Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 348513, 6 pages
http://dx.doi.org/10.1155/2013/348513

Research Article

Well-Posedness of the Two-Dimensional Fractional Quasigeostrophic Equation

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Received 18 July 2013; Accepted 20 October 2013

Academic Editor: Wan-Tong Li

This paper is concerned with the fractional quasigeostrophic equation with modified dissipativity. We prove the local existence of solutions in Sobolev spaces for the general initial data and the global existence for the small initial data when $1/2 \le \alpha < 1$.

1. Introduction

This paper is concerned with the nonlocal quasigeostrophic β-plane model with modified dissipativity [1, 2]

$$\left(\frac{\partial}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\right)q=\frac{1}{R_e}(-\Delta)^{1+\alpha}\psi, \tag{1}$$

where $(x, y) \in \Omega$ can be either the 2D torus $\mathbb{T}^2$ or the whole space $\mathbb{R}^2$, $t \ge 0$, $q = \Delta\psi - F\psi + \beta y$, and $(1/R_e)(-\Delta)^{1+\alpha}\psi$ with $\alpha \in (0,1)$ being the modified dissipative term. Let $J(f,g) = f_x g_y - f_y g_x$ denote the Jacobian operator; (1) can be notationally simplified as

$$\frac{\partial}{\partial t}\left[\Delta\psi-F\psi\right]+J\left(\psi,\Delta\psi\right)+\beta\frac{\partial\psi}{\partial x}=\frac{1}{R_e}(-\Delta)^{1+\alpha}\psi. \tag{2}$$

In this model, ψ is the geostrophic pressure, also called the geostrophic stream function, $\xi = \Delta\psi$ is the vertical component of the relative vorticity, $\nabla^{\perp}\psi = (-\partial\psi/\partial y, \partial\psi/\partial y)$ is a zeroth-order balance in the momentum equation, and F, β, and R_e are the rotational Froude number, the Coriolis parameter, and the Reynolds number, respectively. Usually, $\nu = 1/R_e$ is also called viscosity parameter. It has some features in common with the much studied two-dimensional surface quasigeostrophic equation (SQGE) (see [3–9] and references therein). However the quasi-geostrophic β-plane model has a number of novel and distinctive features.

Recently, this equation has been intensively investigated because of both its mathematical importance and its potential applications in meteorology and oceanography. The quasi-geostrophic β-plane model is a simplified model for the shallow water β-plane model [2, 10, 11] when the Rossby number is small under several assumptions on the magnitude of the bottom topography variations, which is used to understand the atmospheric and oceanic circulation, the gulf stream, and the variability of this circulation on time scales from several months to several years. In this regime, quasi-geostrophic theory is an adequate approximation to describe the flow and is developed for the simulation of large-scale geophysical currents in the middle latitudes.

When $\alpha = 1$, this is the standard quasi-geostrophic model studied in [1], which was put forward as a simplified model of the shallow water model (see also [2] for a review). In [12], the author studied a multilayer quasi-geostrophic model, which is a generalization of the single layer model in the case $\alpha = 1$. The general fractional power α was considered by Pu and Guo [13]. The equation is

$$\frac{\partial}{\partial t}\left[\Delta\psi-F\psi\right]+J\left(\psi,\Delta\psi\right)+\beta\frac{\partial\psi}{\partial x}=\frac{1}{R_e}(-\Delta)^{1+\alpha}\psi+f, \tag{3}$$

$$\psi\left(x,y,0\right)=\psi_0\left(x,y\right). \tag{4}$$

In [13], they proved the global existence of weak solutions by employing the Galerkin approximation method for initial data belonging to the (inhomogeneous) Sobolev space $H^2(\Omega)$. If the initial data is in the (homogeneous) Sobolev space $\dot{H}^s(\Omega)\,(s > 2)$, it is natural for us to ask whether (3) has regular solutions.

In this paper, we only consider the 2D torus $\mathbb{T}^2$ with periodic boundary conditions. And we will prove the well-posedness results of (3) under certain condition on initial data which belong to the (homogeneous) Sobolev space $\dot{H}^s(\mathbb{T}^2)\,(s > 3-2\alpha)$. In Section 3, the local existence and uniqueness of the solutions of the problem are proved in $\dot{H}^s(\mathbb{T}^2)$ when $s > 3-2\alpha$ for $1/2 < \alpha < 1$. That is, for any initial data $\psi_0 \in \dot{H}^s(\mathbb{T}^2)$ and $f \in L^2(0,T;\dot{H}^{s-\alpha-2}(\mathbb{T}^2))$, there exists

$$T = \left(\|f\|_{L^2(0,T;\dot{H}^{s-\alpha-2})}, \|\psi_0\|_{\dot{H}^s}, R_e\right), \tag{5}$$

such that (3) has a uniqueness solution on $[0,T]$, satisfying

$$\psi \in L^\infty\left(0,T;\dot{H}^s\left(\mathbb{T}^2\right)\right) \cap L^2\left(0,T;\dot{H}^{s+\alpha}\left(\mathbb{T}^2\right)\right). \tag{6}$$

