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Research Article

Robustness of Exponential Dissipation with respect to Small Time Delay

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We investigate robustness of exponential dissipation for the following general nonlinear evolutionary equation with small time delay: $\partial_t u + Au = f(u(t), u(t-\tau))$. We firstly obtain a converse Lyapunov theorem. With the help of it, we establish an important result on robustness of exponential dissipation to small time delay assuming that the nonlinearity is globally Lipschitz.

1. Introduction

As is well known, time delays are usually encountered in practical control systems. The stability analysis has received attentions over the last several decades. Mathematically, it is also very important to understand the sensitivity of the dynamical behavior of the system to the introduction of small time delays. For linear systems, we well understand this problem, including both finite dimensional and infinite dimensional situations, see [1–5]. However, for nonlinear systems, the problem is much more difficult, but there are some very nice results in [6–10].

This paper is devoted to the following general nonlinear evolutionary equation with small delay:

$$\partial_t u + Au = f(u(t), u(t-\tau)), \tag{1}$$

where the nonlinearity $f : X^\alpha \times X^\alpha \to X$ is assumed to be globally Lipschitz. Here, A is a sectorial operator on a Banach space X. X^α is a fractional power space. Here, we investigate the effects of small time delay on the exponential dissipation of the corresponding evolutionary equation without delay:

$$\partial_t u + Au = F(u), \tag{2}$$

where $F(u) = f(u, u)$.

In [11], Lyapunov introduced his famous sufficient conditions for asymptotic stability of the following nonautonomous dynamical system:

$$x'(t) = f(t, x(t)), \quad x(t) \in R^n. \tag{3}$$

There, we can also find the first contribution to the converse question, known as converse Lyapunov theorems. In recent years, the answers have proved instrumental in establishing robustness of various stability notions and have served as the starting point for many nonlinear control systems design concepts.

In 2005, Li and Kloeden [8] presented a converse Lyapunov theorem for exponential dissipation of the following general nonlinear differential equations with multiple small time delays:

$$x'(t) = f(x(t-\tau_1), x(t-\tau_2), \dots, x(t-\tau_n)), \tag{4}$$

where f is assumed to be globally Lipschitz. They also prove that exponential dissipation remains under small time delays. This result can be seen as a generalization of some classical ones on global exponential asymptotic stability (e.g., [12]) and was used by the authors to study robustness of exponential dissipation with respect to small time delays.

Recently, Guo and Li [13] gave a nonautonomous analog of the result. They not only present a converse Lyapunov theorem but also prove robustness of the uniform exponential dissipation with respect to unbounded external perturbations.

In the dynamical theory, a basic problem concerns the robustness of global attractors under perturbations [14]. It is known that if a nonlinear system with a global attractor $\mathscr{A}$ is perturbed, then the perturbed one also has an attractor $\mathscr{A}'$ near $\mathscr{A}$, provided that the perturbation is sufficiently small [7, 15]. However, in general, we only know that $\mathscr{A}'$ is a local attractor. Whether the global feature can be preserved is still an open problem. To our great joy, a dissipative system usually implies the existence of the global attractor. So, if one wants to settle the above problem, he only needs to examine the robustness of dissipation under perturbations. In this present work, we will investigate the infinite dimensional situations which are more difficult than the finite ones. With the nonlinearity being globally Lipschitz, we obtain a converse Lyapunov theorem and prove that exponential dissipation has nice robustness properties under small time delay.

2. Preliminaries

In this paper, we study the following delayed initial value problem:

$$u_t + Au = f(u(t), u(t-\tau)), \quad t > 0, \qquad u|_{[-\tau,0]} = u_0(t). \tag{5}$$

For simplicity, we use $\|\cdot\|$ and $\|\cdot\|_\alpha$ to denote the norm on X and X^α, respectively. We write $\mathscr{C} = C([-\tau, 0], X^\alpha)$ with the norm $|||\cdot|||_\alpha$ defined by

$$|||u_0|||_\alpha = \max_{t\in[-\tau,0]} \|u_0(t)\|_\alpha, \quad \forall u_0 \in \mathscr{C}. \tag{6}$$

Next, we will recall some basic definitions and facts.

The upper right Dini derivative of a function $y \in C((\alpha, \beta), X^\alpha)$ is defined as

$$\frac{d^+}{dt} y(t) := \limsup_{h\to 0+} \frac{y(t+h) - y(t)}{h}. \tag{7}$$

Let $x \in X^\alpha$ and $\mathscr{N}$ be an open neighborhood of x. For $V \in C(\mathscr{N}, R^1)$ and $v \in X^\alpha$, we define

$$D_v^+ V(x) := \limsup_{h\to 0+} \frac{V(x+hv) - V(x)}{h}. \tag{8}$$

We will denote by $u(t, x)$ the solution of (2), where $u(0, x) = x$.

