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Research Article

Higher-Order Convergent Iterative Method for Computing the Generalized Inverse over Banach Spaces

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A higher-order convergent iterative method is provided for calculating the generalized inverse over Banach spaces. We also use this iterative method for computing the generalized Drazin inverse a^d in Banach algebra. Moreover, we estimate the error bounds of the iterative methods for approximating $A^{(2)}_{T,S}$ or a^d.

1. Introduction

It is well known that the outer generalized inverse has been widely used in various fields, for instance, in statistics, control theory, power systems, nonlinear equations, optimization and numerical analysis, and so on (see [1–15]). Recently, in [16], the authors discussed the iteration (1) for computing $A^{(2)}_{T,S}$ of a given matrix.

Throughout this paper, let X and Y be arbitrary Banach spaces. Then, the symbol $\mathscr{B}(\mathscr{X},\mathscr{Y})$ denotes the set of all bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$, in particular, $\mathscr{B}(\mathscr{X}) := \mathscr{B}(\mathscr{X},\mathscr{X})$. For any $A \in \mathscr{B}(\mathscr{X},\mathscr{Y})$, we denote its range, null space, and norm by $\mathscr{R}(A)$, $\mathscr{N}(A)$, and $\|A\|$, respectively. Further, we say that A is regular if there exists an $X \in \mathscr{B}(\mathscr{Y},\mathscr{X})$ such that $AXA = A$ and that A has a $\{2\}$ (or outer) inverse if there exists an $X \in \mathscr{B}(\mathscr{Y},\mathscr{X})$ such that $XAX = X$. If $A \in \mathscr{B}(\mathscr{X})$, then we denote its spectrum and spectral radius by $\sigma(A)$ and $\rho(A)$, respectively. Let the symbol $L \subset \mathscr{X}$ denote that L is a subspace of $\mathscr{X}$. If $A \in \mathscr{B}(\mathscr{X},\mathscr{Y})$ and $L \subset \mathscr{X}$, then the restriction $A|_L$ of A on L is defined by $x \mapsto Ax$, $x \in L$. Let $L, M \subset \mathscr{X}$ with $L \oplus M = \mathscr{X}$. Then, the symbol $P_{L,M}$ stands for an operator that is called a projection from $\mathscr{X}$ onto L if it is a bounded linear map from $\mathscr{X}$ onto L and $P^2_{L,M} = P_{L,M}$. It is well known that a closed subspace L of a Banach space $\mathscr{X}$ is complemented in $\mathscr{X}$ if and only if there exists a projection from $\mathscr{X}$ onto L.

Let $A \in \mathscr{B}(\mathscr{X},\mathscr{Y})$ be close; there exists a unique operator $X \in \mathscr{B}(\mathscr{Y},\mathscr{X})$ such that

$$\begin{aligned} &(1)\ \ AXA = A \qquad &(2)\ \ XAX = X \\ &(3)\ \ (AX)^* = AX \qquad &(4)\ \ (XA)^* = XA. \end{aligned} \tag{1}$$

Then, X is called the Moore-Penrose inverse of A, denoted by $X = A^\dagger$. It is well known that A is regular $\Leftrightarrow$ $R(A)$ is closed $\Leftrightarrow$ $A^\dagger$ exists.

Throughout this paper, let $\mathscr{A}$ be a complex Banach algebra with the unit 1. The symbols $ann^l(a)$ and $ann^r(a)$, respectively, stand for the left and right annihilators of a in $\mathscr{A}$. Let $p \in \mathscr{A}$ be idempotent. Then, $p\mathscr{A}p = \{pap : a \in \mathscr{A}\}$ is a subalgebra of $\mathscr{A}$ with unit p. Thus, for $a \in \mathscr{A}$, if there exists an element $b \in p\mathscr{A}p$ such that $ab = ba = p$, then we say that a is invertible in $p\mathscr{A}p$, and b is denoted by $a|^{-1}_{p\mathscr{A}p}$. Recall that an element $b \in \mathscr{A}$ is the generalized Drazin inverse of a (or Koliha-Drazin inverse of a), if the following hold:

$$\begin{gathered} bab = b, \qquad ba = ab, \\ a(1 - ab) \ \text{ is quasinilpotent.} \end{gathered} \tag{2}$$

If the generalized Drazin inverse of a exists, then it is denoted by a^d (see [15] for more details). In particular, if $b = a^d$ and

$a(1-ab)=0$, then b is called the group inverse of a and is denoted by a_g.