However, we may not obtain the global existence of solutions from energy (34), if the initial data has large $\dot{H}^s$ norm. The main reason is that in the $\dot{H}^s$ energy estimate for (3), the integral $(\Lambda^{2(s-1)}, J(\psi,\Delta\psi)) \neq 0$ for $s > 2$, where (u,v) denotes the integral $\int_{\mathbb{T}^2} u(x,y)v(x,y)dx\,dy$ as usual. Thus, it is necessary to control it. To overcome this essential difficulty, we will make use of the properties of the product estimates (Proposition 2) as well as those of the Sobolev embedding inequality.

In Section 4, global existence and uniqueness for small initial data in $\dot{H}^s(\mathbb{T}^2)$ are also proved when $s > 3-2\alpha$. More precisely, we just need the following condition:

$$\begin{aligned}&\|\psi_0\|_{\dot{H}^s_x} + \|f\|_{L^2_t\dot{H}^{s-\alpha-2}_x}\\&\quad + \left(\|\psi_0\|^\gamma_{H^1_x} + \|f\|^\gamma_{L^2_tL^2_x}\right)\left(\|\psi_0\|^{1-\gamma}_{\dot{H}^s_x} + \|f\|^{1-\gamma}_{L^2_t\dot{H}^{s-\alpha-2}_x}\right) < \varepsilon,\end{aligned} \tag{7}$$

where $\gamma = 1-((3-2\alpha)/s)$.

For the cases, $\alpha = 1/2$ and $s > 3$, we also obtain the unique global solution in H^s proved by

$$\begin{aligned}&\left(\|\psi_0\|^{\gamma_1}_{H^1_x} + \|f\|^{\gamma_1}_{L^2_t\dot{H}^{s-5/2}_x}\right)\left(\|\psi_0\|^{1-\gamma_1}_{\dot{H}^s_x} + \|f\|^{1-\gamma_1}_{L^2_t\dot{H}^{s-5/2}_x}\right)\\&\quad + \left(\|\psi_0\|^{\gamma_2}_{H^1_x} + \|f\|^{\gamma_2}_{L^2_tL^2_x}\right)\left(\|\psi_0\|^{1-\gamma_2}_{\dot{H}^s_x} + \|f\|^{1-\gamma_2}_{L^2_t\dot{H}^{s-5/2}_x}\right) < \varepsilon,\end{aligned} \tag{8}$$

where $\gamma_1 = 1-2/s$ and $\gamma_2 = 1-3/s$.

We conclude this introduction by mentioning the global existence result of weak solutions obtained [13].

Proposition 1. *Let $\alpha \in (0,1), T > 0$, $\psi_0 \in H^2(\Omega)$, and $f \in L^2(0,T;L^2(\Omega))$. There exists a weak solution of (3)-(4) which satisfies*

$$\psi \in L^\infty\left(0,T;H^2(\Omega)\right) \cap L^2\left(0,T;H^{2+\alpha}(\Omega)\right). \tag{9}$$

2. Notations and Preliminaries

We now review the notations used throughout the paper. Let us denote $\Lambda = (-\Delta)^{1/2}$. The Fourier transform $\hat{f}$ of a tempered distribution $f(x)$ on $\mathbb{T}^2$ is defined as

$$\hat{f}(k) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(x)e^{-ik\cdot x}dx. \tag{10}$$

Generally, $\Lambda^\beta f$ for $\beta \in \mathbb{R}$ can be identified with the Fourier series

$$\sum_{k\in\mathbb{Z}^2} |k|^\beta \hat{f}(k)e^{ik\cdot x}. \tag{11}$$

$L^p(\mathbb{T}^2)$ denotes the space of the pth-power integrable functions normed by

$$\|f\|_{L^p} = \left(\int_{\mathbb{T}^2} |f|^p dx\right)^{1/p}, \qquad \|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{T}^2} |f(x)|. \tag{12}$$

For any tempered distribution f on Ω and $s \in \mathbb{R}$, we define

$$\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2} = \left(\sum_{k\in\mathbb{Z}^2} |k|^{2s}\left|\hat{f}(k)\right|^2\right)^{1/2}. \tag{13}$$

$\dot{H}^s$ denotes the homogeneous Sobolev space of all f for which $\|f\|_{\dot{H}^s}$ is finite. The homogeneous counterparts of $\dot{H}^s$ are denoted by H^s.