Definition 1. The system (2) is said to be exponentially dissipative, if there exist positive numbers B, λ, and ρ such that

$$\|u(t,x)\|_\alpha \le Be^{-\lambda t}\|x\|_\alpha + \rho, \quad \forall t \ge 0,\ x \in X^\alpha. \tag{9}$$

Lemma 2. *Let $\mathscr{N}$ be an open subset of X^α. Assume that the function $V : \mathscr{N} \to R^+$ is Lipschitz; that is, there exists a $L_V > 0$ such that*

$$|V(x) - V(y)| \le L_V\|x-y\|_\alpha, \quad \forall x, y \in X^\alpha. \tag{10}$$

Let $u(t)$ be a solution of (2). Then,

$$\frac{d^+}{dt} V(u(t)) = D^+_{g(u)} V(u(t)), \quad \text{where } g(u) = F(u) - Au. \tag{11}$$

Proof. The detailed proof is contained in [12, 16]. Here, we give a simple proof for the reader's convenience. Making use of Taylor formula, we observe that

$$\begin{aligned} &V(u(t+h)) - V(u(t)) \\ &\quad = V(u(t) + g(u)h + o(h)) - V(u(t)) \\ &\quad = V(u(t) + g(u)h + o(h)) - V(u(t) + g(u)h) \\ &\qquad + V(u(t) + g(u)h) - V(u(t)). \end{aligned} \tag{12}$$

Since $V(x)$ is Lipschitz, one easily sees that

$$V(u(t) + g(u)h + o(h)) - V(u(t) + g(u)h) = o(h). \tag{13}$$

Therefore, by definition (7), we immediately deduce that

$$\begin{aligned} &\frac{d^+}{dt} V(u(t)) \\ &\quad = \limsup_{h\to 0+} \frac{V(u(t+h)) - V(u(t))}{h} \\ &\quad = \limsup_{h\to 0+} \frac{V(u(t) + g(u)h) - V(u(t))}{h} \\ &\quad = D^+_{g(u)} V(u(t)). \end{aligned} \tag{14}$$

The proof is finished. □

At last, we come to the main theorem on analytic semigroup which is extremely important in the study of the dynamics of nonlinear evolutionary equations [17].

Theorem 3 (fundamental theorem on sectorial operators). *Let A be a positive, sectorial operator on a Banach space X and let e^{-At} be the analytic semigroup generated by $-A$. Then, the following statements hold.*

(1) *For any $\alpha \ge 0$, there is a constant $C_\alpha > 0$ such that for all $t > 0$*

$$\left\|A^\alpha e^{-At}\right\|_{L(X)} \le C_\alpha t^{-\alpha} e^{-at} \quad (a > 0). \tag{15}$$

(2) *For $0 < \alpha \le 1$, there is a constant $C_\alpha > 0$ such that for $t \ge 0$ and $x \in D(A^\alpha)$*

$$\left\|e^{-At}x - x\right\| \le C_\alpha t^\alpha \left\|A^\alpha x\right\|. \tag{16}$$

(3) *For every $\alpha \ge 0$, there is a constant $C_\alpha > 0$ such that for all $t > 0$ and $x \in X$*

$$\left\|\left(e^{-A(t+h)} - e^{-At}\right)x\right\|_\alpha \le C_\alpha |h|\, t^{-(1+\alpha)} \|x\|. \tag{17}$$

3. Main Results

In this section, we will prove our two main results: one is converse Lyapunov theorem, and the other is robustness of exponential dissipation to small time delay.

Theorem 4 (converse Lyapunov theorem). *Suppose that* $F: X^\alpha \to X$ *in* (2) *is globally Lipschitz with Lipschitz constant L. Suppose that the system without delay* (2) *is exponentially dissipative. Then, there exists a function* $V: X^\alpha \to R^+$ *satisfying*

$$\|x\|_\alpha^2 - a \le V(x) \le b\|x\|_\alpha^2 + c, \tag{18}$$

$$D^+_{g(x)} V(x) \le -d\|x\|_\alpha^2 + \sigma, \tag{19}$$

$$|V(x) - V(y)| \le L_V (\|x\|_\alpha + \|y\|_\alpha + 1)\|x - y\|_\alpha \tag{20}$$

for all $x, y \in X^\alpha$, *where* $g(x) = F(x) - Ax$, a, b, c, d, σ, *and* L_V *are appropriate positive constants.*

