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Research Article

On the Mean Values of Certain Character Sums

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Let $q \geq 5$ be an odd number. In this paper, we study the fourth power mean of certain character sums $\sum_{\chi \bmod q,(\chi-1)=-1}^{*} |\sum_{1\leq a\leq q/4} a\chi(a)|^4$ and $\sum_{\chi \bmod q,(\chi-1)=1}^{*} |\sum_{1\leq a\leq q/4} a\chi(a)|^4$, where $\sum^{*}$ denotes the summation over primitive characters modulo q, and give some asymptotic formulae.

1. Introduction

The sum

$$S_\chi(n) = \frac{1}{q^n}\sum_{a=1}^{q} a^n \chi(a) \tag{1}$$

appears frequently in number theory, where χ is a nonprincipal primitive character modulo q, and has been studied by several experts. For example, for $q \equiv 3 \bmod 4$ being a prime p and χ being the Legendre symbol, Ayoub et al. [1] have proved that $S_\chi(n) < 0$ for $n = 1, 2$ and for $n \geq p-2$. Fine [2] has showed that for $n > 2$, there exist infinitely many primes $p \equiv 3 \bmod 4$ with $S_\chi(n) > 0$ and infinitely many with $S_\chi(n) < 0$.

Williams [3] proved that

$$S_\chi(n) = O\left(p^{1/2}\log p\right) \tag{2}$$

for χ being the Legendre symbol modulo p. For primitive character χ modulo q, Toyoizumi [4] used the generalized Bernoulli numbers to express $S_\chi(n)$ in terms of Gauss sums and Dirichlet L-functions as follows:

$$\sum_{a=1}^{q} a^n \chi(a) = \begin{cases} 2q^n\tau(\chi) \\ \quad\times \displaystyle\sum_{1\leq m\leq n/2} \frac{\binom{n}{2m-1}(2m-1)!L(2m,\overline{\chi})}{(-1)^{m+1}(2\pi)^{2m}}, & \text{if } \chi(-1)=1, \\ 2q^n\tau(\chi) \\ \quad\times \displaystyle\sum_{0\leq m\leq (n-1)/2} \frac{\binom{n}{2m}(2m)!L(2m+1,\overline{\chi})}{(-1)^{m+1}(2\pi)^{2m+1}i}, & \text{if } \chi(-1)=-1, \end{cases} \tag{3}$$

where $\tau(\chi) = \sum_{a=1}^{q}\chi(a)e(a/q)$ is the Gauss sum, $e(y) = e^{2\pi i y}$, $L(s,\chi)$ is the Dirichlet L-function corresponding to χ, and $\binom{n}{m}$ denotes the binomial coefficient.

Toyoizumi [4] also gave explicit bounds for $S_\chi(n)$.

Proposition 1. *(a) Assume that $\chi(-1) = 1$ and $n \geq 2$. Then for any primitive character χ mod q, one has*

$$\left|S_\chi(n)\right| < C_1(n)\, q^{1/2}, \tag{4}$$

where

$$C_1(n) = \frac{2\zeta(2)\, n!}{(2\pi)^{n+1}} \sum_{1\leq m\leq n/2} \frac{(2\pi)^{n+1-2m}}{(n+1-2m)!}, \tag{5}$$

and $\zeta(s)$ is the Riemann zeta function.

(b) Assume that $\chi(-1) = -1$ and $n \geq 3$. Then for any primitive character $\chi \bmod q$, one has

$$\left|S_\chi(n)\right| < \left(C_2(n) + \frac{|L(1,\chi)|}{\pi}\right) q^{1/2}, \tag{6}$$

where

$$C_2(n) = \frac{2\zeta(3)\, n!}{(2\pi)^{n+1}} \sum_{1\leq m\leq (n-1)/2} \frac{(2\pi)^{n-2m}}{(n-2m)!}. \tag{7}$$

In [5], Peral used the Gauss sums and adequate Fourier expansion to greatly improve the result in Proposition 1.

Proposition 2. *(a) Assume that $\chi(-1) = 1$ is a primitive nonprincipal character modulo q, and then*

$$\left|S_\chi(n)\right| \leq q^{1/2}\left(\frac{n-1}{2(n+1)}\right). \tag{8}$$

(b) Assume that $\chi(-1) = -1$ is a primitive character modulo q; then,

$$\left|S_\chi(n) + \frac{\tau(\chi) L(1,\overline{\chi})}{\pi i}\right| \leq q^{1/2}\left(\frac{n}{\pi}\int_0^1 \ln\frac{1}{2\sin(\pi t)} t^{n-1}\,dt\right). \tag{9}$$

Furthermore, Liu and Zhang [6] gave an upper bound for $S_\chi(n)$ when χ is a nonprincipal character modulo q.

It may be interesting to consider the mean value of certain character sums. For example, Burgess [7] proved that

$$\sideset{}{^*}\sum_{\chi \bmod q} \sum_{n=1}^{q} \left|\sum_{m=1}^{h} \chi(n+m)\right|^4 \leq 8 d^2(q)\, q^2 h^2, \tag{10}$$

where $\sum^*$ denotes the summation over primitive characters modulo q, and $d(q)$ is the Dirichlet divisor function. Xu and Zhang studied the power mean

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=\pm 1}} \left|\sum_{1\leq a\leq q/4} \chi(a)\right|^4 \tag{11}$$

in [8, 9] and obtained some sharper results.

In this paper, we study the fourth power mean of certain character sums

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \left|\sum_{1\leq a\leq q/4} a\chi(a)\right|^4, \qquad \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \left|\sum_{1\leq a\leq q/4} a\chi(a)\right|^4 \tag{12}$$

and give a few asymptotic formulae.

Theorem 3. *Let $q \geq 5$ be an odd number. Then one has*

$$\begin{aligned}
&\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} \left|\sum_{1\leq a\leq q/4} a\chi(a)\right|^4 \\
&= \frac{7}{2^{17}\cdot 3^2} q^6 J(q) \prod_{p\mid q} \frac{\left(1-1/p^2\right)^3}{1+1/p^2} \\
&+ \frac{35}{2^{16}\cdot 3^2\cdot 17} q^6 J(q) \prod_{p\mid q} \frac{\left(1-1/p^4\right)^3}{1+1/p^4} \\
&- \frac{L^2(3,\chi_4)}{2^{14}\cdot 3^2\cdot \pi\cdot L(5,\chi_4)} \\
&\times q^6 J(q) \prod_{p\mid q} \frac{\left(1-1/p^2\right)^2\left(1-\chi_4(p)/p^3\right)^2}{1-\chi_4(p)/p^5} \\
&+ \frac{L(3,\chi_4)}{2^8\cdot 5\cdot 3^2\cdot \pi^2} q^6 J(q) \\
&\times \prod_{p\mid q}\left(1-\chi_4(p)/p^3\right)\left(1-1/p^4\right) \\
&\times \prod_{p\nmid q}\left(1+\frac{1}{p^2\left(1-\chi_4(p)/p\right)}\right) + O\left(q^{6+\epsilon}\right),
\end{aligned} \tag{13}$$

where $J(q)$ is the number of primitive characters modulo q, χ_4 is the nonprincipal character modulo 4, and ϵ is any fixed positive real number.

Theorem 4. *Let $q \geq 5$ be an odd number. Then one has*

$$\begin{aligned}
&\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \left|\sum_{1\leq a\leq q/4} a\chi(a)\right|^4 \\
&= \frac{385}{2^{20}\cdot 51}\cdot q^6 J(q) \prod_{p\mid q} \frac{\left(1-1/p^4\right)^3}{1+1/p^4} \\
&+ \frac{15 L^4(3,\chi_4)}{\pi^{12}}\cdot q^6 J(q) \prod_{p\mid q} \frac{\left(1-\chi_4(p)/p^3\right)^3}{1+\chi_4(p)/p^3} \\
&- \frac{1423\pi L^2(3,\chi_4)}{2^{16}\cdot 5^2\cdot 3^4\cdot L(7,\chi_4)} \\
&\cdot q^6 J(q) \prod_{p\mid q} \frac{\left(1-\chi_4(p)/p^3\right)^2\left(1-1/p^4\right)^2}{1-\chi_4(p)/p^7} \\
&+ \frac{3}{2^{16}}\cdot q^6 J(q) \prod_{p\mid q} \frac{\left(1-1/p^2\right)^3}{1+1/p^2}
\end{aligned}$$

$$- \frac{L^2(3,\chi_4)}{2^{12}\cdot\pi\cdot L(5,\chi_4)} \cdot q^6 J(q)\prod_{p\mid q}\frac{\left(1-1/p^2\right)^2\left(1-\chi_4(p)/p^3\right)^2}{1-\chi_4(p)/p^5} + \frac{L(3,\chi_4)}{2^{10}\cdot 3\cdot\pi^2}\cdot q^6 J(q) \times\prod_{p\mid q}\left(1-\chi_4(p)/p^3\right)\left(1-1/p^4\right) \times\prod_{p\nmid q}\left(1+\frac{1}{p^2\left(1-\chi_4(p)/p\right)}\right)+O\left(q^{6+\epsilon}\right). \quad (14)$$

From Theorems 3 and 4, we immediately get the following corollaries.

