal{X}_1$  if and only if  $f$  is continuous in the close unit disk  $\bar{D} = \{z : |z| \leq 1\}$  and the boundary function  $f(e^{i\theta})$  such that

$$\left| \frac{f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})}{h} \right| < \infty. \tag{5}$$

When  $\alpha > 1$ , from Proposition 8 of [27], we know that  $\mathcal{X}_\alpha = \beta_{\alpha-1}$ . Then the space  $\mathcal{X}_\alpha$  is called a Zygmund type space if  $0 < \alpha \leq 1$ . However, all results in this paper are valid for all  $\mathcal{X}_\alpha$  spaces ( $\alpha > 0$ ). An analytic function  $f \in H(D)$  is said to belong to the little Zygmund type space  $\mathcal{X}_{\alpha,0}$  which consists of all  $f \in \mathcal{X}_\alpha$  satisfying  $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |f''(z)| = 0$ . It can be easily proved that  $\mathcal{X}_\alpha$  is a Banach space under the norm

$$\|f\|_* = |f(0)| + |f'(0)| + \|f\|_{\mathcal{X}_\alpha}. \tag{6}$$

And the polynomials are norm-dense in closed subspace  $\mathcal{X}_{\alpha,0}$ . For some other information on this space and some operators on it, see, for example, [28–31].

Throughout this paper, constants are denoted by  $C, C(p)$ , they are positive, and  $C(p)$  are only depending on  $p$  and may differ from one occurrence to the another.

## 2. Auxiliary Results

In order to prove the main results of this paper. We need some auxiliary results. The first lemma is well known.

**Lemma 1** (see [24, p. 65]). *For  $p > 1$ , there exists a constant  $C(p)$  such that*

$$\int_0^{2\pi} \frac{d\theta}{|1 - z|^p} \leq \frac{C(p)}{(1 - |z|^2)^{p-1}} \quad \text{for every } z \in D. \tag{7}$$

**Lemma 2.** *Suppose that  $0 < p < \infty, f \in H^p$ ; then*

$$|f^{(n)}(z)| \leq \frac{\|f\|_p}{(1 - |z|^2)^{1/p+n}} \tag{8}$$

for every  $z \in D$  and all nonnegative integer  $n = 0, 1, 2, \dots$

*Proof.* We use induction on  $n$ . The case  $n = 0$  holds because it is Exercise 5 in [25, p. 85]. Assume the case  $n = k$  holds. Fix

$0 < r < 1$  and let  $g(z) = f^{(k)}(rz)$ . Then  $g(z)$  is in  $H^\infty \subset \beta_1$ , and  $\|g\|_{\beta_1} \leq \|g\|_\infty$ . It follows that

$$\begin{aligned} (1 - |z|^2) |g'(z)| &= (1 - |z|^2) |rf^{(k+1)}(rz)| \\ &\leq \frac{\|f\|_p}{(1 - |rz|^2)^{1/p+k}} \\ &\leq \frac{\|f\|_p}{(1 - |z|^2)^{1/p+k}}. \end{aligned} \tag{9}$$

Let  $r \rightarrow 1^-$ ; we have

$$|f^{(k+1)}(z)| \leq \frac{\|f\|_p}{(1 - |z|^2)^{1/p+k+1}}. \tag{10}$$

Then the case  $n = k + 1$  holds. Hence (8) holds.  $\square$

**Lemma 3.** *For  $0 < p < \infty$ , suppose  $uC_\varphi : H^p \rightarrow \mathcal{X}_{\alpha,0}$  is a bounded operator. Then  $uC_\varphi : H^p \rightarrow \mathcal{X}_\alpha$  is a bounded operator.*

This is obvious.

## 3. Boundedness of $uC_\varphi$ from $H^p$ ( $0 < p < \infty$ ) to $\mathcal{X}_\alpha$ and $\mathcal{X}_{\alpha,0}$

In this section we characterize bounded weighted composition operators from the Hardy space  $H^p$  ( $0 < p < \infty$ ) to the Zygmund spaces  $\mathcal{X}_\alpha$ .

**Theorem 4.** *Let  $\alpha > 0, 0 < p < \infty$ , and  $u$  be an analytic function on the unit disc  $D$  and  $\varphi$  an analytic self-map of  $D$ . Then  $uC_\varphi$  is a bounded operator from  $H^p$  to the Zygmund spaces  $\mathcal{X}_\alpha$  if and only if the following are satisfied:*

$$\sup_{z \in D} \frac{(1 - |z|^2)^\alpha |u''(z)|}{(1 - |\varphi(z)|^2)^{1/p}} < \infty, \tag{11}$$

$$\sup_{z \in D} \frac{(1 - |z|^2)^\alpha |u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{1/p+2}} < \infty, \tag{12}$$

$$\sup_{z \in D} \frac{(1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{(1 - |\varphi(z)|^2)^{1/p+1}} < \infty. \tag{13}$$

*Proof.* Suppose  $uC_\varphi$  is bounded from  $H^p$  to the Zygmund spaces  $\mathcal{X}_\alpha$ . Then we can easily obtain the following results by taking  $f(z) = 1$  and  $f(z) = z$  in  $H^p$ , respectively:

$$u \in \mathcal{X}_\alpha,$$

$$\begin{aligned} \sup_{z \in D} (1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z) + \varphi(z)u''(z)| \\ < +\infty. \end{aligned} \tag{14}$$

By (14) and the boundedness of the function  $\varphi(z)$ , we get

$$K_1 = \sup_{z \in D} (1 - |z|^2)^\alpha \left| 2\varphi'(z) u'(z) + \varphi''(z) u(z) \right| < +\infty. \quad (15)$$