Proof. Since the system (2) is exponentially dissipative, there exist positive constants B, λ, and ρ such that

$$\|u(t,x)\|_\alpha \le Be^{-\lambda t}\|x\|_\alpha + \rho, \quad \forall t \ge 0,\ x \in X^\alpha. \tag{21}$$

Let $T = \ln(2B)/\lambda$, and define V_1 as follows:

$$V_1(x) := \int_0^T \|u(s,x)\|_\alpha^2 ds, \quad x \in X^\alpha. \tag{22}$$

By (21) and the elementary inequality, it is easy to check that

$$\begin{aligned} 0 \le V_1(x) &\le \int_0^T \left(Be^{-\lambda s}\|x\|_\alpha + \rho\right)^2 ds \\ &\le 2\int_0^T \left(B^2 e^{-2\lambda s}\|x\|_\alpha^2 + \rho^2\right) ds \\ &\le \frac{B^2}{\lambda}\|x\|_\alpha^2 + 2T\rho^2. \end{aligned} \tag{23}$$

So, $V_1(x)$ satisfies the right inequality of (18).

Next, by the Lipschitz continuity of F, it is easy to verify that there exists a constant $C(T) > 0$ such that

$$\begin{aligned} \|u(t,x) - u(t,y)\|_\alpha \le C(T)\|x - y\|_\alpha, \\ \forall x, y \in X^\alpha,\ t \in [0,T]. \end{aligned} \tag{24}$$

Considering (21) and (23), for any $x, y \in X^\alpha$, we have

$$\begin{aligned} &|V_1(x) - V_1(y)| \\ &= \left|\int_0^T \left(\|u(s,x)\|_\alpha^2 - \|u(s,y)\|_\alpha^2\right) ds\right| \\ &\le C(T)\|x-y\|_\alpha \int_0^T (\|u(s,x)\|_\alpha + \|u(s,y)\|_\alpha) ds \\ &\le C(T)\|x-y\|_\alpha \int_0^T \left[Be^{-\lambda s}(\|x\|_\alpha + \|y\|_\alpha) + 2\rho\right] ds \\ &\le C(T)\|x-y\|_\alpha \left[\frac{B}{\lambda}(\|x\|_\alpha + \|y\|_\alpha) + 2\rho T\right] ds \\ &\le L_1(\|x\|_\alpha + \|y\|_\alpha + 1)\|x-y\|_\alpha. \end{aligned} \tag{25}$$

So, $V_1(x)$ satisfies (20).

Since

$$\begin{aligned} V_1(u(t,x)) &= \int_0^T \|u(s,u(t,x))\|_\alpha^2 ds \\ &= \int_0^T \|u(t+s,x)\|_\alpha^2 ds \\ &= \int_t^{t+T} \|u(s,x)\|_\alpha^2 ds, \end{aligned} \tag{26}$$

by the choice of T and (21), we have that

$$\begin{aligned} &\frac{d}{dt}V_1(u(t,x)) \\ &\quad = \|u(t+T,x)\|_\alpha^2 - \|u(t,x)\|_\alpha^2 \\ &\quad = \|u(T,u(t,x))\|_\alpha^2 - \|u(t,x)\|_\alpha^2 \\ &\quad \le -\|u(t,x)\|_\alpha^2 + \left(Be^{-\lambda T}\|u(t,x)\|_\alpha + \rho\right)^2 \\ &\quad \le -\|u(t,x)\|_\alpha^2 + 2B^2e^{-2\lambda T}\|u(t,x)\|_\alpha^2 + 2\rho^2 \\ &\quad \le -\frac{1}{2}\|u(t,x)\|_\alpha^2 + 2\rho^2. \end{aligned} \tag{27}$$

Consequently, by Lemma 2,

$$D^+_{g(u(t,x))} V_1(u(t,x)) = \frac{d^+}{dt}V_1(u(t,x)) = \frac{d}{dt}V_1(u(t,x)). \tag{28}$$

In particular, setting $t = 0$, one obtains that

$$D^+_{g(x)} V_1(x) = \left.\frac{d}{dt}V_1(u(t,x))\right|_{t=0} \le -\frac{1}{2}\|x\|_\alpha^2 + 2\rho^2, \tag{29}$$

which indicates that V_1 satisfies (19).