In [17], W. G. Li and Z. Li defined the iterative formula

$$X_{k+1} = X_k\left[kI - \frac{k(k-1)}{2}AX_k + \cdots + (-1)^{k-1}(AX_k)^{k-1}\right], \quad k = 2, 3, \ldots. \tag{3}$$

In [18], Chen and Wang extended the iterative method (3) proposed by W. G. Li and Z. Li to compute the Moore-Penrose inverse of a matrix. In [19], Liu et al provided the higher-order convergent iterative method (3) in order to calculate the generalized inverse $A^{(2)}_{T,S}$ of a given matrix. In this paper, we will extend the iterative method proposed by W. G. Li and Z. Li in [17] to compute the {2}-inverse, generalized inverse $A^{(2)}_{T,S}$ over Banach space and also consider the iterative scheme for computing the generalized Drazin inverse a^d in Banach algebra.

The paper is organized as follows. Some lemmas will be presented in the remainder of this section. In Section 2, we consider iterative scheme of [19] to compute the generalized inverses $A^{(2)}_{T,S}$ in Banach space. In Section 3, we present iterative formulas for computing the generalized Drazin inverse a^d of Banach algebra element a.

The following lemmas are needed in what follows.

Lemma 1 (see [14, Chapter 1]). *Let $a \in \mathscr{A}$. Then*

(i) *$\sigma(a)$ is a nonempty closed subset of $\mathbb{C}$.*

(ii) *(Spectral mapping theorem for polynomials) if f is a polynomial, then*

$$\sigma(f(a)) = f(\sigma(a)). \tag{4}$$

(iii) *$\lim_{n\to 0} a^n = 0$ if and only if $\rho(a) < 1$.*

Lemma 2 (see [15, Section 4]). *Let X and Y be Banach spaces, and let $A \in \mathscr{B}(\mathscr{X},\mathscr{Y})$, T and S, respectively, be closed subspaces of X and Y. Then, the following statements are equivalent.*

(i) *A has a {2}-inverse $B \in B(Y,X)$ such that $\mathscr{R}(B) = T$ and $\mathscr{N}(B) = S$.*

(ii) *T is a complemented subspace of X, $A(T)$ is closed, $A|_T : T \to A(T)$ is invertible, and $A(T) \oplus S = Y$.*

In the case when (i) or (ii) holds, B is unique and is denoted by $A^{(2)}_{T,S}$.

Lemma 3. *Suppose that the conditions of Lemma 2 are satisfied. Then, $AA^{(2)}_{T,S} = P_{A(T),S}$ and $A^{(2)}_{T,S}A = P_{T,T_1}$ where $T_1 = \mathscr{N}(A^{(2)}_{T,S}A)$. Moreover, for any $G \in \mathscr{B}(\mathscr{Y},\mathscr{X})$, $P_{T,T_1}G = G \Leftrightarrow \mathscr{R}(G) \subset T$; $GP_{A(T),S} = G \Leftrightarrow \mathscr{N}(G) \supset S$.*

2. Higher-Order Convergent Iterative Method for Computing the Generalized Inverse over Banach Spaces

In this section, we will consider higher-order convergent iterative method for computing the generalized inverse $A^{(2)}_{T,S}$ over Banach spaces. First, we deduce convergent conditions and error bounds of our iterative methods.