Corollary 5. *Let $p\ge 5$ be a prime. Then one has*

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{1\le a\le p/4} a\chi(a)\right|^4 = \frac{21}{2^{17}\cdot 17}p^7 - \frac{L^2(3,\chi_4)}{2^{14}\cdot 3^2\cdot\pi\cdot L(5,\chi_4)}p^7 + \frac{L(3,\chi_4)}{2^8\cdot 5\cdot 3^2\cdot\pi^2}p^7\prod_{p_1}\left(1+\frac{1}{p_1^2\left(1-\chi_4(p_1)/p_1\right)}\right) + O\left(p^{6+\epsilon}\right), \quad (15)$$

where $\prod_{p_1}$ denotes the product over all primes.

Corollary 6. *Let $p\ge 5$ be a prime. Then*

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=1\\ \chi\ne\chi_0}}\left|\sum_{1\le a\le p/4} a\chi(a)\right|^4 = \frac{2833}{2^{20}\cdot 51}p^7 + \frac{15L^4(3,\chi_4)}{\pi^{12}}p^7 - \frac{1423\pi L^2(3,\chi_4)}{2^{16}\cdot 5^2\cdot 3^4\cdot L(7,\chi_4)}p^7 - \frac{L^2(3,\chi_4)}{2^{12}\cdot\pi\cdot L(5,\chi_4)}p^7 + \frac{L(3,\chi_4)}{2^{10}\cdot 3\cdot\pi^2}p^7\prod_{p_1}\left(1+\frac{1}{p_1^2\left(1-\chi_4(p_1)/p_1\right)}\right) + O\left(p^{6+\epsilon}\right). \quad (16)$$

Remark 7. It seems that the contributions of odd and even primitive characters to the fourth power moment of character sums over $[1,q/4]$ are very different.

2. Express the Character Sum in terms of Gauss Sums and L-Functions (I)

Let χ be an odd primitive character modulo q. In this section, we will express $\sum_{1\le a\le q/4} a\chi(a)$ in terms of Gauss sums and Dirichlet L-functions. We need the following lemmas.

Lemma 8. *Suppose that $q\ge 5$ is an odd number, and χ is an odd character modulo q.*

(i) *For $q\equiv 1 \pmod 4$, one has*

$$\sum_{1\le a\le (3q-3)/4} a\chi(a) = \sum_{1\le a\le q} a\chi(a) + q\sum_{1\le a\le (q-1)/4}\chi(a) - \sum_{1\le a\le (q-1)/4} a\chi(a),$$
$$\sum_{1\le a\le (3q-3)/4}\chi(a) = \sum_{1\le a\le (q-1)/4}\chi(a). \quad (17)$$

(ii) *For $q\equiv 3 \pmod 4$, one has*

$$\sum_{1\le a\le (3q-1)/4} a\chi(a) = \sum_{1\le a\le q} a\chi(a) + q\sum_{1\le a\le (q-3)/4}\chi(a) - \sum_{1\le a\le (q-3)/4} a\chi(a),$$
$$\sum_{1\le a\le (3q-1)/4}\chi(a) = \sum_{1\le a\le (q-3)/4}\chi(a). \quad (18)$$

Proof. It is easy to show that

$$\sum_{1\le a\le (3q-3)/4} a\chi(a) = \sum_{1\le a\le q} a\chi(a) - \sum_{(3q+1)/4\le a\le q} a\chi(a) = \sum_{1\le a\le q} a\chi(a) - \sum_{1\le b\le (q-1)/4}(q-b)\chi(q-b) = \sum_{1\le a\le q} a\chi(a) + q\sum_{1\le a\le (q-1)/4}\chi(a) - \sum_{1\le a\le (q-1)/4} a\chi(a),$$

$$\sum_{1\le a\le(3q-3)/4}\chi(a) = \sum_{1\le a\le q}\chi(a) - \sum_{(3q+1)/4\le a\le q}\chi(a) = \sum_{1\le a\le(q-1)/4}\chi(a). \tag{19}$$

This proves (i). Similarly, we can deduce (ii). □

Lemma 9. *Suppose that $q\ge 5$ is an odd number, and χ is an odd character modulo q. Let χ_4 be the nonprincipal character modulo 4. For $q\equiv 1 \pmod 4$, one has*

$$\begin{aligned}\sum_{a=1}^{4q} a^2\chi(a)\chi_4(a) &= 16\chi(4)q\sum_{a=1}^{q} a\chi(a) + 16\chi(4)q^2\sum_{1\le a\le(q-1)/4}\chi(a)\\ &\quad - 64\chi(4)q\sum_{1\le a\le(q-1)/4} a\chi(a).\end{aligned} \tag{20}$$

Proof. Note that $\chi_4(1)=1$ and $\chi_4(3)=-1$, and we get

$$\begin{aligned}\sum_{a=1}^{4q} a^2\chi(a)\chi_4(a) &= \sum_{a=0}^{q-1}(4a+1)^2\chi(4a+1)\\ &\quad - \sum_{a=0}^{q-1}(4a+3)^2\chi(4a+3).\end{aligned} \tag{21}$$

First we have

$$\begin{aligned}\sum_{a=0}^{q-1}(4a+1)^2\chi(4a+1) &= 16\sum_{a=0}^{q-1}a^2\chi(4a+1) + 8\sum_{a=0}^{q-1}a\chi(4a+1)\\ &\quad + \sum_{a=0}^{q-1}\chi(4a+1)\\ &= 16\chi(4)\sum_{a=0}^{q-1}a^2\chi\left(a+\overline{4}\right) + 8\chi(4)\sum_{a=0}^{q-1}a\chi\left(a+\overline{4}\right)\\ &= 16\chi(4)\sum_{a=0}^{q-1}\left(a+\overline{4}\right)^2\chi\left(a+\overline{4}\right)\\ &\quad + 8\chi(4)\left(1-4\cdot\overline{4}\right)\sum_{a=0}^{q-1}\left(a+\overline{4}\right)\chi\left(a+\overline{4}\right),\end{aligned} \tag{22}$$

where $\overline{4}$ is the inverse of 4 modulo q with $4\cdot\overline{4}\equiv 1 \pmod q$ and $1\le\overline{4}\le q$. Since $q\equiv 1 \pmod 4$, we get $\overline{4}=(3q+1)/4$. Then from Lemma 8, we have

$$\begin{aligned}\sum_{a=0}^{q-1}\left(a+\overline{4}\right)^2\chi\left(a+\overline{4}\right) &= \sum_{0\le a\le(q-1)/4}\left(a+\overline{4}\right)^2\chi\left(a+\overline{4}\right)\\ &\quad + \sum_{(q-1)/4<a\le q-1}\left(a+\overline{4}\right)^2\chi\left(a+\overline{4}\right)\\ &= \sum_{0\le a\le(q-1)/4}\left(a+\overline{4}\right)^2\chi\left(a+\overline{4}\right)\\ &\quad + \sum_{(q-1)/4<a\le q-1}\left(a+\overline{4}-q\right)^2\chi\left(a+\overline{4}-q\right)\\ &\quad + 2q\sum_{(q-1)/4<a\le q-1}\left(a+\overline{4}-q\right)\chi\left(a+\overline{4}-q\right)\\ &\quad + q^2\sum_{(q-1)/4<a\le q-1}\chi\left(a+\overline{4}-q\right)\\ &= \sum_{a=1}^{q}a^2\chi(a) + 2q\sum_{1\le a\le(3q-3)/4}a\chi(a)\\ &\quad + q^2\sum_{1\le a\le(3q-3)/4}\chi(a)\\ &= \sum_{a=1}^{q}a^2\chi(a) + 2q\sum_{a=1}^{q}a\chi(a)\\ &\quad + 3q^2\sum_{1\le a\le(q-1)/4}\chi(a)\\ &\quad - 2q\sum_{1\le a\le(q-1)/4}a\chi(a),\end{aligned}$$