Now, let us define another Lyapunov function V_2. We firstly take a nonnegative function $\gamma(s)$ as

$$\gamma(s) = \max\left\{s^2 - \rho_0^2, 0\right\}, \quad s \ge 0, \tag{30}$$

where $\rho_0 = (2B+1)\rho$. It is easy to check that $\gamma(s)$ satisfies

$$|\gamma(s) - \gamma(r)| \le (s+r)|s-r|, \quad \forall s, r \ge 0. \tag{31}$$

Now, we let

$$V_2(x) = \sup_{s\ge 0} \gamma(\|u(s,x)\|_\alpha), \quad \forall x \in X^\alpha. \tag{32}$$

We firstly verify the following fact:

$$V_2(x) = \sup_{0\le s\le T} \gamma(\|u(s,x)\|_\alpha), \quad \forall x \in X^\alpha. \tag{33}$$

Indeed, if $\|x\|_\alpha \le 2\rho$, then by (21)

$$\|u(t,x)\|_\alpha \le Be^{-\lambda t}\|x\|_\alpha + \rho \le 2\rho B + \rho = \rho_0. \tag{34}$$

According to the definition of $\gamma(s)$, we know that $\gamma(\|u(s,x)\|_\alpha) = 0$. Therefore, in case of $\|x\|_\alpha \le 2\rho$, one trivially has

$$V_2(x) = 0 = \sup_{0\le s\le T} \gamma(\|u(s,x)\|_\alpha). \quad (35)$$

If $\|x\|_\alpha \ge 2\rho$, then by the choice of T we find that

$$\|u(s,x)\|_\alpha \le Be^{-\lambda T}\|x\|_\alpha + \rho = \frac{\|x\|_\alpha}{2} + \rho < \|x\|_\alpha, \quad \forall s \ge T. \quad (36)$$

Since $u(0,x) = x$ and $\gamma(s)$ is nondecreasing in s, one can deduce the correctness of (33).

Next, we will check that V_2 also satisfies (20). By (33), (31), (24), and (21)

$$\begin{aligned} V_2(x) &= \sup_{0\le s\le T} \gamma(\|u(s,x)\|_\alpha) \\ &= \sup_{0\le s\le T} [(\gamma(\|u(s,x)\|_\alpha) - \gamma(\|u(s,y)\|_\alpha)) + \gamma(\|u(s,y)\|_\alpha] \\ &\le \sup_{0\le s\le T} [\gamma(\|u(s,x)\|_\alpha) - \gamma(\|u(s,y)\|_\alpha)] + V_2(y) \\ &\le \sup_{0\le s\le T} (\|u(s,x)\|_\alpha + \|u(s,y)\|_\alpha) \\ &\quad \times |\|u(s,x)\|_\alpha - \|u(s,y)\|_\alpha| + V_2(y) \\ &\le C(T)\|x-y\|_\alpha \\ &\quad \times \sup_{0\le s\le T} [Be^{-\lambda s}(\|x\|_\alpha + \|y\|_\alpha) + 2\rho] + V_2(y) \\ &\le C(T)[B(\|x\|_\alpha + \|y\|_\alpha) + 2\rho]\|x-y\|_\alpha + V_2(y) \\ &\le L_2(\|x\|_\alpha + \|y\|_\alpha + 1)\|x-y\|_\alpha + V_2(y). \end{aligned} \quad (37)$$

Next, we will check that for arbitrary $x \in X^\alpha$, $V_2(x)$ is bounded by

$$\|x\|_\alpha^2 - \rho_0^2 \le V_2(x) \le 2B^2\|x\|_\alpha^2 + 2\rho^2. \quad (38)$$

Firstly, according to the definition of $\gamma(s)$, it is obvious to see that

$$s^2 - \rho_0^2 \le \gamma(s) \le s^2, \quad \forall s \ge 0. \quad (39)$$

So, it follows that

$$\begin{aligned} V_2(x) &= \sup_{s\ge 0} \gamma(\|u(t,x)\|_\alpha) \ge \gamma(\|u(0,x)\|_\alpha) \\ &= \gamma(\|x\|_\alpha) \ge \|x\|_\alpha^2 - \rho_0^2. \end{aligned} \quad (40)$$

Recalling (21), we infer

$$\|u(s,x)\|_\alpha \le Be^{-\lambda s}\|x\|_\alpha + \rho \le B\|x\|_\alpha, \quad \forall s \ge 0. \quad (41)$$

Frequently, by the definition of V_2 and the monotonicity property of $\gamma(s)$, we get

$$\begin{aligned} V_2(x) &\le \gamma(B\|x\|_\alpha + \rho) \\ &\le (B\|x\|_\alpha + \rho)^2 \\ &\le 2B^2\|x\|_\alpha^2 + 2\rho^2. \end{aligned} \quad (42)$$

So, we verify the correctness of (38).