Theorem 4. *Let $A \in \mathscr{B}(\mathscr{X},\mathscr{Y})$, $Y \in \mathscr{B}(\mathscr{Y},\mathscr{X})$, and let $T \subset \mathscr{X}$ and $S \subset \mathscr{Y}$ both be complemented subspaces, respectively, with $\mathscr{R}(Y) = T$, $\mathscr{N}(Y) = S$. Define the sequence $\{X_k\}$ in $\mathscr{B}(\mathscr{Y},\mathscr{X})$ in the following way:*

$$X_0 = \alpha Y,$$
$$X_k = \left[C_t^1 I - C_t^2 X_{k-1}A + \cdots + (-1)^{t-1}C_t^t(X_{k-1}A)^{t-1}\right]X_{k-1}; \tag{5}$$

it converges to X_∞ and $X_\infty \in A\{2\}$ with $R(X_\infty) = T$ if and only if $\rho(\alpha YA - P) < 1$ for some scalar $\alpha \in \mathbb{C}\setminus\{0\}$, where $t \ge 2$ is an arbitrary positive integer, $X_\infty = \lim X_k$, and P is projection from $\mathscr{X}$ onto T. Moreover,

(i) *if $\mathscr{N}(X_\infty) = S$, then $A^{(2)}_{T,S}$ exists if and only if $\rho(\alpha YA - P) < 1$ for $\alpha \in \mathbb{C}\setminus\{0\}$;*

(ii) *if $\mathscr{N}(X_\infty) = S$, then $A^{(2)}_{T,S}$ exists.*

In particular, if $A^{(2)}_{T,S}$ exists, $\lim X_k = A^{(2)}_{T,S}$ and $q = \|\alpha YA - P\|$, then

$$\frac{\left\|A^{(2)}_{T,S} - X_k\right\|}{\left\|A^{(2)}_{T,S}\right\|} \le q^{t^k}, \quad k \ge 0. \tag{6}$$

Proof. From (5), we obtain

$$\left[C_t^1 I - C_t^2 X_{k-1}A + \cdots + (-1)^{t-1}C_t^t(X_{k-1}A)^{t-1}\right]X_{k-1} = X_{k-1}\left[C_t^1 I - C_t^2 AX_{k-1} + \cdots + (-1)^{t-1}C_t^t(AX_{k-1})^{t-1}\right]. \tag{7}$$

Note that $\mathscr{R}(X_k) \subset \mathscr{R}(X_{k-1})$, $k \ge 1$ from (7). Similarly, it is easy to prove that $N(X_k) \supseteq N(X_{k-1})$, $k \ge 1$.

Since $\mathscr{R}(X_0) = \mathscr{R}(\alpha Y) = T$ and $\mathscr{N}(X_0) = \mathscr{N}(\alpha Y) = S$, then

$$\mathscr{R}(X_k) \subset T, \qquad \mathscr{N}(X_k) \supset S, \tag{8}$$

for $k \ge 0$.

From (5), we have

$$X_kA - I = (-1)^{t+1}(X_{k-1}A - I)^t = (-1)^{t+1}(X_0A - I)^{t^k}. \tag{9}$$

By (8), we get $PX_k = X_k$. Premultiplying (9) by P, then (9) yields

$$X_kA - P = (-1)^{t+1}(X_0A - P)^{t^k}. \tag{10}$$

Next, we will investigate the necessary and sufficient condition for the convergent property of the iterative scheme (5). Assume that $\lim X_k$ exists; denote by $X_\infty \in A\{2\}$ and $\mathscr{R}(X_\infty A) = T$. Then, $\mathscr{R}(X_\infty) = \mathscr{R}(X_\infty A X_\infty) \subset \mathscr{R}(X_\infty A) \subset \mathscr{R}(X_\infty)$. Thus, $\mathscr{R}(X_\infty A) = T$ and $\mathscr{X} = T \oplus N(X_\infty A)$; we obtain $X_\infty A = P_{T,N(X_\infty A)}$, a projection from $\mathscr{X}$ onto T, and $X_\infty A X_k = X_k$ by (8).