Next, this section contains a few auxiliary results used in the paper. In particular, we recall, by now, the classical, product, and commutator estimates, as well as the Sobolev embedding inequalities. Proofs of these results can be found for instance, in [14–16].

Proposition 2 (product estimate). *If $s > 0$, then, for all $f, g \in H^s \cap L^\infty$, one has the estimates*

$$\|\Lambda^s(fg)\|_{L^p} \leq C\left(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}}\|g\|_{L^{p_4}}\right), \tag{14}$$

where $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$ and p_1, p_2, and $p_3 \in (1,\infty)$. In particular

$$\|\Lambda^s(fg)\|_{L^2} \leq C\left(\|f\|_{L^\infty}\|\Lambda^s g\|_{L^2} + \|\Lambda^s f\|_{L^2}\|g\|_{L^\infty}\right). \tag{15}$$

In the case of a commutator we have the following estimate.

Proposition 3 (commutator estimate). *Suppose that $s > 0$ and $p \in (1,\infty)$. If $f, g \in S$, then*

$$\begin{aligned}&\|\Lambda^s(fg) - f\Lambda^s(g)\|_{L^p}\\&\quad \leq C\left(\|\nabla f\|_{L^{p_1}}\left\|f\Lambda^{s-1}g\right\|_{L^{p_2}} + \|\Lambda^s(f)\|_{L^{p_3}}\|g\|_{L^{p_2}}\right),\end{aligned} \tag{16}$$

where $s > 0$, $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$, and p_1, p_2, and $p_3 \in (1,\infty)$.

We will use as well the following Sobolev inequality.

Proposition 4 (Sobolev inequality). *Suppose that $q > 1$, $p \in [q, \infty)$, and*

$$\frac{1}{p} = \frac{1}{q} - \frac{s}{d}. \tag{17}$$

Suppose that $\Lambda^s f \in L^q$; then $f \in L^p$ and there is a constant $C > 0$ such that

$$\|f\|_{L^p} \le C\|\Lambda^s f\|_{L^q}. \tag{18}$$

The following result is from Henry [17] with extensions for nonintegral order derivatives like in, for example, Triebel [18, 19].

Proposition 5. *If $0 \le a \le 1, 1 \le p, q, r \le \infty$, and m, k are nonnegative with*

$$\begin{gathered} k - \frac{n}{q} = a\left(m - \frac{n}{p}\right) + (1-a)\left(-\frac{n}{r}\right), \\ \frac{1}{q} \le \frac{a}{p} + \frac{1-a}{r} \end{gathered} \tag{19}$$

except that one requires $a \ne 1$ when $m - (n/p) = k, 1 < p < \infty$, then there is a constant C such that

$$\left\|D^k u\right\|_{L^q} \le C\|D^m u\|_{L^p}^a \|u\|_{L^r}^{1-a}, \tag{20}$$

for all $u \in C_c^\infty$.

3. Local Existence and Large Data

In [7], the authors studied and established the existence and uniqueness of local and global solutions to the two-dimensional SQGE. It is natural that (3) is more complex than SQGE. However, we also establish an analogue. In this section we will prove that (3) is locally well-posed in $\dot{H}^s(\mathbb{T}^2)$ when $s > 3 - 2\alpha$ for $1/2 < \alpha < 1$. Regarding arbitrarily large initial data, we obtain the following result.

Theorem 6 (local existence). *Let $\alpha \in (1/2, 1)$ and fix $s > 3 - 2\alpha$. Assume that $\psi_0 \in \dot{H}^s(\mathbb{T}^2)$ and $f \in L^2(0, T; \dot{H}^{s-\alpha-2}(\mathbb{T}^2))$ have zero mean on $\mathbb{T}^2$. Then there exist a time $T > 0$ and a unique smooth solution*

$$\psi \in L^\infty\left(0, T; \dot{H}^s\left(\mathbb{T}^2\right)\right) \cap L^2\left(0, T; \dot{H}^{s+\alpha}\left(\mathbb{T}^2\right)\right) \tag{21}$$

of the Cauchy problem (3)-(4).