Lastly we need to check that $V_2(u(t,x))$ is nonincreasing in t. Note that

$$\begin{aligned} V_2(u(t,x)) &= \sup_{s\ge 0} \gamma(\|u(s,u(t,x))\|_\alpha) \\ &= \sup_{s\ge 0} \gamma(\|u(s+t,x)\|_\alpha) \\ &= \sup_{s\ge t} \gamma(\|u(s,x)\|_\alpha). \end{aligned} \quad (43)$$

It is easy to see the validity of our checking.

Now, let

$$V(x) = V_1(x) + V_2(x). \quad (44)$$

Considering (23), (38), (25), (37), and (31), we can get the validity of (18), (19), and (20). The proof is complete. □

In order to prove the second result, we need to verify the following lemma.

Lemma 5. *Suppose that f is globally Lipschitz with Lipschitz constant $L > 0$, that is,*

$$\|f(x_1,x_2) - f(y_1,y_2)\| \le L(\|x_1-y_1\|_\alpha + \|x_2-y_2\|_\alpha), \quad (45)$$

for any $x_i, y_i \in X^\alpha$, and that the system (2) is exponentially dissipative. Then, there exist $B_0 > 1$ and $\tau_0 > 0$ such that when $\tau \le \tau_0$, any solution of (1) with initial value $u_0 \in \mathscr{C} = C([-\tau,0],X^\alpha)$ satisfies

$$\|u(t,u_0)\|_\alpha < B_0(|\|u_0\||_\alpha + 1), \quad \forall t \ge 0, u_0 \in \mathscr{C}. \quad (46)$$

Proof. According to (45), it is easy to see that there is an $M > 0$ such that

$$\|f(x,y)\| \le M(\|x\|_\alpha + \|y\|_\alpha + 1), \quad \forall x,y \in X^\alpha. \quad (47)$$

Firstly, we prove that for arbitrary $t \in [0,\tau]$, there exists B_1 such that any solution of (1) with initial value $u_0 \in \mathscr{C} = C([-\tau,0],X^\alpha)$ satisfies

$$\|u(t,u_0)\|_\alpha < B_1(|\|u_0\||_\alpha + 1), \quad (48)$$

because $u(t,u_0)$ can be expressed as follows:

$$\begin{aligned} u(t) &= e^{-tA}u_0(0) + \int_0^t e^{-(t-s)A} f(u(s),u(s-\tau))\,ds \\ &= e^{-tA}u_0(0) + \int_0^t e^{-(t-s)A} f(u(s),u_0(s-\tau))\,ds. \end{aligned} \quad (49)$$

By (47) and (15), we can obtain

$$\begin{aligned} \|u(t)\|_\alpha = {} & \|A^\alpha u(t)\| \le \left\|e^{-tA}A^\alpha u_0(0)\right\| + \int_0^t \left\|A^\alpha e^{-(t-s)A}\right\| \\ & \cdot M\left(\|u(s)\|_\alpha + \|u_0(s-\tau)\|_\alpha + 1\right) ds \\ \le {} & C_1\left\|\|u_0\|\right\|_\alpha + MC_\alpha \int_0^t (t-s)^{-\alpha}\|u(s)\|_\alpha ds \\ & + MC_\alpha\left\|\|u_0\|\right\|_\alpha \int_0^t (t-s)^{-\alpha} ds \\ & + M\left\|\|u_0\|\right\|_\alpha \int_0^t (t-s)^{-\alpha} ds \\ \le {} & C_2\left\|\|u_0\|\right\|_\alpha + MC_\alpha \int_0^t (t-s)^{-\alpha}\|u_0(s)\|_\alpha ds. \end{aligned} \tag{50}$$

According to the Gronwall inequality, one easily sees that

$$\|u(t,u_0)\|_\alpha \le C_3\left\|\|u_0\|\right\|_\alpha < B_1\left(\left\|\|u_0\|\right\|_\alpha + 1\right). \tag{51}$$

Now, we choose and fix B_2 and τ_0 with

$$B_2 > \sqrt{\frac{3(bd+ad+cd+b\sigma+c)}{d}}, \qquad \tau_0^\delta < \frac{d}{96bM_0L_VL}. \tag{52}$$

Let $v(t) = u(t,u_0)$. We will show that

$$\|v(t)\|_\alpha < B_2\left(\|v_\tau\|_\alpha + 1\right), \quad \forall t \ge \tau, \tag{53}$$

where $\|v_\tau\|_\alpha = \max_{[0,\tau]}\|v(s)\|_\alpha$.