Since $P_{T,N(X_\infty A)} X_0 = X_0$, and $X_k A - P_{T,N(X_\infty A)} = (-1)^{t+1}(X_0 A - P_{T,N(X_\infty A)})^{t^k}$ by (10). Thus,

$$\begin{aligned} 0 &= \lim X_k A - X_\infty A = \lim X_k A - P_{T,N(X_\infty A)} \\ &= \lim (-1)^{t-1}\left(X_0 A - P_{T,N(X_\infty A)}\right)^{t^k}, \end{aligned} \tag{11}$$

and then $\rho(\alpha YA - P_{T,N(X_\infty A)}) < 1$.

Conversely, suppose that $\rho(\alpha YA - P) < 1$ for some scalar $\alpha \in \mathbb{C} \setminus \{0\}$, where P denotes a projection from $\mathscr{X}$ to T and X is complement. Then, $\lim X_k A = P$ by (10), and therefore $\lim_{k\to\infty} X_k = (A|_T)^{-1}$ and $T = \mathscr{R}(P) \subset \mathscr{R}(\lim X_k)$.

By (8), $R(\lim X_k) \subset T$ because T is close, and then $\mathscr{R}(X_\infty) = T$. Hence, we obtain $\lim X_k A \lim X_k = \lim X_k$. Thus, $\lim X_k \in A\{2\}$. It is easy to know that if $N(\lim X_k) = S$, then $\lim X_k = A^{(2)}_{T,S}$. Thus, $A^{(2)}_{T,S}$ exists.

Assume that $A^{(2)}_{T,S}$ exists. By (8), $N(\lim X_k) \supset S$ because S is closed complement. If $y \in N(\lim X_k) \cup AT$, then $y = Az$ for some $z \in T$. Thus, $0 = \lim X_k y = \lim X_k Az = Pz = z$. Thus, $y = 0$. Therefore, $N(\lim X_k) \cup AT = \{0\}$ and then $N(\lim X_k) = S$ by Lemma 2. Consequently, $\lim X_k = A^{(2)}_{T,S}$.

Since $N(X_k) = S$, $X_k A A^{(2)}_{T,S} = X_k$. Thus, postmultiplying (10) by $A^{(2)}_{T,S}$ yields

$$X_k - A^{(2)}_{T,S} = (-1)^{t+1}(\alpha YA - P)^{t^k} A^{(2)}_{T,S}. \tag{12}$$

Since $A^{(2)}_{T,S} = PA^{(2)}_{T,S}$, we have

$$\begin{aligned} \left\|A^{(2)}_{T,S} - X_k\right\| &= \left\|(\alpha YA - P)^{t^k} A^{(2)}_{T,S}\right\| \\ &\le \|\alpha YA - P\|^{t^k} \left\|A^{(2)}_{T,S}\right\| \\ &= q^{t^k} \left\|A^{(2)}_{T,S}\right\|. \end{aligned} \tag{13}$$

Hence, we get (6). □

Similarly, we have the dual result as below.

Theorem 5. *Let $A \in \mathscr{B}(\mathscr{X},\mathscr{Y})$, $Y \in \mathscr{B}(\mathscr{Y},\mathscr{X})$, and let $T \subset \mathscr{X}$ and $S \subset \mathscr{Y}$ both be closed, respectively, with $\mathscr{R}(Y) = T$, $\mathscr{N}(Y) = S$. Define the sequence $\{X_k\} \in \mathscr{B}(\mathscr{Y},\mathscr{X})$ such that*

$$X_0 = \alpha Y,$$

$$X_k = X_{k-1}\left[C_t^1 I - C_t^2 A X_{k-1} + \cdots + (-1)^{t-1} C_t^t (A X_{k-1})^{t-1}\right]; \tag{14}$$

it converges to X_∞ and $X_\infty \in A\{2\}$ with $\mathscr{N}(X_\infty) = S$ if and only if $\rho(\alpha AY - Q) < 1$ for some scalar $\alpha \in \mathbb{C} \setminus \{0\}$, where $t \ge 2$ is an arbitrary positive integer, $X_\infty = \lim X_k$, and Q is a projection from $\mathscr{Y}$ onto S. Moreover,