$$\begin{aligned}\sum_{a=0}^{q-1}\left(a+\overline{4}\right)\chi\left(a+\overline{4}\right) &= \sum_{0\le a\le(q-1)/4}\left(a+\overline{4}\right)\chi\left(a+\overline{4}\right)\\ &\quad + \sum_{(q-1)/4<a\le q-1}\left(a+\overline{4}\right)\chi\left(a+\overline{4}\right)\\ &= \sum_{0\le a\le(q-1)/4}\left(a+\overline{4}\right)\chi\left(a+\overline{4}\right)\\ &\quad + \sum_{(q-1)/4<a\le q-1}\left(a+\overline{4}-q\right)\chi\left(a+\overline{4}-q\right)\\ &\quad + q\sum_{(q-1)/4<a\le q-1}\chi\left(a+\overline{4}-q\right)\end{aligned}$$

$$= \sum_{a=1}^{q} a\chi(a) + q \sum_{(q-1)/4 \le a \le q-1} \chi\left(a + \overline{4} - q\right) = \sum_{a=1}^{q} a\chi(a) + q \sum_{1 \le a \le (3q-3)/4} \chi(a) = \sum_{a=1}^{q} a\chi(a) + q \sum_{1 \le a \le (q-1)/4} \chi(a). \quad (23)$$

Therefore

$$\sum_{a=0}^{q-1} (4a+1)^2 \chi(4a+1) = 16\chi(4) \sum_{a=0}^{q-1} \left(a + \overline{4}\right)^2 \chi\left(a + \overline{4}\right) + 8\chi(4)\left(1 - 4\cdot\overline{4}\right) \sum_{a=0}^{q-1} \left(a + \overline{4}\right) \chi\left(a + \overline{4}\right) \quad (24)$$
$$= 16\chi(4) \sum_{a=1}^{q} a^2 \chi(a) + 8\chi(4) q \sum_{a=1}^{q} a\chi(a) + 24\chi(4) q^2 \sum_{1 \le a \le (q-1)/4} \chi(a) - 32\chi(4) q \sum_{1 \le a \le (q-1)/4} a\chi(a).$$

On the other hand, we get

$$\sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) = 16 \sum_{a=0}^{q-1} a^2 \chi(4a+3) + 24 \sum_{a=0}^{q-1} a\chi(4a+3) + 9 \sum_{a=0}^{q-1} \chi(4a+3) \quad (25)$$
$$= 16\chi(4) \sum_{a=0}^{q-1} a^2 \chi\left(a + 3\cdot\overline{4}\right) + 24\chi(4) \sum_{a=0}^{q-1} a\chi\left(a + 3\cdot\overline{4}\right).$$

Since $\overline{4} = (3q+1)/4$ and $3\cdot\overline{4} = 2q + (q+3)/4$, we have

$$\sum_{a=0}^{q-1} (4a+3)^2 \chi(4a+3) = 16\chi(4) \sum_{a=0}^{q-1} a^2 \chi\left(a + \frac{q+3}{4}\right) + 24\chi(4) \sum_{a=0}^{q-1} a\chi\left(a + \frac{q+3}{4}\right)$$
$$= 16\chi(4) \sum_{a=0}^{q-1} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) + 24\left(1 - \frac{q+3}{3}\right)\chi(4) \times \sum_{a=0}^{q-1} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right). \quad (26)$$

Note that

$$\sum_{a=0}^{q-1} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) = \sum_{0 \le a \le (3q-3)/4} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) + \sum_{(3q+1)/4 \le a \le q-1} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right)$$
$$= \sum_{0 \le a \le (3q-3)/4} \left(a + \frac{q+3}{4}\right)^2 \chi\left(a + \frac{q+3}{4}\right) + \sum_{(3q+1)/4 \le a \le q-1} \left(a + \frac{q+3}{4} - q\right)^2 \chi\left(a + \frac{q+3}{4} - q\right) + 2q \sum_{(3q+1)/4 \le a \le q-1} \left(a + \frac{q+3}{4} - q\right) \chi\left(a + \frac{q+3}{4} - q\right) + q^2 \sum_{(3q+1)/4 \le a \le q-1} \chi\left(a + \frac{q+3}{4} - q\right)$$
$$= \sum_{a=1}^{q} a^2 \chi(a) + q^2 \sum_{1 \le a \le (q-1)/4} \chi(a) + 2q \sum_{1 \le a \le (q-1)/4} a\chi(a),$$

$$\sum_{a=0}^{q-1} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right) = \sum_{0 \le a \le (3q-3)/4} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right) + \sum_{(3q+1)/4 \le a \le q-1} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right)$$
$$= \sum_{0 \le a \le (3q-3)/4} \left(a + \frac{q+3}{4}\right) \chi\left(a + \frac{q+3}{4}\right)$$

$$+\sum_{(3q+1)/4\le a\le q-1}\left(a+\frac{q+3}{4}-q\right)\chi\left(a+\frac{q+3}{4}-q\right)$$
$$+q\sum_{(3q+1)/4\le a\le q-1}\chi\left(a+\frac{q+3}{4}-q\right)$$
$$=\sum_{a=1}^{q}a\chi(a)+q\sum_{1\le a\le (q-1)/4}\chi(a), \tag{27}$$

so we get

$$\sum_{a=0}^{q-1}(4a+3)^2\chi(4a+3)$$
$$=16\chi(4)\sum_{a=0}^{q-1}\left(a+\frac{q+3}{4}\right)^2\chi\left(a+\frac{q+3}{4}\right)$$
$$+24\left(1-\frac{q+3}{3}\right)\chi(4)$$
$$\times\sum_{a=0}^{q-1}\left(a+\frac{q+3}{4}\right)\chi\left(a+\frac{q+3}{4}\right)$$
$$=16\chi(4)\sum_{a=1}^{q}a^2\chi(a)-8\chi(4)q\sum_{a=1}^{q}a\chi(a)$$
$$+8\chi(4)q^2\sum_{1\le a\le (q-1)/4}\chi(a)+32\chi(4)q\sum_{1\le a\le (q-1)/4}a\chi(a). \tag{28}$$

Now combine (21)–(28); we have

$$\sum_{a=1}^{4q}a^2\chi(a)\chi_4(a)$$
$$=\sum_{a=0}^{q-1}(4a+1)^2\chi(4a+1)$$
$$-\sum_{a=0}^{q-1}(4a+3)^2\chi(4a+3) \tag{29}$$
$$=16\chi(4)q\sum_{a=1}^{q}a\chi(a)$$
$$+16\chi(4)q^2\sum_{1\le a\le (q-1)/4}\chi(a)$$
$$-64\chi(4)q\sum_{1\le a\le (q-1)/4}a\chi(a).$$

□

Lemma 10. *Suppose that $q\ge 5$ is an odd number, and χ is an odd character modulo q. Let χ_4 be the nonprincipal character modulo 4. For $q\equiv 3 \pmod 4$, one has*

$$\sum_{a=1}^{4q}a^2\chi(a)\chi_4(a)=-16\chi(4)q\sum_{a=1}^{q}a\chi(a)$$
$$-16\chi(4)q^2\sum_{1\le a\le (q-1)/4}\chi(a) \tag{30}$$
$$+64\chi(4)q\sum_{1\le a\le (q-1)/4}a\chi(a).$$