Let  $f(z) = z^2$  in  $H^p$  again; in the same way we have

$$\begin{aligned} \sup_{z \in D} (1 - |z|^2)^\alpha \left| 4\varphi(z) \varphi'(z) u'(z) + \varphi^2(z) u''(z) \right. \\ \left. + 2u(z) \left( \varphi(z) \varphi''(z) + (\varphi'(z))^2 \right) \right| < \infty. \end{aligned} \quad (16)$$

Using these facts and the boundedness of the function  $\varphi(z)$  again, we get

$$K_2 = \sup_{z \in D} (1 - |z|^2)^\alpha \left| (\varphi'(z))^2 u(z) \right| < +\infty. \quad (17)$$

Fix  $a \in D$ ; we take the test functions

$$f_a(z) = -\frac{1 - |a|^2}{(1 - \bar{a}z)^{1+1/p}} + 2\frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{2+1/p}} - \frac{(1 - |a|^2)^3}{(1 - \bar{a}z)^{3+1/p}} \quad (18)$$

for  $z \in D$ . From Lemma 1 we obtain that  $f_a \in H^p$  and  $\sup_a \|f_a\|_p \leq C(p) < \infty$  with a direct calculation. Since  $f_a(a) = 0$ ,  $f'_a(a) = 0$ , and  $f''_a(a) = -2\bar{a}^2/(1 - |a|^2)^{2+1/p}$ , it follows that, for all  $\lambda \in D$  with  $|\varphi(\lambda)| > 1/2$ , we have

$$\begin{aligned} C\|f_a\|_* &\geq \|uC_\varphi f_a\|_* \geq \sup_{z \in D} (1 - |z|^2)^\alpha \left| (uC_\varphi f_a)''(z) \right| \\ &= \sup_{z \in D} (1 - |z|^2)^\alpha \\ &\quad \times \left| (2\varphi'(z) u'(z) + \varphi''(z) u(z)) f'_a(\varphi(z)) \right. \\ &\quad \left. + f''_a(\varphi(z)) (\varphi'(z))^2 u(z) \right. \\ &\quad \left. + u''(z) f_a(\varphi(z)) \right|. \end{aligned} \quad (19)$$

Let  $a = \varphi(\lambda)$ ; it follows that

$$\begin{aligned} C\|f_a\|_* &\geq (1 - |\lambda|^2)^\alpha \\ &\quad \times \left| (2\varphi'(\lambda) u'(\lambda) + \varphi''(\lambda) u(\lambda)) f'_{\varphi(\lambda)}(\varphi(\lambda)) \right. \\ &\quad \left. + f''_{\varphi(\lambda)}(\varphi(\lambda)) (\varphi'(\lambda))^2 u(\lambda) \right. \\ &\quad \left. + u''(\lambda) f_{\varphi(\lambda)}(\varphi(\lambda)) \right| \end{aligned}$$

$$\begin{aligned} &= (1 - |\lambda|^2)^\alpha \left| (\varphi'(\lambda))^2 u(\lambda) \frac{2(\overline{\varphi(\lambda)})^2}{(1 - |\varphi(\lambda)|^2)^{2+1/p}} \right| \\ &\geq \frac{1}{2} \frac{(1 - |\lambda|^2)^\alpha |u(\lambda) (\varphi'(\lambda))^2|}{(1 - |\varphi(\lambda)|^2)^{2+1/p}}. \end{aligned} \quad (20)$$

For all  $\lambda \in D$  with  $|\varphi(\lambda)| \leq 1/2$ , by (17), we have

$$\begin{aligned} &\sup_{\lambda \in D} \frac{(1 - |\lambda|^2)^\alpha |u(\lambda) (\varphi'(\lambda))^2|}{1 - |\varphi(\lambda)|^{1/p+2}} \\ &\leq \left(\frac{4}{3}\right)^{1/p+2} \sup_{\lambda \in D} (1 - |\lambda|^2)^\alpha |u(\lambda) (\varphi'(\lambda))^2| \\ &< +\infty. \end{aligned} \quad (21)$$

Hence (12) holds.

Next, fix  $a \in D$ ; we take another test functions

$$g_a(z) = -3\frac{1 - |a|^2}{(1 - \bar{a}z)^{1/p+1}} + 5\frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{1/p+2}} - 2\frac{(1 - |a|^2)^3}{(1 - \bar{a}z)^{1/p+3}} \quad (22)$$

for  $z \in D$ . From Lemma 1 we obtain that  $g_a \in H^p$  and  $\sup_a \|g_a\|_p \leq C(p) < \infty$  with a direct calculation. Since  $g_a(a) = 0$ ,  $g''_a(a) = 0$ , and  $g'_a(a) = \bar{a}/(1 - |a|^2)^{1/p+1}$ , it follows that, for all  $\lambda \in D$  with  $|\varphi(\lambda)| > 1/2$ , we obtain that

$$\begin{aligned} C\|g_a\|_* &\geq \|uC_\varphi g_a(z)\|_* \\ &\geq \sup_{z \in D} (1 - |z|^2)^\alpha \left| (uC_\varphi g_a)''(z) \right| \\ &= \sup_{z \in D} (1 - |z|^2)^\alpha \\ &\quad \times \left| (2\varphi'(z) u'(z) + \varphi''(z) u(z)) g'_a(\varphi(z)) \right. \\ &\quad \left. + g''_a(\varphi(z)) (\varphi'(z))^2 u(z) \right. \\ &\quad \left. + u''(z) g_a(\varphi(z)) \right| \\ &\geq (1 - |\lambda|^2)^\alpha \end{aligned}$$

$$\begin{aligned}
& \times \left| (2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda))g'_{\varphi(\lambda)}(\varphi(\lambda)) \right. \\
& \quad \left. + g''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda) \right. \\
& \quad \left. + u''(\lambda)g_{\varphi(\lambda)}(\varphi(\lambda)) \right| \\
& = (1 - |\lambda|^2)^\alpha \\
& \times \left| (2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)) \right. \\
& \quad \left. \times \frac{\overline{\varphi(\lambda)}}{(1 - |\varphi(\lambda)|^2)^{1/p+1}} \right| \\
& \geq \frac{1}{2} \frac{(1 - |\lambda|^2)^\alpha |2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{1/p+1}}.
\end{aligned} \tag{23}$$