(i) *if $\mathscr{R}(\Psi) = T$, then $A^{(2)}_{T,S}$ exists if and only if $\rho(\alpha AY - Q) < 1$ for $\alpha \in \mathbb{C} \setminus \{0\}$;*

(ii) *if $\mathscr{R}(\Psi) = T$, then $A^{(2)}_{T,S}$ exists.*

In particular, if $A^{(2)}_{T,S}$ exists, $X_\infty = A^{(2)}_{T,S}$ and $q = \|\alpha AY - Q\|$, then $\|A^{(2)}_{T,S} - X_k\| / \|A^{(2)}_{T,S}\| \le q^{t^k}$, $k \ge 0 \cdots$.

Remark 6. Now, we consider how to choose a suitable scalar $\alpha \in \mathbb{C} \setminus \{0\}$ for the iterative scheme (5) such that it converges more faster to $A^{(2)}_{T,S}$.

Since $\mathscr{R}(YA) \subset T$ and for any $\alpha \in \mathbb{C} \setminus \{0\}$, $\rho(P - \alpha YA) = \rho(P - \alpha (YA)|_T) = \max|1 - \alpha\mu|(\mu \in \sigma(YA)|_T)$. Therefore, $\rho(P - \alpha YA) < 1$ if and only if $0 \notin \sigma((YA)|_T)$ and $\max_{\mu \in (YA)\setminus\{0\}}|1 - \alpha\mu| < 1$. Thus, there exists $\lambda_0 \in (YA) \setminus \{0\}$ with $|1 - \alpha\lambda_0| = \rho(P - \alpha YA)$.

Let $\lambda_0 = |\lambda_0|(\cos\theta + i\sin\theta)$ and $\alpha = |\alpha|(\cos\varphi + i\sin\varphi)$, where $\theta = \arg(\lambda_0)$, $\varphi = \arg(\alpha)$. Then, $\rho(P - \alpha YA) = [|\alpha\lambda_0|^2 + 1 - 2|\alpha\lambda_0|\cos(\theta + \varphi)]^{1/2}$. Thus, $\rho(P - \alpha YA) < 1$ if and only if $0 < |\alpha\lambda_0| < 2\cos(\theta + \varphi)$ and $0 \notin \sigma((YA)|_T)$.

Hence, by $0 \notin \sigma((YA)|_T)$ and α satisfing $0 < |\alpha| < 2\cos(\theta + \varphi)/\rho(YA)$, we have $\rho(P - \alpha YA) < 1$. In practice, once such a λ_0 is determined, α is taken to satisfy $\arg(\alpha) = -\arg(\lambda_0)$ and $0 < |\alpha| < 2/\rho(YA)$. If $\sigma(YA)$ is a subset of $\mathbb{R}$, then we take α satisfying $0 < |\alpha| < 2/\rho(YA)$ and $\operatorname{sgn}\alpha = \operatorname{sgn}\lambda_0$, where $\lambda_0 \in \sigma(YA)$, so as to ensure that $\rho(P - \alpha YA) < 1$.

Assume that $0 \notin \sigma((YA)|_T)$ hold. In the following, we will obtain the best value α_{opt} such that $\rho(P - \alpha YA)$ minimizes for achieving good convergence. Unfortunately, it may be rather difficult. If $\sigma(YA)$ is a subset of $\mathbb{R}$ and $\lambda_{\min} = \min\{\lambda : \lambda \in \sigma(YA)|_T\} > 0$ analogous to [8, Example 4.1], we can have

$$\alpha_{\text{opt}} = \frac{2}{\lambda_{\min} + \rho(YA)}. \tag{15}$$

In practice, because $\rho(YA)$ is not easily obtained, we often utilize $\|YA\|$ instead of it in the above inequations and (15) to choose α, which is followed from $\rho(YA) \le \|YA\|$.

3. Higher-Order Convergent Iterative Method for Computing the Generalized Inverse over Banach Algebra

In the section, we will investigate a higher-order convergent iterative method for computing the generalized Drazin inverse a^d over Banach algebra.