Proof. First of all, multiplying (3) by $\Lambda^{2(s-1)}\psi$, we get the following energy inequality:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|\Lambda^s\psi\|_{L^2}^2 + F\left\|\Lambda^{s-1}\psi\right\|_{L^2}^2\right] + \frac{1}{R_e}\left\|\Lambda^{s+\alpha}\psi\right\|_{L^2}^2 \\ &\quad \le \left|\left(f, \Lambda^{2(s-1)}\psi\right)\right| + \left|\left(J\left(\psi, \Delta\psi\right), \Lambda^{2(s-1)}\psi\right)\right| \\ &\qquad + \left|\left(\beta\frac{\partial\psi}{\partial x}, \Lambda^{2(s-1)}\psi\right)\right|. \end{aligned} \tag{22}$$

Integration by parts gives us the following estimate:

$$\left(\beta\frac{\partial\psi}{\partial x}, \Lambda^{2(s-1)}\psi\right) = 0. \tag{23}$$

Then we get the inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|\Lambda^s\psi\|_{L^2}^2 + F\left\|\Lambda^{s-1}\psi\right\|_{L^2}^2\right] + \frac{1}{R_e}\left\|\Lambda^{s+\alpha}\psi\right\|_{L^2}^2 \\ &\quad \le \left|\left(f, \Lambda^{2(s-1)}\psi\right)\right| + \left|\left(J\left(\psi, \Delta\psi\right), \Lambda^{2(s-1)}\psi\right)\right|. \end{aligned} \tag{24}$$

We estimate the first term on the right side by

$$\left|\left(f, \Lambda^{2(s-1)}\psi\right)\right| \le \frac{R_e}{2}\left\|\Lambda^{s-\alpha-2} f\right\|_{L^2}^2 + \frac{1}{2R_e}\left\|\Lambda^{s+\alpha}\psi\right\|_{L^2}^2. \tag{25}$$

To handle the second term, we proceed as follows.

First note that

$$\begin{aligned} &\left|\left(J\left(\psi, \Delta\psi\right), \Lambda^{2(s-1)}\psi\right)\right| \\ &\quad = \left|\left(\nabla^\perp\psi \cdot \nabla\Delta\psi, \Lambda^{2(s-1)}\psi\right)\right| \\ &\quad = \left|\Lambda^{s-\alpha-2}\left(\nabla^\perp\psi \cdot \nabla\Delta\psi\right), \Lambda^{s+\alpha}\psi\right| \\ &\quad \le \left\|\Lambda^{s-\alpha-2}\left(\nabla^\perp\psi \cdot \nabla\Delta\psi\right)\right\|_{L^2}\left\|\Lambda^{s+\alpha}\psi\right\|_{L^2} \\ &\quad \le \left\|\Lambda^{s-\alpha-1}\left(\nabla^\perp\psi \cdot \Delta\psi\right)\right\|_{L^2}\left\|\Lambda^{s+\alpha}\psi\right\|_{L^2}. \end{aligned} \tag{26}$$

The estimate of the product term follows from Proposition 2. Hence, we have

$$\begin{aligned} &\left\|\Lambda^{s-\alpha-1}\left(\nabla^\perp\psi \cdot \Delta\psi\right)\right\|_{L^2} \\ &\quad \le C\left(\left\|\Lambda^{s-\alpha}\psi\right\|_{L^2}\|\Delta\psi\|_{L^2} + \left\|\Lambda^{s-\alpha+1}\psi\right\|_{L^p}\|\nabla\psi\|_{L^{2p/(p-2)}}\right). \end{aligned} \tag{27}$$

We now fix an arbitrary p such that

$$\frac{2}{s-1} < p < \frac{2}{1+(1-2\alpha)} = \frac{1}{1-\alpha}. \tag{28}$$

Note that $p > 2$ since $s > 2$ and the range for p is nonempty since $s > 3 - 2\alpha$. For $\alpha \in (1/2, 1)$, our choice of p and Proposition 5 give

$$\left\|\Lambda^{s-\alpha+1}\psi\right\|_{L^p} \le \|\Lambda^s\psi\|_{L^2}^{1-\xi}\left\|\Lambda^{s+\alpha}\psi\right\|_{L^2}^{\xi}, \tag{29}$$

where $\xi \in (0, 1)$ may be computed explicitly from $\xi\alpha = 2 - \alpha - (2/p)$.

In order to estimate $\|\Lambda^{s-\alpha}\psi\|_{L^2}\|\Delta\psi\|_{L^2}$ in (27), we split it into two cases.

Case 1 ($3 - 2\alpha < 2 < s$). From Proposition 5 and Sobolev inequality, we have

$$\begin{aligned} \left\|\Lambda^{s-\alpha}\psi\right\|_{L^2} &\le \left\|\Lambda^{s+\alpha}\psi\right\|_{L^2}^{\theta}\left\|\Lambda^{s-1}\psi\right\|_{L^2}^{1-\theta} \\ &\le \left\|\Lambda^{s+\alpha}\psi\right\|_{L^2}^{\theta}\left\|\Lambda^{s}\psi\right\|_{L^2}^{1-\theta}, \end{aligned} \tag{30}$$

where $\theta = (1-\alpha)/(1+\alpha)$. In addition, since ψ has zero mean and $p > 2/(s-1)$, from the Sobolev embedding we obtain

$$\begin{gathered}\|\Lambda\psi\|_{L^{2p/(p-2)}} \le C\|\Lambda^s\psi\|_{L^2},\\ \|\Delta\psi\|_{L^2} \le C\|\Lambda^s\psi\|_{L^2}.\end{gathered} \tag{31}$$