For all  $\lambda \in D$  with  $|\varphi(\lambda)| \leq 1/2$ , by (15), we have

$$\begin{aligned}
& \sup_{|\varphi(\lambda)| \leq 1/2} \frac{(1 - |\lambda|^2)^\alpha |2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{1/p+1}} \\
& \leq \left(\frac{4}{3}\right)^{1/p+1} \sup_{|\varphi(\lambda)| \leq 1/2} (1 - |\lambda|^2)^\alpha \\
& \quad \times |2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda)| \\
& < \infty.
\end{aligned} \tag{24}$$

Hence (13) holds.

Finally, fix  $a \in D$ , and, for all  $z \in D$ , let

$$\begin{aligned}
h_a(z) & = -\frac{p+3}{p+1} \frac{1 - |a|^2}{(1 - \bar{a}z)^{1/p+1}} \\
& \quad + \frac{2(p+3)}{p+2} \frac{(1 - |a|^2)^2}{(1 - \bar{a}z)^{1/p+2}} - \frac{(1 - |a|^2)^3}{(1 - \bar{a}z)^{1/p+3}}.
\end{aligned} \tag{25}$$

From Lemma 1 we obtain that  $h_a \in H^p$  and  $\sup_a \|h_a\|_p \leq C(p) < \infty$  with a direct calculation. Since  $h'_a(a) = 0$ ,

$h''_a(a) = 0$ , and  $h_a(a) = -2/(p+1)(p+2)(1 - |a|^2)^{1/p}$ , it follows that, for all  $\lambda \in D$ , we obtain that

$$\begin{aligned}
C\|h_a\|_* & \geq \|uC_\varphi h_a(z)\|_* \\
& \geq \sup_{z \in D} (1 - |z|^2)^\alpha \left| (uC_\varphi h_a)''(z) \right| \\
& = \sup_{z \in D} (1 - |z|^2)^\alpha \\
& \quad \times \left| (2\varphi'(z)u'(z) + \varphi''(z)u(z))h'_a(\varphi(z)) \right. \\
& \quad \left. + h''_a(\varphi(z))(\varphi'(z))^2u(z) \right. \\
& \quad \left. + u''(z)h_a(\varphi(z)) \right| \\
& \geq (1 - |\lambda|^2)^\alpha \\
& \quad \times \left| (2\varphi'(\lambda)u'(\lambda) + \varphi''(\lambda)u(\lambda))h'_{\varphi(\lambda)}(\varphi(\lambda)) \right. \\
& \quad \left. + h''_{\varphi(\lambda)}(\varphi(\lambda))(\varphi'(\lambda))^2u(\lambda) \right. \\
& \quad \left. + u''(\lambda)h_{\varphi(\lambda)}(\varphi(\lambda)) \right| \\
& = (1 - |\lambda|^2)^\alpha |u''(\lambda)h_{\varphi(\lambda)}(\varphi(\lambda))| \\
& = \frac{2}{(p+1)(p+2)} \frac{(1 - |\lambda|^2)^\alpha |u''(\lambda)|}{(1 - |\varphi(\lambda)|^2)^{1/p}}.
\end{aligned} \tag{26}$$

Then (11) holds.

Conversely, suppose that (11), (12), and (13) hold. For  $f \in H^p$ , by Lemma 2, we have the following inequality:

$$\begin{aligned}
& (1 - |z|^2)^\alpha \left| (uC_\varphi f)''(z) \right| \\
& = (1 - |z|^2)^\alpha \\
& \quad \times \left| (2\varphi'(z)u'(z) + \varphi''(z)u(z))f'(\varphi(z)) \right. \\
& \quad \left. + f''(\varphi(z))(\varphi'(z))^2u(z) \right. \\
& \quad \left. + u''(z)f(\varphi(z)) \right| \\
& \leq (1 - |z|^2)^\alpha \\
& \quad \times \left| (2\varphi'(z)u'(z) + \varphi''(z)u(z))f'(\varphi(z)) \right| \\
& \quad + (1 - |z|^2)^\alpha \left| f''(\varphi(z))(\varphi'(z))^2u(z) \right| \\
& \quad + (1 - |z|^2)^\alpha |u''(z)f(\varphi(z))|
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{(1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{(1 - |\varphi(z)|^2)^{1/p+1}} \|f\|_p \\
 &\quad + \frac{(1 - |z|^2)^\alpha |(\varphi'(z))^2 u(z)|}{(1 - |\varphi(z)|^2)^{1/p+2}} \|f\|_p \\
 &\quad + \frac{(1 - |z|^2)^\alpha |u''(z)|}{(1 - |\varphi(z)|^2)^{1/p}} \|f\|_p \\
 &\leq C \|f\|_p, \\
 &|u(0)f(\varphi(0))| + |u'(0)f(\varphi(0))| + |u(0)f'(\varphi(0))| \\
 &\leq \left( \frac{|u(0)|}{(1 - |\varphi(0)|^2)^{1/p}} + \frac{|u'(0)|}{(1 - |\varphi(0)|^2)^{1/p}} \right. \\
 &\quad \left. + \frac{|u(0)|}{(1 - |\varphi(0)|^2)^{1/p+1}} \right) \|f\|_p. \tag{27}
 \end{aligned}$$

This shows that  $uC_\varphi$  is bounded. This completes the proof of Theorem 4.  $\square$