We argue by contradiction and suppose that for some solution $v(t) = u(t,u_0)$ of (1), it holds that

$$\|v(t_1)\|_\alpha \ge B_2\left(\|v_\tau\|_\alpha + 1\right) \tag{54}$$

for some $t_1 > \tau$. Observing that $B_2 > 1$, we deduce that there exists a $t_0 > \tau$ such that

$$\|v(t)\|_\alpha < B_2\left(\|v_\tau\|_\alpha + 1\right), \quad \text{for } t \in [\tau, t_0), \tag{55}$$

$$\|v(t_0)\|_\alpha = B_2\left(\|v_\tau\|_\alpha + 1\right). \tag{56}$$

Thanks to Theorem 4, there is a Lyapunov function V satisfying (18)–(20). By Lemma 2, we find that

$$\begin{aligned} & \frac{d^+}{dt}V(v(t)) \\ & = \limsup_{h\to 0+}\frac{1}{h}\left[V\left(v(t) + hf(v(t), v(t-\tau)) - Av(t)\right) - V(v(t))\right] \\ & \le \limsup_{h\to 0+}\frac{1}{h}\left[V(v(t) + hF(v(t)) - Av(t)) - V(v(t))\right] \\ & \quad + \limsup_{h\to 0+}\frac{1}{h}\left[V\left(v(t) + hf(v(t), v(t-\tau))\right) - V\left(v(t) + hF(v(t))\right)\right] \\ & = D^+_{g(v(t))}V(v(t)) \\ & \quad + \limsup_{h\to 0+}\frac{1}{h}\left[V\left(v(t) + hf(v(t), v(t-\tau))\right) - V\left(v(t) + hF(v(t))\right)\right]. \end{aligned} \tag{57}$$

By (47) and (55), we see that for $t \in [\tau, t_0]$

$$\begin{aligned} & \|F(v(t))\|, \|f(v(t), v(t-\tau))\| \\ & \quad \le M\left[2B_2\left(\|v_\tau\|_\alpha + 1\right) + 1\right] = R. \end{aligned} \tag{58}$$

Denote by L_h the Lipschitz constant of V on $\overline{\mathscr{B}}(v(t), hR)$. Then, we infer from (20) that

$$\limsup_{h\to 0+} L_h \le L_V\left(2\|v(t)\|_\alpha + 1\right). \tag{59}$$

At the same time, from Lemma 3.3.2 of [18], we can show that $v(t)$ is locally Hölder. That is to say,

$$\|v(t-\tau) - v(t)\|_\alpha \le M_0\tau^\delta, \quad \delta \in (0, 1-\alpha). \tag{60}$$

Therefore, on $[\tau, t_0]$ we have that

$$\begin{aligned} & \frac{d^+}{dt}V(v(t)) \\ & \le D^+_{g(v(t))}V(v(t)) \\ & \quad + \limsup_{h\to 0+} L_h \left\|f(v(t), v(t-\tau)) - f(v(t), v(t))\right\| \\ & \le D^+_{g(v(t))}V(v(t)) + \limsup_{h\to 0+} L_h L\|v(t-\tau) - v(t)\|_\alpha \\ & \le -d\|v(t)\|_\alpha^2 + \sigma + M_0\tau^\delta LL_V\left(2\|v(t)\|_\alpha + 1\right) \\ & \le -d\|v(t)\|_\alpha^2 + \sigma + M_0\tau^\delta LL_V\left[2B_2\left(\|v_\tau\|_\alpha + 1\right) + 1\right], \end{aligned} \tag{61}$$

because

$$\begin{aligned} & \left[2B_2\left(\|v_\tau\|_\alpha + 1\right) + 1\right] \\ & \quad \le \left[2B_2(\|v_\tau\|_\alpha + 1) + 1\right]^2 \\ & \quad \le \left[2B_2\left(\|v_\tau\|_\alpha + 1\right) + 2B_2\right]^2 \\ & \quad = 4B_2^2(\|v_\tau\|_\alpha + 2)^2 \\ & \quad \le 16B_2^2(\|v_\tau\|_\alpha + 1)^2. \end{aligned} \tag{62}$$

According to (18) and (61), we find that

$$\frac{d^+}{dt}V(v(t)) \le -\frac{d}{b}V(v(t)) + \frac{dc}{b} + \sigma + 32M_0\tau^\delta LL_V B_2^2(\|v_\tau\|_\alpha + 1)^2, \quad \forall t \in [\tau, t_0]. \tag{63}$$