Proof. For $q\equiv 3 \pmod 4$, we get $\overline{4}=(q+1)/4$ and $3\cdot\overline{4}=(3q+3)/4$. Using the methods of proving Lemma 9, we have

$$\sum_{a=1}^{4q}a^2\chi(a)\chi_4(a)$$
$$=\sum_{a=0}^{q-1}(4a+1)^2\chi(4a+1)-\sum_{a=0}^{q-1}(4a+3)^2\chi(4a+3), \tag{31}$$

$$\sum_{a=0}^{q-1}(4a+1)^2\chi(4a+1)$$
$$=16\chi(4)\sum_{a=0}^{q-1}\left(a+\overline{4}\right)^2\chi\left(a+\overline{4}\right) \tag{32}$$
$$+8\chi(4)\left(1-4\cdot\overline{4}\right)\sum_{a=0}^{q-1}\left(a+\overline{4}\right)\chi\left(a+\overline{4}\right),$$

$$\sum_{a=0}^{q-1}(4a+3)^2\chi(4a+3)$$
$$=16\sum_{a=0}^{q-1}a^2\chi(4a+3)+24\sum_{a=0}^{q-1}a\chi(4a+3)$$
$$+9\sum_{a=0}^{q-1}\chi(4a+3)$$
$$=16\chi(4)\sum_{a=0}^{q-1}a^2\chi\left(a+3\cdot\overline{4}\right)$$
$$+24\chi(4)\sum_{a=0}^{q-1}a\chi\left(a+3\cdot\overline{4}\right)$$
$$=16\chi(4)\sum_{a=0}^{q-1}\left(a+\frac{3q+3}{4}\right)^2\chi\left(a+\frac{3q+3}{4}\right)$$
$$-24\chi(4)q\sum_{a=0}^{q-1}\left(a+\frac{3q+3}{4}\right)\chi\left(a+\frac{3q+3}{4}\right). \tag{33}$$

It is not hard to show that

$$\begin{aligned}
&\sum_{a=0}^{q-1}\left(a+\bar{4}\right)^2\chi\left(a+\bar{4}\right)\\
&=\sum_{0\le a\le(3q-1)/4}\left(a+\bar{4}\right)^2\chi\left(a+\bar{4}\right)\\
&\quad+\sum_{(3q-1)/4<a\le q-1}\left(a+\bar{4}\right)^2\chi\left(a+\bar{4}\right)\\
&=\sum_{0\le a\le(3q-1)/4}\left(a+\bar{4}\right)^2\chi\left(a+\bar{4}\right)\\
&\quad+\sum_{(3q-1)/4<a\le q-1}\left(a+\bar{4}-q\right)^2\chi\left(a+\bar{4}-q\right)\\
&\quad+2q\sum_{(3q-1)/4<a\le q-1}\left(a+\bar{4}-q\right)\chi\left(a+\bar{4}-q\right)\\
&\quad+q^2\sum_{(3q-1)/4<a\le q-1}\chi\left(a+\bar{4}-q\right)\\
&=\sum_{a=1}^{q}a^2\chi(a)+q^2\sum_{1\le a\le(q-3)/4}\chi(a)\\
&\quad+2q\sum_{1\le a\le(q-3)/4}a\chi(a),\\
&\sum_{a=0}^{q-1}\left(a+\bar{4}\right)\chi\left(a+\bar{4}\right)\\
&=\sum_{0\le a\le(3q-1)/4}\left(a+\bar{4}\right)\chi\left(a+\bar{4}\right)\\
&\quad+\sum_{(3q-1)/4<a\le q-1}\left(a+\bar{4}\right)\chi\left(a+\bar{4}\right)\\
&=\sum_{0\le a\le(3q-1)/4}\left(a+\bar{4}\right)\chi\left(a+\bar{4}\right)\\
&\quad+\sum_{(3q-1)/4<a\le q-1}\left(a+\bar{4}-q\right)\chi\left(a+\bar{4}-q\right)\\
&\quad+q\sum_{(3q-1)/4<a\le q-1}\chi\left(a+\bar{4}-q\right)\\
&=\sum_{a=1}^{q}a\chi(a)+q\sum_{1\le a\le(q-3)/4}\chi(a).
\end{aligned}\tag{34}$$

Then by (32), we have

$$\begin{aligned}
&\sum_{a=0}^{q-1}(4a+1)^2\chi(4a+1)\\
&\quad=16\chi(4)\sum_{a=0}^{q-1}\left(a+\bar{4}\right)^2\chi\left(a+\bar{4}\right)\\
&\qquad+8\chi(4)\left(1-4\times\bar{4}\right)\sum_{a=0}^{q-1}\left(a+\bar{4}\right)\chi\left(a+\bar{4}\right)\\
&\quad=16\chi(4)\sum_{a=1}^{q}a^2\chi(a)\\
&\qquad-8\chi(4)q\sum_{a=1}^{q}a\chi(a)+8\chi(4)q^2\sum_{1\le a\le(q-3)/4}\chi(a)\\
&\qquad+32\chi(4)q\sum_{1\le a\le(q-3)/4}a\chi(a).
\end{aligned}\tag{35}$$

On the other hand, by Lemma 8, we get

$$\begin{aligned}
&\sum_{a=0}^{q-1}\left(a+\frac{3q+3}{4}\right)^2\chi\left(a+\frac{3q+3}{4}\right)\\
&=\sum_{0\le a\le(q-3)/4}\left(a+\frac{3q+3}{4}\right)^2\chi\left(a+\frac{3q+3}{4}\right)\\
&\quad+\sum_{(q+1)/4\le a\le q-1}\left(a+\frac{3q+3}{4}\right)^2\chi\left(a+\frac{3q+3}{4}\right)\\
&=\sum_{0\le a\le(q-3)/4}\left(a+\frac{3q+3}{4}\right)^2\chi\left(a+\frac{3q+3}{4}\right)\\
&\quad+\sum_{(q+1)/4\le a\le q-1}\left(a+\frac{3q+3}{4}-q\right)^2\chi\left(a+\frac{3q+3}{4}-q\right)\\
&\quad+2q\sum_{(q+1)/4\le a\le q-1}\left(a+\frac{3q+3}{4}-q\right)\chi\left(a+\frac{3q+3}{4}-q\right)\\
&\quad+q^2\sum_{(q+1)/4\le a\le q-1}\chi\left(a+\frac{3q+3}{4}-q\right)\\
&=\sum_{a=1}^{q}a^2\chi(a)+2q\sum_{1\le a\le(3q-1)/4}a\chi(a)\\
&\quad+q^2\sum_{1\le a\le(3q-1)/4}\chi(a)\\
&=\sum_{a=1}^{q}a^2\chi(a)+2q\sum_{a=1}^{q}a\chi(a)\\
&\quad+3q^2\sum_{1\le a\le(q-3)/4}\chi(a)-2q\sum_{1\le a\le(q-3)/4}a\chi(a),\\
&\sum_{a=0}^{q-1}\left(a+\frac{3q+3}{4}\right)\chi\left(a+\frac{3q+3}{4}\right)\\
&=\sum_{0\le a\le(q-3)/4}\left(a+\frac{3q+3}{4}\right)\chi\left(a+\frac{3q+3}{4}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{(q+1)/4\le a\le q-1}\left(a+\frac{3q+3}{4}\right)\chi\left(a+\frac{3q+3}{4}\right)\\
&=\sum_{0\le a\le (q-3)/4}\left(a+\frac{3q+3}{4}\right)\chi\left(a+\frac{3q+3}{4}\right)\\
&\quad+\sum_{(q+1)/4\le a\le q-1}\left(a+\frac{3q+3}{4}-q\right)\chi\left(a+\frac{3q+3}{4}-q\right)\\
&\quad+q\sum_{(q+1)/4\le a\le q-1}\chi\left(a+\frac{3q+3}{4}-q\right)\\
&=\sum_{a=1}^{q}a\chi(a)+q\sum_{1\le a\le (3q-1)/4}\chi(a)\\
&=\sum_{a=1}^{q}a\chi(a)+q\sum_{1\le a\le (q-3)/4}\chi(a).
\end{aligned}\tag{36}
$$

Then from (33), we have

$$
\begin{aligned}
&\sum_{a=0}^{q-1}(4a+3)^2\chi(4a+3)\\
&\quad=16\chi(4)\sum_{a=0}^{q-1}\left(a+\frac{3q+3}{4}\right)^2\chi\left(a+\frac{3q+3}{4}\right)\\
&\qquad-24\chi(4)q\sum_{a=0}^{q-1}\left(a+\frac{3q+3}{4}\right)\chi\left(a+\frac{3q+3}{4}\right)\\
&\quad=16\chi(4)\sum_{a=1}^{q}a^2\chi(a)+8\chi(4)q\sum_{a=1}^{q}a\chi(a)\\
&\qquad+24\chi(4)q^2\sum_{1\le a\le (q-3)/4}\chi(a)\\
&\qquad-32\chi(4)q\sum_{1\le a\le (q-3)/4}a\chi(a).
\end{aligned}\tag{37}
$$

Combining (31), (35), and (37), we have

$$
\begin{aligned}
&\sum_{a=1}^{4q}a^2\chi(a)\chi_4(a)\\
&\quad=\sum_{a=0}^{q-1}(4a+1)^2\chi(4a+1)\\
&\qquad-\sum_{a=0}^{q-1}(4a+3)^2\chi(4a+3)\\
&\quad=-16\chi(4)q\sum_{a=1}^{q}a\chi(a)\\
&\qquad-16\chi(4)q^2\sum_{1\le a\le (q-3)/4}\chi(a)+64\chi(4)q\sum_{1\le a\le (q-3)/4}a\chi(a).
\end{aligned}\tag{38}
$$

□

Now we can express $\sum_{1\le a\le q/4}a\chi(a)$ in terms of Gauss sums and Dirichlet L-functions.