**Theorem 5.** Let  $\alpha > 0$ ,  $0 < p < \infty$ , and  $u$  be an analytic function on the unit disc  $D$  and  $\varphi$  an analytic self-map of  $D$ . Then  $uC_\varphi : H^p \rightarrow \mathcal{F}_{\alpha,0}$  is a bounded operator provided that the following are satisfied:

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |u''(z)|}{(1 - |\varphi(z)|^2)^{1/p}} = 0, \tag{28}$$

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{1/p+2}} = 0, \tag{29}$$

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{(1 - |\varphi(z)|^2)^{1/p+1}} = 0. \tag{30}$$

Conversely, if  $uC_\varphi : H^p \rightarrow \mathcal{F}_{\alpha,0}$  is a bounded operator, then  $u \in \mathcal{F}_{\alpha,0}$ , (11), (12), and (13) hold, and the following are satisfied:

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |u(z)(\varphi'(z))^2| = 0, \tag{31}$$

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)| = 0. \tag{32}$$

*Proof.* Assume that (28), (29), and (30) hold. Then for any  $\epsilon > 0$ , there is a constant  $\delta$ ,  $0 < \delta < 1$ , such that  $\delta < |z| < 1$  implies

$$\begin{aligned}
 &\frac{(1 - |z|^2)^\alpha |u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{1/p+2}} < \epsilon, \\
 &\frac{(1 - |z|^2)^\alpha |u''(z)|}{(1 - |\varphi(z)|^2)^{1/p}} < \epsilon, \tag{33}
 \end{aligned}$$

$$\frac{(1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{(1 - |\varphi(z)|^2)^{1/p+1}} < \epsilon.$$

Then, for any  $f \in H^p$ , from Lemma 2 we obtain that

$$\begin{aligned}
 &(1 - |z|^2)^\alpha |(uC_\varphi f)''(z)| \\
 &= (1 - |z|^2)^\alpha \\
 &\quad \times \left| (2\varphi'(z)u'(z) + \varphi''(z)u(z))f'(z) \right. \\
 &\quad \left. + f''(z)(\varphi'(z))^2 u(z) \right. \\
 &\quad \left. + u''(z)f(\varphi(z)) \right| \\
 &\leq (1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)| \\
 &\quad \times \frac{1}{(1 - |\varphi(z)|^2)^{1/p+1}} \|f\|_p \\
 &\quad + (1 - |z|^2)^\alpha |(\varphi'(z))^2 u(z)| \frac{1}{(1 - |\varphi(z)|^2)^{1/p+2}} \|f\|_p \\
 &\quad + \frac{(1 - |z|^2)^\alpha |u''(z)|}{(1 - |\varphi(z)|^2)^{1/p}} \|f\|_p \leq 3\|f\|_p \epsilon. \tag{34}
 \end{aligned}$$

Hence  $uC_\varphi f \in \mathcal{F}_{\alpha,0}$  for all  $f \in \mathcal{F}_{\alpha,0}$ . On the other hand, (25), (28), and (29) imply that (11), (12), and (13) hold; then  $uC_\varphi : H^p \rightarrow \mathcal{F}_\alpha$  is bounded by Theorem 4. So  $uC_\varphi : H^p \rightarrow \mathcal{F}_{\alpha,0}$  is bounded.

Conversely, assume that  $uC_\varphi$  is bounded from  $H^p$  to the little Zygmund type space  $\mathcal{F}_{\alpha,0}$ . Then  $u = uC_\varphi 1 \in \mathcal{F}_{\alpha,0}$ . Also  $u\varphi = uC_\varphi z \in \mathcal{F}_{\alpha,0}$ ; thus

$$\begin{aligned}
 &(1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z) + \varphi(z)u''(z)| \rightarrow 0 \\
 &\hspace{20em} (|z| \rightarrow 1^-). \tag{35}
 \end{aligned}$$

Since  $|\varphi| \leq 1$  and  $u \in \mathcal{F}_{\alpha,0}$ , we have  $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)| = 0$ . Hence (32) holds.

Similarly,  $uC_\varphi z^2 \in \mathcal{X}_{\alpha,0}$ ; then

$$\begin{aligned} & (1 - |z|^2)^\alpha \\ & \times \left| 4\varphi(z) \varphi'(z) u'(z) + \varphi^2(z) u''(z) \right. \\ & \left. + 2u(z) \left( \varphi(z) \varphi''(z) + (\varphi'(z))^2 \right) \right| \longrightarrow 0 \\ & (|z| \longrightarrow 1^-). \end{aligned} \quad (36)$$

By (32),  $|\varphi| \leq 1$ , and  $u \in \mathcal{X}_{\alpha,0}$ , we get that  $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |u(z)(\varphi'(z))^2| = 0$ ; that is, (31) holds.

On the other hand, from Lemma 3 and Theorem 4, we obtain that (11), (12), and (13) hold.  $\square$

#### 4. Compactness of $uC_\varphi$

In order to prove the compactness of  $uC_\varphi$  from  $H^p$  to the Zygmund spaces  $\mathcal{X}_\alpha$ , we require the following lemmas.

**Lemma 6.** *Let  $\alpha > 0$ ,  $0 < p < \infty$ , and  $u$  be an analytic function on the unit disc  $D$  and  $\varphi$  an analytic self-map of  $D$ . Suppose that  $uC_\varphi$  is a bounded operator from  $H^p$  to  $\mathcal{X}_\alpha$ . Then  $uC_\varphi$  is compact if and only if, for any bounded sequence  $\{f_n\}$  in  $H^p$  which converges to 0 uniformly on compact subsets of  $D$ , one has  $\|uC_\varphi(f_n)\|_* \rightarrow 0$  as  $n \rightarrow \infty$ .*

The proof is similar to that of Proposition 3.11 in [32]. The details are omitted.