If we denote that

$$\lambda_1 = \frac{d}{b}, \qquad \sigma_1 = \frac{cd}{b}, \qquad C(\|v_\tau\|_\alpha) = 32M_0\tau^\delta LL_V B_2^2(\|v_\tau\|_\alpha + 1)^2, \tag{64}$$

then by the Gronwall inequality

$$\begin{aligned} V(v(t)) &\le V(v(\tau))e^{-\lambda_1(t-\tau)} + \frac{1}{\lambda_1}[\sigma_1 + C(\|v_\tau\|_\alpha)](1 - e^{-\lambda_1(t-\tau)}) \\ &\le V(v(\tau)) + \frac{1}{\lambda_1}[\sigma_1 + C(\|v_\tau\|_\alpha)]. \end{aligned} \tag{65}$$

Utilizing (18) again, we conclude that for $t \in [\tau, t_0]$,

$$\|v(t)\|_\alpha^2 \le b\|v_\tau\|_\alpha^2 + \left(a + c + \frac{\sigma_1}{\lambda_1}\right) + \frac{32}{\lambda_1}M_0\tau^\delta LL_V B_2^2(\|v_\tau\|_\alpha + 1)^2. \tag{66}$$

By the choice of B_1 and τ_0, one easily checks that

$$b < \frac{1}{3}B_2^2, \qquad a + c + \frac{\sigma_1}{\lambda_1} < \frac{1}{3}B_2^2, \qquad \frac{32}{\lambda_1}M_0\tau^\delta LL_V < \frac{1}{3}. \tag{67}$$

Hence, in particular, for $t = t_0$, we find that

$$\|v(t_0)\|_\alpha^2 < B_2^2(\|v_\tau\|_\alpha + 1)^2. \tag{68}$$

This contradicts (56).

Now, the conclusion of the theorem follows immediately from (48) and (53). And the proof is complete. □

Theorem 6. *Assume that f is globally Lipschitz and the system* (2) *without delay is exponentially dissipative. Then, the system* (1) *with time delay is also exponentially dissipative.*

Proof. Let $u_0 \in \mathscr{C}$ and $v(t) = u(t, u_0)$ be the solution of (1). According to Lemma 5, $\|v(t)\|_\alpha \le B_1(\|v_\tau\|_\alpha + 1)$ for all $t \ge \tau$, repeating the same argument as in (65), one easily sees that the first inequality in (65) remains valid for all $t \ge \tau$. Furthermore, making use of (18), we deduce that

$$\begin{aligned} \|v(t)\|_\alpha^2 - a &\le V(v(t)) \\ &\le V(v(\tau))e^{-\lambda_1(t-\tau)} + \frac{1}{\lambda_1}[\sigma_1 + C(\|v_\tau\|_\alpha)] \\ &\le (b\|v_\tau\|_\alpha^2 + c)e^{-\lambda_1(t-\tau)} + \frac{1}{\lambda_1}[\sigma_1 + C(\|v_\tau\|_\alpha)]. \end{aligned} \tag{69}$$

Frequently,

$$\|v(t)\|_\alpha^2 \le (b\|v_\tau\|_\alpha^2 + c)e^{-\lambda_1(t-\tau)} + a + \frac{1}{\lambda_1}[\sigma_1 + C(\|v_\tau\|_\alpha)], \quad \forall t \ge \tau. \tag{70}$$

By (46), it can be easily seen that

$$\begin{aligned} \|v(t)\|_\alpha^2 &\le C_1(|\|u_0\||_\alpha + 1)^2 e^{-\lambda_1}(t - \tau) + \tau^\delta C_2(|\|u_0\||_\alpha + 1)^2 + C_3 \\ &\le C_1 e^{\lambda_1\tau_0}(|\|u_0\||_\alpha + 1)^2 e^{-\lambda_1 t} + \tau^\delta C_2(|\|u_0\||_\alpha + 1)^2 + C_3, \quad \forall t \ge \tau, \end{aligned} \tag{71}$$

where C_i $(i = 1, 2, 3)$ are appropriate positive constants independent of τ and u_0. For $t \in [0, \tau]$, we have by (46) that

$$\|v(t)\|_\alpha^2 \le B_0^2 e^{\lambda_1\tau_0}(|\|u_0\||_\alpha + 1)^2 e^{-\lambda_1 t}. \tag{72}$$

Therefore, taking $C_1' = (C_1 + B_0^2)e^{\lambda_1\tau_0}$, one concludes that

$$\|v(t)\|_\alpha^2 \le C_1'(|\|u_0\||_\alpha + 1)^2 e^{-\lambda_1 t} + \tau^\delta C_2(|\|u_0\||_\alpha + 1)^2 + C_3, \quad \forall t \ge 0. \tag{73}$$

Now, we fix a $T > 0$ and $\tau \le \tau_0$ independent of u_0 such that

$$C_1' e^{-\lambda_1 T} < \frac{1}{8}, \qquad \tau^\delta C_2 < \frac{1}{8}. \tag{74}$$