Theorem 11. *Let χ be an odd primitive character modulo odd integer $q\ge 5$, and let χ_4 be the nonprincipal character modulo 4. Then one has*

$$
\begin{aligned}
&\sum_{1\le a\le q/4}a\chi(a)\\
&\quad=\frac{q}{8\pi i}\tau(\chi)\Big(\overline{\chi}(2)L(1,\overline{\chi})-\overline{\chi}(4)L(1,\overline{\chi})\\
&\qquad+\frac{4}{\pi}L(2,\overline{\chi}\chi_4)\Big).
\end{aligned}\tag{39}
$$

Proof. By Lemmas 9 and 10, we get

$$
\begin{aligned}
&16\chi(4)q\sum_{a=1}^{q}a\chi(a)+16\chi(4)q^2\sum_{1\le a\le q/4}\chi(a)\\
&\quad-64\chi(4)q\sum_{1\le a\le q/4}a\chi(a)\\
&\quad=\begin{cases}\displaystyle\sum_{a=1}^{4q}a^2\chi(a)\chi_4(a), & \text{if } q\equiv 1 \pmod 4\\ \displaystyle-\sum_{a=1}^{4q}a^2\chi(a)\chi_4(a), & \text{if } q\equiv 3 \pmod 4\end{cases}\\
&\quad=\chi_4(q)\sum_{a=1}^{4q}a^2\chi\chi_4(a).
\end{aligned}\tag{40}
$$

From the Fourier expansion for primitive character sums (see [10] or [11])

$$
\sum_{a<\lambda q}\chi(a)=\begin{cases}\displaystyle\frac{\tau(\chi)}{\pi}\sum_{n=1}^{+\infty}\frac{\overline{\chi}(n)\sin(2\pi n\lambda)}{n}, & \text{if } \chi(-1)=1,\\ \displaystyle\frac{\tau(\chi)L(1,\overline{\chi})}{\pi i}-\frac{\tau(\chi)}{\pi i}\times\sum_{n=1}^{+\infty}\frac{\overline{\chi}(n)\cos(2\pi n\lambda)}{n}, & \text{if } \chi(-1)=-1,\end{cases}\tag{41}
$$

we easily have

$$
\sum_{1\le a\le q/4}\chi(a)=\frac{2+\overline{\chi}(2)-\overline{\chi}(4)}{2\pi i}\tau(\chi)L(1,\overline{\chi}).\tag{42}
$$

Note that $\chi\chi_4$ is a primitive character modulo $4q$ satisfying $\chi\chi_4(-1)=1$, and

$$
\begin{aligned}
\tau(\chi\chi_4)&=\sum_{a=1}^{4q}\chi\chi_4(a)\,e\left(\frac{a}{4q}\right)\\
&=\sum_{a=1}^{4}\sum_{b=1}^{q}\chi(4b+qa)\chi_4(4b+qa)\,e\left(\frac{4b+qa}{4q}\right)\\
&=\sum_{a=1}^{4}\sum_{b=1}^{q}\chi(4b)\chi_4(qa)\,e\left(\frac{b}{q}+\frac{a}{4}\right)
\end{aligned}
$$

$$= \chi(4)\chi_4(q)\left(\sum_{a=1}^{4}\chi_4(a)\,e\left(\frac{a}{4}\right)\right) \times\left(\sum_{b=1}^{q}\chi(b)\,e\left(\frac{b}{q}\right)\right) = 2i\chi(4)\chi_4(q)\tau(\chi), \tag{43}$$

then from (3) we have

$$\sum_{a=1}^{q} a\chi(a) = -\frac{q}{\pi i}\tau(\chi)L(1,\overline{\chi}),$$
$$\sum_{a=1}^{4q} a^2\chi\chi_4(a) = \frac{16q^2}{\pi^2}\tau(\chi\chi_4)L(2,\overline{\chi}\chi_4) = \frac{32i\chi(4)\chi_4(q)}{\pi^2}q^2\tau(\chi)L(2,\overline{\chi}\chi_4). \tag{44}$$

Therefore

$$-\frac{16\chi(4)}{\pi i}q^2\tau(\chi)L(1,\overline{\chi}) + \frac{8\chi(4)(2+\overline{\chi}(2)-\overline{\chi}(4))}{\pi i}q^2\tau(\chi)L(1,\overline{\chi}) - 64\chi(4)q\sum_{1\le a\le q/4}a\chi(a) = \frac{32i\chi(4)}{\pi^2}q^2\tau(\chi)L(2,\overline{\chi}\chi_4). \tag{45}$$

Then we have

$$\sum_{1\le a\le q/4}a\chi(a) = \frac{q}{8\pi i}\tau(\chi)\Big(\overline{\chi}(2)L(1,\overline{\chi}) - \overline{\chi}(4)L(1,\overline{\chi}) + \frac{4}{\pi}L(2,\overline{\chi}\chi_4)\Big). \tag{46}$$

□

3. Express the Character Sum in terms of Gauss Sums and L-Functions (II)

Let χ be an even primitive character modulo q. In this section, we express $\sum_{1\le a\le q/4}a\chi(a)$ in terms of Gauss sums and Dirichlet L-functions.

Lemma 12. *Let $q > 2$ be an odd number, and let χ be a nonprincipal character modulo q. Then*

$$4\chi(2)q\sum_{a=1}^{(q-1)/2}a\chi(a) = (2\chi(2)+1)q\sum_{a=1}^{q}a\chi(a) - (4\chi(2)-1)\sum_{a=1}^{q}a^2\chi(a),$$
$$4\chi(2)q\sum_{a=(q+1)/2}^{q}a\chi(a) = (2\chi(2)-1)q\sum_{a=1}^{q}a\chi(a) + (4\chi(2)-1)\sum_{a=1}^{q}a^2\chi(a). \tag{47}$$

Proof. We have

$$\begin{aligned}\sum_{a=1}^{q}(2a)^2\chi(2a) &= \sum_{a=1}^{(q-1)/2}(2a)^2\chi(2a) + \sum_{a=(q+1)/2}^{q}(2a)^2\chi(2a)\\ &= \sum_{a=1}^{(q-1)/2}(2a)^2\chi(2a) + \sum_{b=1}^{(q+1)/2}(2b+q-1)^2\chi(2b+q-1)\\ &= \sum_{a=1}^{(q-1)/2}(2a)^2\chi(2a) + \sum_{b=1}^{(q+1)/2}(2b-1)^2\chi(2b-1) + 2q\sum_{b=1}^{(q+1)/2}(2b-1)\chi(2b-1) + q^2\sum_{b=1}^{(q+1)/2}\chi(2b-1)\\ &= \sum_{a=1}^{q}a^2\chi(a) + 2q\sum_{a=1}^{(q+1)/2}(2a-1)\chi(2a-1) + q^2\sum_{a=1}^{(q+1)/2}\chi(2a-1).\end{aligned} \tag{48}$$