**Theorem 7.** *Let  $\alpha > 0$ ,  $0 < p < \infty$ ,  $u$  be an analytic function on the unit disc  $D$  and  $\varphi$  an analytic self-map of  $D$ . Then  $uC_\varphi$  is a compact operator from  $H^p$  to  $\mathcal{X}_\alpha$  if and only if  $uC_\varphi$  is a bounded operator and the following are satisfied:*

$$\begin{aligned} (i) \quad & \lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |u''(z)|}{(1 - |\varphi(z)|^2)^{1/p}} = 0, \\ (ii) \quad & \lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{1/p+2}} = 0, \\ (iii) \quad & \lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{(1 - |\varphi(z)|^2)^{1/p+1}} = 0. \end{aligned} \quad (37)$$

*Proof.* Suppose that  $uC_\varphi$  is compact from  $H^p$  to the Zygmund type space  $\mathcal{X}_\alpha$ . Let  $\{z_n\}$  be a sequence in  $D$  such that  $|\varphi(z_n)| \rightarrow 1$  as  $n \rightarrow \infty$ . If such a sequence does not exist, then (37) are automatically satisfied. Without loss of

generality we may suppose that  $|\varphi(z_n)| > 1/2$  for all  $n$ . We take the test functions

$$\begin{aligned} f_n(z) = & -\frac{1 - |\varphi(z_n)|^2}{(1 - \overline{\varphi(z_n)}z)^{1/p+1}} + 2\frac{(1 - |\varphi(z_n)|^2)^2}{(1 - \overline{\varphi(z_n)}z)^{1/p+2}} \\ & - \frac{(1 - |\varphi(z_n)|^2)^3}{(1 - \overline{\varphi(z_n)}z)^{1/p+3}}. \end{aligned} \quad (38)$$

By a direct calculation, we may easily prove that  $\{f_n\}$  converges to 0 uniformly on compact subsets of  $D$  and  $\sup_n \|f_n\|_p \leq C(p) < \infty$ . Then  $\{f_n\}$  is a bounded sequence in  $H^p$  which converges to 0 uniformly on compact subsets of  $D$ . Then  $\lim_{n \rightarrow \infty} \|uC_\varphi(f_n)\|_* = 0$  by Lemma 6. Note that

$$\begin{aligned} f_n(\varphi(z_n)) &= 0, \quad f'_n(\varphi(z_n)) = 0, \\ f''_n(\varphi(z_n)) &= -\frac{\overline{2\varphi(z_n)}^2}{(1 - |\varphi(z_n)|^2)^{1/p+2}}. \end{aligned} \quad (39)$$

It follows that

$$\begin{aligned} \|uC_\varphi f_n\|_* &\geq \|uC_\varphi f_n\|_{\mathcal{X}_\alpha} \\ &\geq (1 - |z_n|^2)^\alpha \\ &\quad \times \left| (2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)) \right. \\ &\quad \times f'_n(\varphi(z_n)) + u(z_n)f''_n(\varphi(z_n))(\varphi'(z_n))^2 \\ &\quad \left. + u''(z_n)f_n(\varphi(z_n)) \right| \\ &= (1 - |z_n|^2)^\alpha \\ &\quad \times \left| (\varphi'(z_n))^2 u(z_n) \frac{\overline{2\varphi(z_n)}^2}{(1 - |\varphi(z_n)|^2)^{1/p+2}} \right|. \end{aligned} \quad (40)$$

Then

$$\lim_{n \rightarrow \infty} \frac{(1 - |z_n|^2)^\alpha |u(z_n)(\varphi'(z_n))^2|}{(1 - |\varphi(z_n)|^2)^{1/p+2}} = 0. \quad (41)$$

Next, let

$$\begin{aligned} g_n(z) = & -3\frac{1 - |\varphi(z_n)|^2}{(1 - \overline{\varphi(z_n)}z)^{1/p+1}} + 5\frac{(1 - |\varphi(z_n)|^2)^2}{(1 - \overline{\varphi(z_n)}z)^{1/p+2}} \\ & - 2\frac{(1 - |\varphi(z_n)|^2)^3}{(1 - \overline{\varphi(z_n)}z)^{1/p+3}}. \end{aligned} \quad (42)$$

By a direct calculation we obtain that  $g_n \rightrightarrows 0$  ( $n \rightarrow \infty$ ) on compact subsets of  $D$  and  $\sup_n \|g_n\|_p \leq C(p) < +\infty$ . Consequently,  $\{g_n\}$  is a bounded sequence in  $H^p$  which converges to 0 uniformly on compact subsets of  $D$ . Then  $\lim_{n \rightarrow \infty} \|uC_\varphi(g_n)\|_* = 0$  by Lemma 6. Note that  $g_n(\varphi(z_n)) \equiv 0$ ,  $g_n''(\varphi(z_n)) \equiv 0$  and  $g_n'(\varphi(z_n)) = \overline{\varphi(z_n)}/(1 - |\varphi(z_n)|^2)^{1/p+1}$ ; it follows that

$$\begin{aligned} \|uC_\varphi g_n\|_* &\geq \|uC_\varphi g_n\|_{\mathcal{X}_\alpha} \\ &\geq (1 - |z_n|^2)^\alpha \\ &\quad \times \left| (2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)) \right. \\ &\quad \times g_n'(\varphi(z_n)) + u(z_n)g_n''(\varphi(z_n))(\varphi'(z_n))^2 \\ &\quad \left. + u''(z_n)g_n(\varphi(z_n)) \right| \\ &= (1 - |z_n|^2)^\alpha \\ &\quad \times \left| (2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)) \right. \\ &\quad \left. \times \frac{\overline{\varphi(z_n)}}{(1 - |\varphi(z_n)|^2)^{1/p+1}} \right|. \end{aligned} \tag{43}$$

Then  $\lim_{n \rightarrow \infty} (1 - |z_n|^2)^\alpha (2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n))/(1 - |\varphi(z_n)|^2)^{1/p+1} = 0$ .