Theorem 7. *Let $a \in \mathscr{A}$, $p \in \mathscr{A}$ be idempotents with $ap = pa$, and $y \in \mathscr{A}$ with $(1-p)y = y(1-p) = y$. Define the sequence $\{x_k\}$ in $\mathscr{A}$ such that*

$$x_0 = \alpha y, \quad \forall x_0 \in \mathscr{A},$$

$$x_k = \left[C_t^1 - C_t^2 x_{k-1} a + \cdots + (-1)^{t-1} C_t^t (x_{k-1} a)^{t-1}\right] x_{k-1}, \tag{16}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ and $t \geq 2$. Then the iteration (16) converges to $\lim x_k$ and $px_0 = 0$ if and only if $\rho(1-p-\alpha ya) < 1$. In this case, assume that $ann^l(y) \cap (1-p)\mathscr{A}(1-p) = \{0\}$. Then

(i) *a^d exists and the iteration (16) converges to a^d if and only if ap is quasinilpotent in $\mathscr{A}$;*

(ii) *if $q = (1-p-\alpha ya) < 1$, then $\|a^d - x_k\| \leq q^{t^k}\|y\| \cdot \|\alpha\|/\|p+\alpha ya\|$.*

Proof. (i) By $(1-p)y = y(1-p) = y$ and $x_0 = \alpha y$, it implies that $(1-p)x_0 = x_0$. By induction on k, we have

$$\begin{aligned}(1-p)\,x_k &\\ &= (1-p)\left[C_t^1 - C_t^2 x_{k-1}a + \cdots + (-1)^{t-1}C_t^t(x_{k-1}a)^{t-1}\right]x_{k-1} \\ &= x_k.\end{aligned} \tag{17}$$

By (16), we obtain

$$\begin{aligned} x_k a - 1 &= (-1)^{t-1}(x_{k-1}a-1)^t \\ &= (-1)^{(t-1)^k}(x_0 a - 1)^{t^k}.\end{aligned} \tag{18}$$

From (17) and (18), we get

$$\begin{aligned}(1-p)(x_k a - 1) &= x_k a - (1-p) \\ &= (-1)^{(t-1)^k}(x_0 a - (1-p))^{t^k}.\end{aligned} \tag{19}$$

The right-hand side of the last equality of (19) implies that

$$0 = \lim_{t\to\infty}(-1)^{t-1}(x_0 a - (1-p))^{t^k}. \tag{20}$$

By (20), we easily have $\rho(x_0 a - (1-p)) = \rho(1-p-\alpha ya) < 1$.

Conversely, assume that $\rho(1-p-\alpha ya) < 1$. Since $pa = ap$ and $(1-p)y = y(1-p) = y$, $(1-p-\alpha ya) \in (1-p)\mathscr{A}(1-p)$, and then ya is invertible in $(1-p)\mathscr{A}(1-p)$. We will show that ay is invertible in $(1-p)\mathscr{A}(1-p)$. Clearly, $ay \in (1-p)\mathscr{A}(1-p)$, if $ayc = 0$ for some $c \in (1-p)\mathscr{A}(1-p)$, then $yc = [(ya)]^{-1}_{(1-p)\mathscr{A}} yac$. Hence, $c \in ann^r(y) \cap (1-p)\mathscr{A}(1-p) = \{0\}$ and $c = 0$. Hence, $0 \notin [(ay)]_{(1-p)\mathscr{A}(1-p)}$, and then ay is invertible in $(1-p)\mathscr{A}(1-p)$.