Since

$$\sum_{a=1}^{(q+1)/2}\chi(2a-1) + \sum_{a=1}^{(q-1)/2}\chi(2a) = \sum_{a=1}^{q}\chi(a) = 0,$$
$$\sum_{a=1}^{(q+1)/2}(2a-1)\chi(2a-1) + \sum_{a=1}^{(q-1)/2}2a\chi(2a) = \sum_{a=1}^{q}a\chi(a), \tag{49}$$

we have

$$\sum_{a=1}^{q}(2a)^2\chi(2a) = \sum_{a=1}^{q}a^2\chi(a)$$

$$+ 2q\sum_{a=1}^{q} a\chi(a) - 4\chi(2)q\sum_{a=1}^{(q-1)/2} a\chi(a) - \chi(2)q^2\sum_{a=1}^{(q-1)/2}\chi(a). \tag{50}$$

It is not hard to show that

$$\chi(2)q\sum_{a=1}^{(q-1)/2}\chi(a) = (1-2\chi(2))\sum_{a=1}^{q} a\chi(a), \qquad \chi(2)q\sum_{a=(q+1)/2}^{q}\chi(a) = (2\chi(2)-1)\sum_{a=1}^{q} a\chi(a). \tag{51}$$

Therefore

$$4\chi(2)q\sum_{a=1}^{(q-1)/2} a\chi(a) = (2\chi(2)+1)q \times \sum_{a=1}^{q} a\chi(a) - (4\chi(2)-1)\sum_{a=1}^{q} a^2\chi(a). \tag{52}$$

Note that

$$\sum_{a=1}^{(q-1)/2} a\chi(a) + \sum_{a=(q+1)/2}^{q} a\chi(a) = \sum_{a=1}^{q} a\chi(a), \tag{53}$$

we have

$$4\chi(2)q\sum_{a=(q+1)/2}^{q} a\chi(a) = (2\chi(2)-1)q \times \sum_{a=1}^{q} a\chi(a) + (4\chi(2)-1)\sum_{a=1}^{q} a^2\chi(a). \tag{54}$$

□

Lemma 13. *Let $q \geq 5$ be an odd number, and let χ be an nonprincipal even character modulo q. If $q \equiv 1 \pmod 4$, then*

$$16\chi(4)\sum_{a=1}^{q-1} a^2\chi(a) = (8\chi(4) - 2\chi(2) + 1) \times \sum_{a=1}^{q} a^2\chi(a) - 16q\chi(4) \times \sum_{a=1}^{(q-1)/4} a\chi(a) + 4q^2\chi(4)\sum_{a=1}^{(q-1)/4}\chi(a). \tag{55}$$

While if $q \equiv 3 \pmod 4$, we have

$$16\chi(4)\sum_{a=1}^{q-1} a^2\chi(a) = (8\chi(4) - 2\chi(2) + 1)\sum_{a=1}^{q} a^2\chi(a) - 16q\chi(4) \times \sum_{a=1}^{(q-3)/4} a\chi(a) + 4q^2\chi(4)\sum_{a=1}^{(q-3)/4}\chi(a). \tag{56}$$

Proof. First suppose that $q \equiv 1 \pmod 4$. Then $\overline{4} = (3q+1)/4$. We have

$$\begin{aligned}
16\chi(4)\sum_{a=1}^{q-1} a^2\chi(a) &= \sum_{a=1}^{q-1}(4a)^2\chi(4a) = \sum_{a=1}^{(q-1)/4}(4a)^2\chi(4a) \\
&\quad + \sum_{a=(q+3)/4}^{(2q-2)/4}(4a)^2\chi(4a) + \sum_{a=(2q+2)/4}^{(3q-3)/4}(4a)^2\chi(4a) \\
&\quad + \sum_{a=(3q+1)/4}^{q-1}(4a)^2\chi(4a) \\
&= \sum_{a=1}^{(q-1)/4}(4a)^2\chi(4a) \\
&\quad + \sum_{a=1}^{(q-1)/4}(4a+q-1)^2\chi(4a-1) \\
&\quad + \sum_{a=1}^{(q-1)/4}(4a+2q-2)^2\chi(4a-2) \\
&\quad + \sum_{a=1}^{(q-1)/4}(4a+3q-3)^2\chi(4a-3) \\
&= \sum_{a=1}^{(q-1)/4}(4a)^2\chi(4a) \\
&\quad + \sum_{a=1}^{(q-1)/4}(4a-1)^2\chi(4a-1) \\
&\quad + \sum_{a=1}^{(q-1)/4}(4a-2)^2\chi(4a-2) \\
&\quad + \sum_{a=1}^{(q-1)/4}(4a-3)^2\chi(4a-3)
\end{aligned}$$

$$+ 2q \sum_{a=1}^{(q-1)/4} (4a-1)\chi(4a-1)$$
$$+ 4q \sum_{a=1}^{(q-1)/4} (4a-2)\chi(4a-2)$$
$$+ 6q \sum_{a=1}^{(q-1)/4} (4a-3)\chi(4a-3)$$
$$+ q^2 \sum_{a=1}^{(q-1)/4} \chi(4a-1) + 4q^2 \sum_{a=1}^{(q-1)/4} \chi(4a-2)$$
$$+ 9q^2 \sum_{a=1}^{(q-1)/4} \chi(4a-3)$$
$$= \sum_{a=1}^{q-1} a^2\chi(a) + 8q\chi(4) \sum_{a=1}^{(q-1)/4} \left(a+\frac{q-1}{4}\right)\chi\left(a+\frac{q-1}{4}\right)$$
$$- 2q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a+\frac{q-1}{4}\right)$$
$$+ 16q\chi(4) \sum_{a=1}^{(q-1)/4} \left(a+\frac{q-1}{2}\right)\chi\left(a+\frac{q-1}{2}\right)$$
$$- 8q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a+\frac{q-1}{2}\right)$$
$$+ 24q\chi(4) \sum_{a=1}^{(q-1)/4} \left(a+\frac{3q-3}{4}\right)\chi\left(a+\frac{3q-3}{4}\right)$$
$$- 18q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a+\frac{3q-3}{4}\right)$$
$$+ q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a+\frac{q-1}{4}\right)$$
$$+ 4q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a+\frac{q-1}{2}\right)$$
$$+ 9q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi\left(a+\frac{3q-3}{4}\right)$$
$$= \sum_{a=1}^{q-1} a^2\chi(a) + 8q\chi(4) \sum_{a=(q+3)/4}^{(q-1)/2} a\chi(a)$$
$$+ 16q\chi(4) \sum_{a=(q+1)/2}^{(3q-3)/4} a\chi(a)$$
$$+ 24q\chi(4) \sum_{a=(3q+1)/4}^{q-1} a\chi(a)$$
$$- q^2\chi(4) \sum_{a=(q+3)/4}^{(q-1)/2} \chi(a) - 4q^2\chi(4) \sum_{a=(q+1)/2}^{(3q-3)/4} \chi(a)$$
$$- 9q^2\chi(4) \sum_{a=(3q+1)/4}^{q-1} \chi(a). \tag{57}$$

Note that $\sum_{1\le a\le (q-1)/2} \chi(a) = 0$ and $\sum_{a=1}^{q} a\chi(a) = 0$ for even character χ. By Lemma 12 we have

$$\begin{aligned}
16\chi(4)\sum_{a=1}^{q-1} a^2\chi(a)
&= \sum_{a=1}^{q} a^2\chi(a) - 16q\chi(4) \\
&\quad \times \sum_{a=1}^{(q-1)/4} a\chi(a) - 8q\chi(4) \sum_{a=1}^{(q-1)/2} a\chi(a) \\
&\quad + 4q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi(a) \\
&= \left(8\chi(4) - 2\chi(2) + 1\right) \\
&\quad \times \sum_{a=1}^{q} a^2\chi(a) - 16q\chi(4) \sum_{a=1}^{(q-1)/4} a\chi(a) \\
&\quad + 4q^2\chi(4) \sum_{a=1}^{(q-1)/4} \chi(a).
\end{aligned} \tag{58}$$