Finally, let

$$\begin{aligned} h_n(z) &= -\frac{p+3}{p+1} \frac{1 - |\varphi(z_n)|^2}{(1 - \overline{\varphi(z_n)}z)^{1/p+1}} \\ &\quad + \frac{2(p+3)}{p+2} \frac{(1 - |\varphi(z_n)|^2)^2}{(1 - \overline{\varphi(z_n)}z)^{1/p+2}} \\ &\quad - \frac{(1 - |\varphi(z_n)|^2)^3}{(1 - \overline{\varphi(z_n)}z)^{1/p+3}}. \end{aligned} \tag{44}$$

By a direct calculation we obtain that  $h_n \rightrightarrows 0$  ( $n \rightarrow \infty$ ) on compact subsets of  $D$  and  $\sup_n \|h_n\|_p < \infty$ . Consequently,  $\{h_n\}$  is a bounded sequence in  $H^p$  which converges to 0 uniformly on compact subsets of  $D$ . Then  $\lim_{n \rightarrow \infty} \|uC_\varphi(h_n)\|_* = 0$  by Lemma 6. Note that

$h_n(\varphi(z_n)) = 2/(p+1)(p+2)(1 - |\varphi(z_n)|^2)^{1/p}$ ,  $h_n'(\varphi(z_n)) \equiv 0$ , and  $h_n''(\varphi(z_n)) \equiv 0$ ; it follows that

$$\begin{aligned} \|uC_\varphi h_n\|_* &\geq \|uC_\varphi h_n\|_{\mathcal{X}_\alpha} \\ &\geq (1 - |z_n|^2)^\alpha \\ &\quad \times \left| (2u'(z_n)\varphi'(z_n) + \varphi''(z_n)u(z_n)) \right. \\ &\quad \times h_n'(\varphi(z_n)) + u(z_n)h_n''(\varphi(z_n))(\varphi'(z_n))^2 \\ &\quad \left. + u''(z_n)h_n(\varphi(z_n)) \right| \\ &= \frac{2(1 - |z_n|^2)^\alpha |u''(z_n)|}{(p+1)(p+2)(1 - |\varphi(z_n)|^2)^{1/p}}. \end{aligned} \tag{45}$$

Then  $\lim_{n \rightarrow \infty} ((1 - |z_n|^2)^\alpha |u''(z_n)|/(1 - |\varphi(z_n)|^2)^{1/p}) = 0$ . The proof of the necessary is completed.

Conversely, Suppose that (37) hold. Since  $uC_\varphi$  is a bounded operator, from Theorem 4, we have

$$M_1 \triangleq \sup_{z \in D} (1 - |z|^2)^\alpha |u(z)(\varphi'(z))^2| < \infty,$$

$$M_2 \triangleq \sup_{z \in D} (1 - |z|^2)^\alpha |u''(z)| < \infty,$$

$$M_3 \triangleq \sup_{z \in D} (1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)| < \infty. \tag{46}$$

Let  $\{f_n\}$  be a bounded sequence in  $H^p$  with  $\|f_n\|_p \leq 1$  and  $f_n \rightarrow 0$  uniformly on compact subsets of  $D$ . We only prove  $\lim_{n \rightarrow \infty} \|uC_\varphi(f_n)\|_* = 0$  by Lemma 6. By the assumption, for any  $\epsilon > 0$ , there is a constant  $\delta$ ,  $0 < \delta < 1$ , such that  $\delta < |\varphi(z)| < 1$  implies

$$\begin{aligned} \frac{(1 - |z|^2)^\alpha |u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^{1/p+2}} &< \epsilon, \\ \frac{(1 - |z|^2)^\alpha |u''(z)|}{(1 - |\varphi(z)|^2)^{1/p}} &< \epsilon, \\ \frac{(1 - |z|^2)^\alpha |2\varphi'(z)u'(z) + \varphi''(z)u(z)|}{(1 - |\varphi(z)|^2)^{1/p+1}} &< \epsilon. \end{aligned} \tag{47}$$



Let  $K = \{w \in D : |w| \leq \delta\}$ . Note that  $K$  is a compact subset of  $D$ . Then from Lemma 2 it follows that

$$\begin{aligned} \|uC_\varphi f_n\|_{\mathcal{Z}_\alpha} &= \sup_{z \in D} (1 - |z|^2)^\alpha \left| (uC_\varphi f_n)''(z) \right| \\ &= (1 - |z|^2)^\alpha \\ &\quad \times \left| \left( 2\varphi'(z)u'(z) + \varphi''(z)u(z) \right) f_n'(\varphi(z)) \right. \\ &\quad \left. + f_n''(\varphi(z))(\varphi'(z))^2 u(z) \right. \\ &\quad \left. + u''(z)f_n(\varphi(z)) \right| \\ &\leq \sup_{z \in D} (1 - |z|^2)^\alpha \\ &\quad \times \left| \left( 2\varphi'(z)u'(z) + \varphi''(z)u(z) \right) f_n'(\varphi(z)) \right| \\ &\quad + \sup_{z \in D} (1 - |z|^2)^\alpha \left| f_n''(\varphi(z))(\varphi'(z))^2 u(z) \right| \\ &\quad + \sup_{z \in D} (1 - |z|^2)^\alpha \left| u''(z)f_n(\varphi(z)) \right| \\ &\leq 3\epsilon + \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^\alpha \\ &\quad \times \left| \left( 2\varphi'(z)u'(z) + \varphi''(z)u(z) \right) f_n'(\varphi(z)) \right| \\ &\quad + \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^\alpha \left| f_n''(\varphi(z))(\varphi'(z))^2 u(z) \right| \\ &\quad + \sup_{|\varphi(z)| \leq \delta} (1 - |z|^2)^\alpha \left| u''(z)f_n(\varphi(z)) \right| \\ &\leq 3\epsilon + M_3 \sup_{w \in K} |f_n'(w)| + M_1 \sup_{w \in K} |f_n''(w)| \\ &\quad + M_2 \sup_{w \in K} |f_n(w)|. \end{aligned} \tag{48}$$

As  $n \rightarrow \infty$ ,

$$\|uC_\varphi f_n\|_{\mathcal{Z}_\alpha} \rightarrow 0. \tag{49}$$

Hence  $uC_\varphi$  is compact. This completes the proof of Theorem 7.  $\square$

In order to prove the compactness of  $uC_\varphi$  on the little Zygmund spaces  $\mathcal{Z}_{\alpha,0}$ , we require the following lemma.