Combining estimates (27)–(31) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\Lambda^s\psi\|_{L^2}^2 + F\|\Lambda^{s-1}\psi\|_{L^2}^2\right] + \frac{1}{R_e}\|\Lambda^{s+\alpha}\psi\|_{L^2}^2\\ &\quad\le R_e\|\Lambda^{s-\alpha-2}f\|_{L^2}^2 + C\left(\|\Lambda^{s+\alpha}\psi\|_{L^2}^{1+\theta}\|\Lambda^s\psi\|_{L^2}^{2-\theta}\right.\\ &\qquad\left.+\|\Lambda^{s+\alpha}\psi\|_{L^2}^{1+\xi}\|\Lambda^s\psi\|_{L^2}^{2-\xi}\right),\end{aligned} \tag{32}$$

where $0 < \xi < 1$ and $\theta = (1-\alpha)/(1+\alpha)$ is as defined earlier. The second term on the right side of (32) is bounded using the ε-Young inequality as

$$\begin{aligned}&\frac{1}{2R_e}\|\Lambda^{s+\alpha}\psi\|_{L^2}^2 + CR_e^{-(1+\theta)/(1-\theta)}\|\Lambda^s\psi\|_{L^2}^{(2-\theta)/(1-\theta)}\\ &\quad+ CR_e^{-(1+\xi)/(1-\xi)}\|\Lambda^s\psi\|_{L^2}^{(2-\xi)/(1-\xi)}\end{aligned} \tag{33}$$

and we finally obtain the following estimate:

$$\begin{aligned}&\frac{d}{dt}\left[\|\Lambda^s\psi\|_{L^2}^2 + F\|\Lambda^{s-1}\psi\|_{L^2}^2\right] + \frac{1}{2R_e}\|\Lambda^{s+\alpha}\psi\|_{L^2}^2\\ &\quad\le R_e\|\Lambda^{s-\alpha-2}f\|_{L^2}^2\\ &\qquad+ C\left(\|\Lambda^s\psi\|_{L^2}^{2(2-\theta)/(1-\theta)} + \|\Lambda^s\psi\|_{L^2}^{2(2-\xi)/(1-\xi)}\right).\end{aligned} \tag{34}$$

Using Gronwall's inequality, from estimate (34) we may deduce the existence of a positive time

$$T = \left(\|f\|_{L^2(0,T;\dot{H}^{s-\alpha-2})}, \|\psi_0\|_{\dot{H}^s}, R_e\right) \tag{35}$$

such that

$$\psi \in L^\infty\left(0,T;\dot{H}^s\left(\mathbb{T}^2\right)\right) \cap L^2\left(0,T;\dot{H}^{s+\alpha}\left(\mathbb{T}^2\right)\right). \tag{36}$$

Note that we have $2(2-\theta)/(1-\theta) > 2, 2(2-\xi)/(1-\xi) > 2$, and hence we may not obtain the global existence of solutions from the energy (34), if the initial data has large H^s norm. These a priori estimates can be made formal using a standard approximation procedure. We omit further details

Case 2 ($3-2\alpha < s < 2$). Using Proposition 5, we obtain

$$\|\Delta\psi\|_{L^2} \le C\|\Lambda^s\psi\|_{L^2}^{1-\theta_1}\|\Lambda^{s+\alpha}\psi\|_{L^2}^{\theta_1}, \tag{37}$$

where $\theta_1 = (2-s)/\alpha$. From Sobolev embedding, we have

$$\|\Lambda^{s-\alpha}\psi\|_{L^2} \le C\|\Lambda^s\psi\|_{L^2}. \tag{38}$$

Then, using the same method as in Case 1, we can complete Theorem 6. □

4. Global Existence and Small Data

The main result of this section concerns global well-posedness in case of small initial data.