Now assume that $q \equiv 3 \pmod 4$. Then $\overline{4} = (q+1)/4$. We have

$$16\chi(4)\sum_{a=1}^{q-1} a^2\chi(a)$$
$$= \sum_{a=1}^{q-1} (4a)^2\chi(4a)$$
$$= \sum_{a=1}^{(q-3)/4} (4a)^2\chi(4a) + \sum_{a=(q+1)/4}^{(2q-2)/4} (4a)^2\chi(4a)$$
$$+ \sum_{a=(2q+2)/4}^{(3q-1)/4} (4a)^2\chi(4a) + \sum_{a=(3q+3)/4}^{q-1} (4a)^2\chi(4a)$$
$$= \sum_{a=1}^{(q-3)/4} (4a)^2\chi(4a)$$
$$+ \sum_{a=1}^{(q+1)/4} (4a+q-3)^2\chi(4a-3)$$
$$+ \sum_{a=1}^{(q+1)/4} (4a+2q-2)^2\chi(4a-2)$$

$$\begin{aligned}
&+\sum_{a=1}^{(q-3)/4}(4a+3q-1)^2\chi(4a-1)\\
&=\sum_{a=1}^{(q-3)/4}(4a)^2\chi(4a)+\sum_{a=1}^{(q-3)/4}(4a-1)^2\chi(4a-1)\\
&+\sum_{a=1}^{(q+1)/4}(4a-2)^2\chi(4a-2)\\
&+\sum_{a=1}^{(q+1)/4}(4a-3)^2\chi(4a-3)\\
&+6q\sum_{a=1}^{(q-3)/4}(4a-1)\chi(4a-1)\\
&+4q\sum_{a=1}^{(q+1)/4}(4a-2)\chi(4a-2)\\
&+2q\sum_{a=1}^{(q+1)/4}(4a-3)\chi(4a-3)\\
&+9q^2\sum_{a=1}^{(q-3)/4}\chi(4a-1)+4q^2\sum_{a=1}^{(q+1)/4}\chi(4a-2)\\
&+q^2\sum_{a=1}^{(q+1)/4}\chi(4a-3)\\
&=\sum_{a=1}^{q-1}a^2\chi(a)\\
&+24q\chi(4)\sum_{a=1}^{(q-3)/4}\left(a+\frac{3q-1}{4}\right)\chi\left(a+\frac{3q-1}{4}\right)\\
&-18q^2\chi(4)\sum_{a=1}^{(q-3)/4}\chi\left(a+\frac{3q-1}{4}\right)\\
&+16q\chi(4)\sum_{a=1}^{(q+1)/4}\left(a+\frac{q-1}{2}\right)\chi\left(a+\frac{q-1}{2}\right)\\
&-8q^2\chi(4)\sum_{a=1}^{(q+1)/4}\chi\left(a+\frac{q-1}{2}\right)\\
&+8q\chi(4)\sum_{a=1}^{(q+1)/4}\left(a+\frac{q-3}{4}\right)\chi\left(a+\frac{q-3}{4}\right)\\
&-2q^2\chi(4)\sum_{a=1}^{(q+1)/4}\chi\left(a+\frac{q-3}{4}\right)\\
&+9q^2\chi(4)\sum_{a=1}^{(q-3)/4}\chi\left(a+\frac{3q-1}{4}\right)\\
&+4q^2\chi(4)\sum_{a=1}^{(q+1)/4}\chi\left(a+\frac{q-1}{2}\right)\\
&+q^2\chi(4)\sum_{a=1}^{(q+1)/4}\chi\left(a+\frac{q-3}{4}\right)\\
&=\sum_{a=1}^{q-1}a^2\chi(a)+24q\chi(4)\sum_{a=(3q+3)/4}^{q-1}a\chi(a)\\
&+16q\chi(4)\sum_{a=(q+1)/2}^{(3q-1)/4}a\chi(a)\\
&+8q\chi(4)\sum_{a=(q+1)/4}^{(q-1)/2}a\chi(a)\\
&-9q^2\chi(4)\sum_{a=(3q+3)/4}^{q-1}\chi(a)\\
&-4q^2\chi(4)\sum_{a=(q+1)/2}^{(3q-1)/4}\chi(a)\\
&-q^2\chi(4)\sum_{a=(q+1)/4}^{(q-1)/2}\chi(a).
\end{aligned}\tag{59}$$

Note that $\sum_{1\le a\le (q-1)/2}\chi(a)=0$ and $\sum_{a=1}^{q}a\chi(a)=0$ for even character χ. By Lemma 12 we have

$$\begin{aligned}
16\chi(4)&\sum_{a=1}^{q-1}a^2\chi(a)\\
&=\sum_{a=1}^{q}a^2\chi(a)-16q\chi(4)\sum_{a=1}^{(q-3)/4}a\chi(a)\\
&-8q\chi(4)\sum_{a=1}^{(q-1)/2}a\chi(a)+4q^2\chi(4)\sum_{a=1}^{(q-3)/4}\chi(a)\\
&=(8\chi(4)-2\chi(2)+1)\sum_{a=1}^{q}a^2\chi(a)\\
&-16q\chi(4)\sum_{a=1}^{(q-3)/4}a\chi(a)\\
&+4q^2\chi(4)\sum_{a=1}^{(q-3)/4}\chi(a).
\end{aligned}\tag{60}$$

□

Now we express $\sum_{1\le a\le q/4}a\chi(a)$ in terms of Gauss sums and Dirichlet L-functions.

Theorem 14. *Let χ be an even primitive character modulo odd integer $q\ge 5$. Then one has*

$$\begin{aligned}
\sum_{1\le a\le q/4}a\chi(a)&=\frac{q}{16\pi^2}\left(\overline{\chi}(4)-2\overline{\chi}(2)-8\right)\tau(\chi)L\left(2,\overline{\chi}\right)\\
&+\frac{q}{4\pi}\tau(\chi)L\left(1,\overline{\chi}\chi_4\right).
\end{aligned}\tag{61}$$

Proof. By Lemma 13, (3), and (37), we have

$$\begin{aligned}16q\chi(4)\sum_{1\le a\le q/4}a\chi(a) &= (1-2\chi(2)-8\chi(4))\sum_{a=1}^{q}a^2\chi(a) \\ &\quad + 4q^2\chi(4)\sum_{1\le a\le q/4}\chi(a) \\ &= \frac{q^2}{\pi^2}(1-2\chi(2)-8\chi(4))\tau(\chi)L(2,\overline{\chi}) \\ &\quad + \frac{4q^2}{\pi}\chi(4)\tau(\chi)L(1,\overline{\chi}\chi_4).\end{aligned} \tag{62}$$

Therefore

$$\begin{aligned}\sum_{1\le a\le q/4}a\chi(a) &= \frac{q}{16\pi^2}(\overline{\chi}(4)-2\overline{\chi}(2)-8) \\ &\quad \times\tau(\chi)L(2,\overline{\chi})+\frac{q}{4\pi}\tau(\chi)L(1,\overline{\chi}\chi_4).\end{aligned} \tag{63}$$

□

4. Mean Values of Dirichlet L-Functions

In this section, we will study the mean values of Dirichlet L-functions, which will be used to prove Theorems 3 and 4.

Lemma 15. *Let q and r be integers with $q\ge 2$ and $(r,q)=1$. Then one has the identities*

$$\begin{aligned}\sum_{\chi \bmod q}^{*}\chi(r) &= \sum_{d\mid(q,r-1)}\mu\left(\frac{q}{d}\right)\phi(d), \\ J(q) &= \sum_{d\mid q}\mu\left(\frac{q}{d}\right)\phi(d),\end{aligned} \tag{64}$$

where $\sum_{\chi \bmod q}^{*}$ denotes the summation over all primitive characters modulo q, and $J(q)$ is the number of primitive characters modulo q.