**Lemma 8.** *Let  $U \subset \mathcal{Z}_{\alpha,0}$ . Then  $U$  is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z| \rightarrow 1^-} \sup_{f \in U} (1 - |z|^2)^\alpha |f''(z)| = 0. \tag{50}$$

The proof is similar to that of Lemma 1 in [1], but we omit it.

**Theorem 9.** *Let  $\alpha > 0$ ,  $0 < p < \infty$ ,  $u$  be an analytic function on the unit disc  $D$  and  $\varphi$  an analytic self-map of  $D$ . Then  $uC_\varphi$*

*is compact from  $H^p$  to the little Zygmund type spaces  $\mathcal{Z}_{\alpha,0}$  if and only if (28), (29), and (30) hold.*

*Proof.* Assume that (28), (29), and (30) hold. By Theorem 5, we know that  $uC_\varphi$  is bounded from  $H^p$  to the little Zygmund type spaces  $\mathcal{Z}_{\alpha,0}$ . Suppose that  $f \in H^p$  with  $\|f\|_p \leq 1$ . From Lemmas 1 and 2 we obtain that

$$\begin{aligned} &(1 - |z|^2)^\alpha \left| (uC_\varphi f)''(z) \right| \\ &\leq (1 - |z|^2)^\alpha \left| \left( 2\varphi'(z)u'(z) + \varphi''(z)u(z) \right) f'(\varphi(z)) \right| \\ &\quad + (1 - |z|^2)^\alpha \left| f''(\varphi(z))(\varphi'(z))^2 u(z) \right| \\ &\quad + (1 - |z|^2)^\alpha \left| u''(z)f(\varphi(z)) \right| \\ &\leq (1 - |z|^2)^\alpha \left| 2\varphi'(z)u'(z) + \varphi''(z)u(z) \right| \\ &\quad \times \frac{1}{(1 - |\varphi(z)|^2)^{1/p+1}} \|f\|_p \\ &\quad + \frac{(1 - |z|^2)^\alpha \left| (\varphi'(z))^2 u(z) \right|}{(1 - |\varphi(z)|^2)^{1/p+2}} \|f\|_p \\ &\quad + \frac{(1 - |z|^2)^\alpha \left| u''(z) \right|}{(1 - |\varphi(z)|^2)^{1/p}} \|f\|_p, \end{aligned} \tag{51}$$

thus

$$\begin{aligned} &\sup \left\{ \left| (1 - |z|^2)^\alpha (uC_\varphi f)''(z) \right| : f \in H^p, \|f\|_p \leq 1 \right\} \\ &\leq (1 - |z|^2)^\alpha \left| 2\varphi'(z)u'(z) + \varphi''(z)u(z) \right| \frac{1}{1 - |\varphi(z)|^2} \\ &\quad + \frac{(1 - |z|^2)^\alpha \left| (\varphi'(z))^2 u(z) \right|}{(1 - |\varphi(z)|^2)^{1/p+2}} + \frac{(1 - |z|^2)^\alpha \left| u''(z) \right|}{(1 - |\varphi(z)|^2)^{1/p}}, \end{aligned} \tag{52}$$

and it follows that

$$\begin{aligned} &\lim_{|z| \rightarrow 1^-} \sup \left\{ \left| (1 - |z|^2)^\alpha (uC_\varphi f)''(z) \right| : f \in H^p, \|f\|_p \leq 1 \right\} \\ &= 0. \end{aligned} \tag{53}$$

Hence  $uC_\varphi : H^p \rightarrow \mathcal{Z}_{\alpha,0}$  is compact by Lemma 8.

Conversely, suppose that  $uC_\varphi : H^p \rightarrow \mathcal{Z}_{\alpha,0}$  is compact.

Firstly, it is obvious  $uC_\varphi$  is bounded, from Theorem 5 we have  $u \in \mathcal{Z}_{\alpha,0}$ , and (31), (32) hold. On the other hand, we have



$$\lim_{|z| \rightarrow 1^-} \sup \left\{ \left| (1 - |z|^2)^\alpha (uC_\varphi f)''(z) \right| : f \in H^p, \|f\|_p \leq M \right\} = 0, \tag{54}$$

for some  $M > 0$  by Lemma 6.