Theorem 7 (global existence). *Let $\alpha \in (1/2, 1)$, $f \in L^2(0,\infty;\dot{H}^{s-\alpha-2}(\mathbb{T}^2)) \cap L^2(0,\infty;L^2(\mathbb{T}^2))$ and let $\psi_0 \in \dot{H}^s(\mathbb{T}^2)$ have zero mean on $\mathbb{T}^2$, where $s > 3-2\alpha$. There exists a small enough constant $\varepsilon > 0$ depending on R_e, such that if*

$$\begin{aligned}&\|\psi_0\|_{\dot{H}_x^s} + \|f\|_{L_t^2\dot{H}_x^{s-\alpha-2}}\\ &\quad+\left(\|\psi_0\|_{H_x^1}^\gamma + \|f\|_{L_t^2L_x^2}^\gamma\right)\left(\|\psi_0\|_{\dot{H}_x^s}^{1-\gamma} + \|f\|_{L_t^2\dot{H}_x^{s-\alpha-2}}^{1-\gamma}\right) < \varepsilon,\end{aligned} \tag{39}$$

where $\gamma = 1-(3-2\alpha)/s$, then the unique smooth solution ψ of the Cauchy problem (3)-(4) is global in time; that is, $\psi \in L^\infty(0,\infty;\dot{H}^s(\mathbb{T}^2))$.

Proof. We proceed as in the proof of Theorem 6. The product term in (27) is now estimated by

$$\begin{aligned}&\left\|\Lambda^{s-\alpha-1}\left(\nabla^\perp\psi\cdot\Delta\psi\right)\right\|_{L^2}\\ &\quad\le C\left(\|\Lambda^{s-\alpha}\psi\|_{L^2}\|\Delta\psi\|_{L^2} + \|\Lambda^{s-\alpha+1}\psi\|_{L^p}\|\nabla\psi\|_{L^{2p/(p-2)}}\right),\end{aligned} \tag{40}$$

where $p = 1/(1-\alpha)$, so that

$$\|\Lambda^{s-\alpha+1}\psi\|_{L^p} \le \|\Lambda^{s+\alpha}\psi\|_{L^2}. \tag{41}$$

Similarly, in order to estimate $\|\Lambda^{s-\alpha}\psi\|_{L^2}\|\Delta\psi\|_{L^2}$ in (40), we split it into two cases.

Case 3 ($3-2\alpha < 2 < s$). From Sobolev imbedding, we have

$$\begin{gathered}\|\Lambda^{s-\alpha}\psi\|_{L^2} \le C\|\Lambda^{s+\alpha}\psi\|_{L^2},\\ \|\Delta\psi\|_{L^2} \le C\|\Lambda^s\psi\|_{L^2}.\end{gathered} \tag{42}$$

Case 4 ($3-2\alpha < s < 2$). Using Sobolev imbedding, we have

$$\begin{gathered}\|\Lambda^{s-\alpha}\psi\|_{L^2} \le C\|\Lambda^s\psi\|_{L^2},\\ \|\Delta\psi\|_{L^2} \le C\|\Lambda^{s+\alpha}\psi\|_{L^2}.\end{gathered} \tag{43}$$

So, we can always obtain the following estimate

$$\|\Lambda^{s-\alpha}\psi\|_{L^2}\|\Delta\psi\|_{L^2} \le C\|\Lambda^s\psi\|_{L^2}\|\Lambda^{s+\alpha}\psi\|_{L^2}. \tag{44}$$

With this choice of p and the above embedding, the product estimate gives us

$$\begin{aligned}&\left\|\Lambda^{s-\alpha-1}\left(\nabla^\perp\psi\cdot\Delta u\right)\right\|_{L^2}\\ &\quad\le C\left(\|\Lambda^{s+\alpha}\psi\|_{L^2}\|\Lambda^s\psi\|_{L^2} + \|\Lambda^{s+\alpha}\psi\|_{L^2}\|\nabla\psi\|_{L^{2p/(p-2)}}\right)\\ &\quad\le C\|\Lambda^{s+\alpha}\psi\|_{L^2}\left(\|\Lambda^s\psi\|_{L^2} + \|\nabla\psi\|_{L^{2p/(p-2)}}\right).\end{aligned} \tag{45}$$

Combining (24) with (45) and proceeding as in (34) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|\Lambda^s\psi\|_{L^2}^2+F\left\|\Lambda^{s-1}\psi\right\|_{L^2}^2\right]+\frac{1}{R_e}\|\Lambda^{s+\alpha}\psi\|_{L^2}^2\\ \le\left\|\Lambda^{s-\alpha-2}f\right\|_{L^2}\|\Lambda^{s+\alpha}\psi\|_{L^2}\\ +C\|\Lambda^{s+\alpha}\psi\|_{L^2}^2\left(\|\Lambda^s\psi\|_{L^2}+\|\nabla\psi\|_{L^{2p/(p-2)}}\right)\end{aligned} \tag{46}$$

which in turn implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|\Lambda^s\psi\|_{L^2}^2+F\left\|\Lambda^{s-1}\psi\right\|_{L^2}^2\right]+\frac{1}{2R_e}\|\Lambda^{s+\alpha}\psi\|_{L^2}^2\\ \le\frac{R_e}{2}\left\|\Lambda^{s-\alpha-2}f\right\|_{L^2}^2\\ +C\|\Lambda^{s+\alpha}\psi\|_{L^2}^2\left(\|\Lambda^s\psi\|_{L^2}+\|\nabla\psi\|_{L^{2p/(p-2)}}\right).\end{aligned} \tag{47}$$