(i) In the following, we will consider the result (i). It is similar to the deduction of (10), we can write (16) as

$$x_k a = (1-p) + (-1)^{t-1}(x_0 a - (1-p))^{t^k}. \tag{21}$$

Thus, postmultiplying (21) by y yields to

$$x_k ay = (1-p)\,y + (-1)^{t-1}(x_0 a - (1-p))^{t^k} y. \tag{22}$$

By Lemma 1 and (22), we prove that x_k converges to $y[(ay)]^{-1}_{(1-p)\mathscr{A}(1-p)}$ and is denoted by $x_\infty = y[(ay)]^{-1}_{(1-p)\mathscr{A}(1-p)}$. Therefore, $y[(ay)]^{-1} = [(ya)]^{-1}y$ in $(1-p)\mathscr{A}(1-p)$; then

$$\begin{aligned} x_\infty a &= y[(ay)]^{-1}_{(1-p)\mathscr{A}(1-p)}a \\ &= (ya)^{-1}_{(1-p)\mathscr{A}(1-p)}ya \\ &= 1-p = ay[(ay)]^{-1}_{(1-p)\mathscr{A}(1-p)} \\ &= ax_\infty.\end{aligned} \tag{23}$$

Thus, we obtain $a - a^2 x_\infty = ap$. Since $x_\infty a x_\infty = ay[(ay)]^{-1}_{(1-p)\mathscr{A}(1-p)}x_\infty = x_\infty$, we have that $x_\infty = a^d$ if and only if ap is quasinilpotent in $\mathscr{A}$.

(ii) Since p is idempotent and $ap = pa$, and

$$\begin{gathered}(1-p)\,y = y\,(1-p) = y, \\ p\,(p+\alpha ay)\,ay = p\,(p+\alpha ya),\end{gathered} \tag{24}$$

then

$$\begin{aligned}\alpha(p+\alpha ay)^{-1}ay &= 1-(p+\alpha ay)^{-1}p = 1-p \\ &= 1-p\,(p+\alpha ay) \\ &= \alpha ay(p+\alpha ay)^{-1}.\end{aligned} \tag{25}$$

Therefore, we obtain $(ay)^{-1} = \alpha(p+\alpha ay)^{-1}$ in $(1-p)\mathscr{A}(1-p)$. By (10), we have

$$x_k ay = \left[(1-p) + (-1)^{t+1}[\alpha ay - (1-p)]^{t^k}\right]y. \tag{26}$$

Hence, by the argument in (i) and (26), we have

$$\begin{aligned} a^d - x_k &= x_\infty - x_k \\ &= y[(ay)]^{-1}_{(1-p)\mathscr{A}(1-p)} \\ &\quad - \left[(1-p) + (-1)^{t+1}[\alpha ay - (1-p)]^{t^k}\right]y \\ &\quad \times [(ay)]^{-1}_{(1-p)\mathscr{A}(1-p)} \\ &= (-1)^{t+2}[\alpha ay - (1-p)]^{t^k}y \\ &\quad \times [(ay)]^{-1}_{(1-p)\mathscr{A}(1-p)}.\end{aligned} \tag{27}$$

Taking limit in (27), then it reduces to (ii). □

Similarly, we have the following.

Theorem 8. *Let $a \in \mathscr{A}$, $p \in \mathscr{A}$ be idempotents with $ap = pa$, and $y \in \mathscr{A}$ with $(1-p)y = y(1-p) = y$. Define the sequence $\{x_k\}$ in $\mathscr{A}$ such that*

$$\begin{gathered} x_0 = \alpha y, \quad \forall x_0 \in \mathscr{A}, \\ x_k = x_{k-1}\left[C_t^1 - C_t^2 a x_{k-1} + \cdots + (-1)^{t-1}C_t^t(ax_{k-1})^{t-1}\right],\end{gathered} \tag{28}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $t \geq 2$. *Then, the iteration* (28) *converges to* $\lim x_k$ *and* $px_0 = 0$ *if and only if* $\rho(1 - p - a\alpha y) < 1$. *In this case, assume that* $ann^l(y) \cap (1-p)\mathscr{A}(1-p) = \{0\}$. *Then*

(i) a^d *exists and the iteration* (28) *converges to* a^d *if and only if ap is quasinilpotent in* $\mathscr{A}$;

(ii) *if* $q = (1 - p - \alpha ya) < 1$, *then* $\|a^d - x_k\| \leq q^{t^k}\|y\| \cdot \|\alpha\|/\|p + \alpha ya\|$.

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