Next, note that the proof of Theorem 4 and the fact that the functions given in (18) are in  $H^p$  and have norms bounded independently of  $a$ ; we obtain that

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| u(z) (\varphi'(z))^2 \right|}{(1 - |\varphi(z)|^2)^{1/p+2}} = 0 \tag{55}$$

for  $|\varphi(z)| > 1/2$ . However, if  $|\varphi(z)| \leq 1/2$ , by (31), we easily have

$$\begin{aligned} & \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| u(z) (\varphi'(z))^2 \right|}{(1 - |\varphi(z)|^2)^{1/p+2}} \\ & \leq \lim_{|z| \rightarrow 1^-} \left( \frac{4}{3} \right)^{1/p+2} (1 - |z|^2)^\alpha \left| u(z) (\varphi'(z))^2 \right| \\ & = 0. \end{aligned} \tag{56}$$

Similarly, note that the functions given in (22) and (25) are in  $H^p$  and have norms bounded independently of  $a$ , we obtain that

$$\begin{aligned} & \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| 2\varphi'(z) u'(z) + \varphi''(z) u(z) \right|}{(1 - |\varphi(z)|^2)^{1/p+1}} = 0, \\ & \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| u''(z) \right|}{(1 - |\varphi(z)|^2)^{1/p}} = 0 \end{aligned} \tag{57}$$

for  $|\varphi(z)| > 1/2$ . However, if  $|\varphi(z)| \leq 1/2$ , from  $u \in \mathcal{X}_{\alpha,0}$  and (32), we easily have

$$\begin{aligned} & \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| 2\varphi'(z) u'(z) + \varphi''(z) u(z) \right|}{(1 - |\varphi(z)|^2)^{1/p+1}} \\ & \leq \left( \frac{4}{3} \right)^{1/p+1} \lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha \\ & \quad \times \left| 2\varphi'(z) u'(z) + \varphi''(z) u(z) \right| \\ & = 0, \\ & \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| u''(z) \right|}{(1 - |\varphi(z)|^2)^{1/p}} \\ & \leq \lim_{|z| \rightarrow 1^-} \left( \frac{4}{3} \right)^{1/p} (1 - |z|^2)^\alpha \left| u''(z) \right| \\ & = 0. \end{aligned} \tag{58}$$

This completes the proof of Theorem 9.  $\square$

*Remark 10.* From Theorems 5 and 9, we conjecture that  $uC_\varphi : H^p \rightarrow \mathcal{X}_{\alpha,0}$  is compact if and only if  $uC_\varphi : H^p \rightarrow \mathcal{X}_{\alpha,0}$  is bounded.

Taking  $u(z) = 1$  from Theorems 4, 7, and 9, we obtain the following results about the characterization of the boundedness and compactness of the composition operator  $C_\varphi : H^p \rightarrow \mathcal{X}_\alpha$  (or  $\mathcal{X}_{\alpha,0}$ ).

**Corollary 11.** *Let  $\alpha > 0$ ,  $0 < p < \infty$ , and  $\varphi$  be an analytic self-map of  $D$ . Then  $C_\varphi : H^p \rightarrow \mathcal{X}_\alpha$  is a bounded operator if and only if the following are satisfied:*

$$\begin{aligned} & \sup_{z \in D} \frac{(1 - |z|^2)^\alpha \left| (\varphi'(z))^2 \right|}{(1 - |\varphi(z)|^2)^{1/p+2}} < \infty, \\ & \sup_{z \in D} \frac{(1 - |z|^2)^\alpha \left| \varphi''(z) \right|}{(1 - |\varphi(z)|^2)^{1/p+1}} < \infty. \end{aligned} \tag{59}$$

**Corollary 12.** *Let  $\alpha > 0$ ,  $0 < p < \infty$ , and  $\varphi$  be an analytic self-map of  $D$ . Then  $C_\varphi : H^p \rightarrow \mathcal{X}_\alpha$  is a compact operator if and only if  $C_\varphi$  is bounded and the following are satisfied:*

$$\begin{aligned} & \lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| (\varphi'(z))^2 \right|}{(1 - |\varphi(z)|^2)^{1/p+2}} = 0, \\ & \lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| \varphi''(z) \right|}{(1 - |\varphi(z)|^2)^{1/p+1}} = 0. \end{aligned} \tag{60}$$

**Corollary 13.** *Let  $\alpha > 0$ ,  $0 < p < \infty$ , and  $\varphi$  be an analytic self-map of  $D$ . Then  $C_\varphi : H^p \rightarrow \mathcal{X}_{\alpha,0}$  is a compact operator if and only if*

$$\begin{aligned} & \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| (\varphi'(z))^2 \right|}{(1 - |\varphi(z)|^2)^{1/p+2}} = 0, \\ & \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\alpha \left| \varphi''(z) \right|}{(1 - |\varphi(z)|^2)^{1/p+1}} = 0. \end{aligned} \tag{61}$$

In the formulation of corollary, we use the notation  $M_u$  on  $H(D)$  defined by  $M_u f = uf$  for  $f \in H(D)$ . Taking  $\varphi(z) = z$  from Theorems 4, 5, 7, and 9, we obtain the following results about the characterization of the boundedness and compactness of pointwise multiplier  $M_u : H^p \rightarrow \mathcal{X}_\alpha$  (or  $\mathcal{X}_{\alpha,0}$ ).

**Corollary 14.** *Let  $\alpha > 0$ ,  $0 < p < \infty$ . Then the pointwise multiplier  $M_u : H^p \rightarrow \mathcal{X}_\alpha$  is a bounded operator if and only if*

- (i)  $u = 0$  if  $\alpha < 2 + 1/p$ ;
- (ii)  $u \in H^\infty$  if  $\alpha = 2 + 1/p$ ;
- (iii)  $\sup_{z \in D} (1 - |z|^2)^{\alpha-2-1/p} |u(z)| < \infty$  if  $\alpha > 2 + 1/p$ .

**Corollary 15.** Let  $\alpha > 0$ ,  $0 < p < \infty$ . Then the pointwise multiplier  $M_u : H^p \rightarrow \mathcal{X}_{\alpha,0}$  is a bounded operator if and only if  $M_u : H^p \rightarrow \mathcal{X}_{\alpha,0}$  is a compact operator if and only if  $M_u : H^p \rightarrow \mathcal{X}_\alpha$  is a compact operator if and only if

$$(i) u = 0 \text{ if } \alpha \leq 2 + 1/p,$$

$$(ii) \lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{\alpha-2-1/p} |u(z)| = 0 \text{ if } \alpha > 2 + 1/p.$$

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