Observe that

$$\|\nabla\psi\|_{L^{2p/(p-2)}}\le C\|\psi\|_{L^2}^{\gamma}\|\Lambda^s\psi\|_{L^2}^{1-\gamma}, \tag{48}$$

where $\gamma=1-(3-2\alpha)/s$. Therefore, if

$$C\left(\|\Lambda^s\psi\|_{L^2}+\|\psi\|_{L^2}^{\gamma}\|\Lambda^s\psi\|_{L^2}^{1-\gamma}\right)\le\frac{1}{4CR_e} \tag{49}$$

estimate (47) combined with Sobolev imbedding inequality $\|\Lambda^{s+\alpha}\psi\|_{L^2}\ge\|\Lambda^s\psi\|_{L^2}$ shows that

$$\begin{aligned}\frac{d}{dt}\left[\|\Lambda^s\psi\|_{L^2}^2+F\left\|\Lambda^{s-1}\psi\right\|_{L^2}^2\right]\\ +\frac{1}{2R_e}\|\Lambda^s\psi\|_{L^2}^2\le R_e\|f\|_{\dot H_x^{s-\alpha-2}}^2\end{aligned} \tag{50}$$

and hence

$$\begin{aligned}\sup_{0\le t<\infty}\left\{\|\Lambda^s\psi\|_{L^2}^2+F\left\|\Lambda^{s-1}\psi\right\|_{L^2}^2\right\}\\ \le\|\Lambda^s\psi_0\|_{L^2}^2+F\left\|\Lambda^{s-1}\psi_0\right\|_{L^2}^2+R_e\|f\|_{L_t^2\dot H_x^{s-\alpha-2}}^2.\end{aligned} \tag{51}$$

By Sobolev imbedding, we have

$$\left\|\Lambda^{s-1}\psi_0\right\|_{L^2}\le C\|\Lambda^s\psi_0\|_{L^2}. \tag{52}$$

Combining (51) and (52), we get

$$\|\Lambda^s\psi\|_{L^2}^2\le C\|\Lambda^s\psi_0\|_{L^2}^2+R_e\|f\|_{L_t^2\dot H_x^{s-\alpha-2}}^2. \tag{53}$$

Note that taking the L^2-product of (3) with ψ gives for any $t>0$

$$\frac{d}{dt}\left[\|\nabla\psi\|_{L^2}^2+F\left\|\Lambda^{s-1}\psi\right\|_{L^2}^2\right]+\frac{1}{R_e}\int_0^t\left\|\Lambda^{1+\alpha}\psi\right\|_{L^2}^2\le R_e\|f\|_{L_x^2}^2. \tag{54}$$

Thus, there exists some constant K (dependent on F) such that

$$\sup_{0\le t<\infty}\left\{\|\nabla\psi\|_{L^2}^2+F\|\psi\|_{L^2}^2\right\}\le K\|\psi_0\|_{H^1}^2+K\|f\|_{L_t^2L_x^2} \tag{55}$$

which gives us a basic uniform estimate of ψ in $L_t^\infty H_x^1$.

Hence, from (53) and (55) we obtain that condition (49) is satisfied for all $t>0$ as long as we have

$$\begin{aligned}\|\psi_0\|_{\dot H_x^s}+\|f\|_{L_t^2\dot H_x^{s-\alpha-2}}+\left(\|\psi_0\|_{H_x^1}^{\gamma}+\|f\|_{L_t^2L_x^2}^{\gamma}\right)\\ \times\left(\|\psi_0\|_{\dot H_x^s}^{1-\gamma}+\|f\|_{L_t^2\dot H_x^{s-\alpha-2}}^{1-\gamma}\right)<\varepsilon,\end{aligned} \tag{56}$$

where ε is sufficiently small, thereby concluding the proof of the theorem. □

Note also that the proof of Theorem 7 fails for the value $\alpha=1/2$. Thus, $\alpha=1/2$ indeed is the limit of the local well-posedness theory. Nonetheless, we still can prove that the considered system is globally well-posed for small data.