So,

$$\begin{aligned} \|v(t)\|_\alpha^2 &\le \frac{1}{4}(|\|u_0\||_\alpha + 1)^2 + C_3 \\ &\le \left(\frac{|\|u_0\||_\alpha}{2} + \frac{1}{2} + \sqrt{C_3}\right)^2, \quad \forall t \ge T. \end{aligned} \tag{75}$$

Setting $C_0 = 1/2 + \sqrt{C_3}$, we find that

$$\|v(t)\|_\alpha \le \frac{1}{2}|\|u_0\||_\alpha + C_0, \quad \forall t \ge T,\ u_0 \in \mathscr{C}. \tag{76}$$

Next, we will use mathematical induction to prove that

$$\|v(t)\|_\alpha \le \frac{1}{2^k}|\|u_0\||_\alpha + C_0\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}}\right), \quad \forall t \ge T_k,\ u_0 \in \mathscr{C}, \tag{77}$$

where $T_k = k(T + 1)$.

Indeed, let $u_0 \in \mathscr{C}$. If $k = 1$, then (77) clearly holds true. Suppose that (77) holds for $k = m$; that is,

$$\|v(t)\|_\alpha \le \frac{1}{2^m}\||u_0\||_\alpha + C_0\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-1}}\right), \quad \forall t \ge T_m, u_0 \in \mathscr{C}. \tag{78}$$

Then, in particular,

$$\|v(t)\|_\alpha \le \frac{1}{2^m}\||u_0\||_\alpha + C_0\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-1}}\right), \quad \forall t \in [T_m, T_m + \tau]. \tag{79}$$

From (76), we know that

$$\|v(t + T_m + \tau)\|_\alpha \le \frac{1}{2}\max_{t\in[-\tau,0]}\|v(t + T_m + \tau)\|_\alpha + C_0, \quad \forall t \ge T. \tag{80}$$

If we consider $t + T_m + \tau$ as t, then the above can be rewritten as

$$\|v(t)\|_\alpha \le \frac{1}{2}\max_{t\in[T_m,T_m+\tau]}\|v(t)\|_\alpha + C_0, \quad \forall t \ge T_m + \tau + T. \tag{81}$$

From (79),

$$\begin{aligned}\|v(t)\|_\alpha &\le \frac{1}{2}\left[\frac{1}{2^m}\||u_0\||_\alpha + C_0\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^m - 1}\right)\right] + \rho_0 \\ &= \frac{1}{2^{m+1}}\||u_0\||_\alpha + C_0\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^m}\right), \\ &\qquad \forall t \ge T_m + \tau + T.\end{aligned} \tag{82}$$

Choosing $\tau \le 1$, we conclude that

$$\|v(t)\|_\alpha \le \frac{1}{2^{m+1}}\||u_0\||_\alpha + C_0\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^m}\right), \quad \forall t \ge T_m + 1 + T = T_{m+1}. \tag{83}$$

Thus, we see that (77) holds for $m + 1$.

By (77), we know that

$$\|v(t)\|_\alpha \le \frac{1}{2^k}\||u_0\||_\alpha + 2C_0, \quad \forall t \in [T_k, T_{k+1}], u_0 \in \mathscr{C}, k = 1, 2, \ldots. \tag{84}$$

Furthermore, by Lemma 5, we see that

$$\|v(t)\|_\alpha \le \frac{B_0}{2^k}\||u_0\||_\alpha + \rho_0, \quad \forall t \in [T_k, T_{k+1}], u_0 \in \mathscr{C}, k = 0, 1, 2, \ldots, \tag{85}$$

where $\rho_0 = 2C_0 + 1$.

We observe that $t/(T + 1) \in [k, k + 1]$ when $t \in [T_k, T_{k+1}]$, so we infer from (85) that

$$\begin{aligned}\|v(t)\|_\alpha &\le \frac{B_0}{2^k}\||u_0\||_\alpha + \rho_0 \\ &= \frac{2B_0}{2^{k+1}}\||u_0\||_\alpha + \rho_0 \\ &\le 2B_0 2^{-\alpha t}\||u_0\||_\alpha + \rho_0, \\ &\forall t \in [T_k, T_{k+1}], u_0 \in \mathscr{C}, k = 0, 1, 2, \ldots,\end{aligned} \tag{86}$$

where $\alpha = 1/(T + 1)$. So, we easily see that

$$\|v(t)\|_\alpha \le 2B_0 2^{-\alpha t}\||u_0\||_\alpha + \rho_0, \quad \forall t \ge 0, u_0 \in \mathscr{C}. \tag{87}$$

This completes the proof of the theorem. □

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