Theorem 8 (global existence for small data). *Let $s>3$ and assume that the initial data $\psi_0\in\dot H^s(\mathbb{T}^2)$ and $f\in L^2(0,\infty;\dot H^{s-(5/2)}(\mathbb{T}^2))\cap L^2(0,\infty;L^2(\mathbb{T}^2))$ have zero mean on $\mathbb{T}^2$. There exists a sufficiently small constant $\varepsilon>0$ depending on R_e, such that if*

$$\begin{aligned}\left(\|\psi_0\|_{H_x^1}^{\gamma_1}+\|f\|_{L_t^2\dot H_x^{s-(5/2)}}^{\gamma_1}\right)\left(\|\psi_0\|_{\dot H_x^s}^{1-\gamma_1}+\|f\|_{L_t^2\dot H_x^{s-(5/2)}}^{1-\gamma_1}\right)\\ +\left(\|\psi_0\|_{H_x^1}^{\gamma_2}+\|f\|_{L_t^2L_x^2}^{\gamma_2}\right)\left(\|\psi_0\|_{\dot H_x^s}^{1-\gamma_2}+\|f\|_{L_t^2\dot H_x^{s-(5/2)}}^{1-\gamma_2}\right)<\varepsilon,\end{aligned} \tag{57}$$

where $\gamma_1=1-(2/s)$ and $\gamma_2=1-(3/s)$, then the unique smooth solution

$$\psi\in L^\infty\left(0,\infty;\dot H^s\left(\mathbb{T}^2\right)\right) \tag{58}$$

of the Cauchy problem (3)-(4) *is global in time.*

Proof. We proceed as in the proof of Theorem 7 and obtain the energy estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|\Lambda^s\psi\|_{L^2}^2+\left\|\Lambda^{s-1}\psi\right\|_{L^2}^2\right]+\frac{1}{R_e}\left\|\Lambda^{s+(1/2)}\psi\right\|_{L^2}^2\\ \le\left\|\Lambda^{s-(5/2)}f\right\|_{L^2}\left\|\Lambda^{s+(1/2)}\psi\right\|_{L^2}\\ +\left\|\Lambda^{s-(3/2)}\left(\nabla^\perp\psi\cdot\Delta\psi\right)\right\|_{L^2}\left\|\Lambda^{s+(1/2)}\psi\right\|_{L^2}.\end{aligned} \tag{59}$$

The second term on the right side is estimated using the product estimate in Proposition 2. Thus we obtain, similar to (45),

$$\begin{aligned}\left\|\Lambda^{s-(3/2)}\left(\nabla^\perp\psi\cdot\Delta\psi\right)\right\|_{L^2}\\ \le C\left(\|\nabla\psi\|_{L^\infty}\left\|\Lambda^{s+(1/2)}\psi\right\|_{L^2}+\left\|\Lambda^{s-(1/2)}\psi\right\|_{L^2}\|\Delta\psi\|_{L^\infty}\right)\\ \le C\left(\|\nabla\psi\|_{L^\infty}\left\|\Lambda^{s+(1/2)}\psi\right\|_{L^2}+\left\|\Lambda^{s+(1/2)}\psi\right\|_{L^2}\|\Delta\psi\|_{L^\infty}\right).\end{aligned} \tag{60}$$

By interpolation inequality, we have

$$\begin{aligned}\|\nabla\psi\|_{L^\infty}\le\|\Lambda^s\psi\|_{L^2}^{1-\gamma_1}\|\psi\|_{L^2}^{\gamma_1},\\ \|\Delta\psi\|_{L^\infty}\le\|\Lambda^s\psi\|_{L^2}^{1-\gamma_2}\|\psi\|_{L^2}^{\gamma_2},\end{aligned} \tag{61}$$

where $\gamma_1 = 1 - (2/s)$ and $\gamma_2 = 1 - (3/s)$. Combining estimates (61) gives

$$\frac{1}{2}\frac{d}{dt}\left[\left\|\Lambda^{s}\psi\right\|_{L^2}^{2} + \left\|\Lambda^{s-1}\psi\right\|_{L^2}^{2}\right] + \frac{1}{R_e}\left\|\Lambda^{s+(1/2)}\psi\right\|_{L^2}^{2} \leq \frac{1}{2R_e}\left\|\Lambda^{s-(5/2)}f\right\|_{L^2}^{2} + \frac{CR_e}{2}\left\|\Lambda^{s+(1/2)}\psi\right\|_{L^2}^{2} \times \left(\left\|\Lambda^{s}\psi\right\|_{L^2}^{1-\gamma_1}\left\|\psi\right\|_{L^2}^{\gamma_1} + \left\|\Lambda^{s}\psi\right\|_{L^2}^{1-\gamma_2}\left\|\psi\right\|_{L^2}^{\gamma_2}\right). \quad (62)$$

We obtain the desired result as in the proof of Theorem 7. □

Remark 9. When $2 < s \leq 3$, the result of Theorem 8 is still open.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the Doctoral Starting-up Foundation of Minnan Normal University, China-NSAF (Grant no. L21228).

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