Proof. This is Lemma 3 of [12]. □

Lemma 16. *Let $q\ge 2$ be an odd number, and let $k\ge 0$ be an integer. Then one has*

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{d(n)d(2^kn)}{n^2} = \frac{5+3k}{5}\cdot\frac{\zeta^4(2)}{\zeta(4)}\prod_{p\mid q}\frac{(1-1/p^2)^3}{1+1/p^2},$$

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{d(n)d(2^kn)}{n^4} = \frac{15k+17}{17}\cdot\frac{\zeta^4(4)}{\zeta(8)}\prod_{p\mid q}\frac{(1-1/p^4)^3}{1+1/p^4},$$

$$\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{d^2(n)}{n^2} = \frac{27}{80}\cdot\frac{\zeta^4(2)}{\zeta(4)}\prod_{p\mid q}\frac{(1-1/p^2)^3}{1+1/p^2},$$

$$\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{d^2(n)}{n^4} = \frac{3375}{4352}\cdot\frac{\zeta^4(4)}{\zeta(8)}\prod_{p\mid q}\frac{(1-1/p^4)^3}{1+1/p^4},$$

$$\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{d^2(n)\chi_4(n)}{n^3} = \frac{1}{(1-1/2^6)}\cdot\frac{L^4(3,\chi_4)}{\zeta(6)}\prod_{p\mid q}\frac{(1-\chi_4(p)/p^3)^3}{1+\chi_4(p)/p^3},$$

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{d(n)\tau_1(2^kn)}{n^2} = \frac{\zeta^2(2)L^2(3,\chi_4)}{L(5,\chi_4)}\times\prod_{p\mid q}\frac{(1-1/p^2)^2(1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5},$$

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\tau_1(n)\tau_1(2^kn)}{n^2} = \zeta(4)L(3,\chi_4)\prod_{p\mid q}\left(1-\frac{\chi_4(p)}{p^3}\right)\left(1-\frac{1}{p^4}\right)\times\prod_{p\nmid q}\left(1+\frac{1}{p^2(1-\chi_4(p)/p)}\right),$$

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{d(2^kn)\tau_2(n)}{n^3} = \left(\frac{15}{16}k+1\right)\cdot\frac{\zeta^2(4)L^2(3,\chi_4)}{L(7,\chi_4)}\times\prod_{p\mid q}\frac{(1-\chi_4(p)/p^3)^2(1-1/p^4)^2}{1-\chi_4(p)/p^7},$$

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{d(n)\tau_2(2^kn)}{n^3} = \frac{1}{2^k}\cdot\frac{\zeta^2(4)L^2(3,\chi_4)}{L(7,\chi_4)}\times\prod_{p\mid q}\frac{(1-\chi_4(p)/p^3)^2(1-1/p^4)^2}{1-\chi_4(p)/p^7},$$

$$\sum_{\substack{n=1\\(n,2q)=1}}^{\infty}\frac{\chi_4(n)d(n)\tau_2(n)}{n^2} = \frac{9}{16}\cdot\frac{\zeta^2(2)L^2(3,\chi_4)}{L(5,\chi_4)} \times\prod_{p|q}\frac{(1-1/p^2)^2(1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5},$$

$$\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{\tau_2(n)\tau_2(2^kn)}{n^2} = \frac{4}{2^k5}\cdot\zeta(4)L(3,\chi_4) \times\prod_{p|q}\left(1-\frac{\chi_4(p)}{p^3}\right)\left(1-\frac{1}{p^4}\right) \times\prod_{p\nmid q}\left(1+\frac{1}{p^2(1-\chi_4(p)/p)}\right), \tag{65}$$

where $d(n)=\sum_{d|n}1$, $\tau_1(n)=\sum_{d|n}(\chi_4(d)/d)$, and $\tau_2(n)=\sum_{d|n}(\chi_4(n/d)/d)$.

Proof. By the Euler product, we have

$$\begin{aligned}\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\frac{d(n)d(2^kn)}{n^2} &= \sum_{i=0}^{\infty}\sum_{\substack{m=1\\(m,2q)=1}}^{\infty}\frac{d(2^im)d(2^{k+i}m)}{(2^im)^2} \\ &= \sum_{i=0}^{\infty}\frac{d(2^i)d(2^{k+i})}{2^{2i}}\sum_{\substack{m=1\\(m,2q)=1}}^{\infty}\frac{d^2(m)}{m^2} \\ &= \left(\sum_{i=0}^{\infty}\frac{(i+1)(k+i+1)}{2^{2i}}\right)\prod_{p\nmid 2q}\left(\sum_{j=0}^{\infty}\frac{(j+1)^2}{p^{2j}}\right) \\ &= \frac{5/4+(3/4)k}{(1-1/2^2)^3}\prod_{p\nmid 2q}\frac{1+1/p^2}{(1-1/p^2)^3} \\ &= \frac{5+3k}{5}\prod_{p}\frac{1+1/p^2}{(1-1/p^2)^3}\prod_{p|q}\frac{(1-1/p^2)^3}{1+1/p^2} \\ &= \frac{5+3k}{5}\cdot\frac{\zeta^4(2)}{\zeta(4)}\prod_{p|q}\frac{(1-1/p^2)^3}{1+1/p^2}.\end{aligned} \tag{66}$$

Similarly, we can deduce the other identities. □

Lemma 17. *Let $q\ge 2$ be an odd number. For integers $k\ge 0$ and $l\ge 1$, one has*

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\chi(2^k)|L(1,\chi)|^4 = \frac{5+3k}{2^{k+1}\cdot 5}\cdot\frac{\zeta^4(2)}{\zeta(4)}J(q)\prod_{p\nmid q}\frac{(1-1/p^2)^3}{1+1/p^2}+O(q^{\epsilon}),$$

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}|L(2,\chi\chi_4)|^4 = \frac{3375}{8704}\cdot\frac{\zeta^4(4)}{\zeta(8)}J(q)\prod_{p|q}\frac{(1-1/p^4)^3}{1+1/p^4}+O(q^{\epsilon}),$$

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(2^k)L^2(1,\overline{\chi})L(1,\chi)L(2,\chi\chi_4) = \frac{\zeta^2(2)L^2(3,\chi_4)}{2^{k+1}L(5,\chi_4)}J(q) \times\prod_{p|q}\frac{(1-1/p^2)^2(1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5}+O(q^{\epsilon}),$$

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(2^l)L^2(1,\overline{\chi})L^2(2,\chi\chi_4)\ll q^{\epsilon},$$

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(2^l)L(1,\overline{\chi})L(2,\overline{\chi}\chi_4)L^2(2,\chi\chi_4)\ll q^{\epsilon},$$

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\chi(2^k)|L(1,\chi)|^2|L(2,\chi\chi_4)|^2 = \frac{\zeta(4)L(3,\chi_4)}{2^{k+1}}J(q)\prod_{p|q}\left(1-\frac{\chi_4(p)}{p^3}\right)\left(1-\frac{1}{p^4}\right) \times\prod_{p\nmid q}\left(1+\frac{1}{p^2(1-\chi_4(p)/p)}\right)+O(q^{\epsilon}). \tag{67}$$

Proof. We only prove the first formula since, similarly, we can get the others. Let $d(n)=\sum_{d|n}1$ be the divisor function. For $N\ge q^2$, by Abel's identity, we get

$$L^2(1,\chi) = \left(\sum_{n=1}^{\infty}\frac{\chi(n)}{n}\right)^2 = \sum_{1\le n\le N}\frac{\chi(n)d(n)}{n}+O\left(\frac{q^{1/2}\log q\log N}{N^{1/2}}\right). \tag{68}$$

For $(r, q) = 1$, from Lemma 15, we have

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \chi(r) = \frac{1}{2} \sum_{\chi \bmod q}^{*} \left(1 - \chi(-1)\right) \chi(r) \tag{69}$$

$$= \frac{1}{2} \sum_{\chi \bmod q}^{*} \chi(r) - \frac{1}{2} \sum_{\chi \bmod q}^{*} \chi(-r) = \frac{1}{2} \sum_{d|(q,r-1)} \mu\left(\frac{q}{d}\right) \phi(d) - \frac{1}{2} \sum_{d|(q,r+1)} \mu\left(\frac{q}{d}\right) \phi(d). \tag{70}$$

Then from Lemma 16 we get

$$\begin{aligned}
\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \chi\left(2^k\right) \left|L(1,\chi)\right|^4 &= \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \chi\left(2^k\right) \left| \sum_{1 \le n \le N} \frac{\chi(n) d(n)}{n} + O\left(\frac{q^{1/2} \log q \log N}{N^{1/2}}\right) \right|^2 \\
&= \sum_{1 \le n \le N} \sum_{1 \le m \le N} \frac{d(n) d(m)}{nm} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \chi\left(2^k n\right) \overline{\chi}(m) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\
&= \frac{1}{2} \sum_{\substack{1 \le n \le N \\ (n,q)=1}} \sum_{\substack{1 \le m \le N \\ (m,q)=1}} \frac{d(n) d(m)}{nm} \sum_{d|(q,2^k n - m)} \mu\left(\frac{q}{d}\right) \phi(d) \\
&\quad - \frac{1}{2} \sum_{\substack{1 \le n \le N \\ (n,q)=1}} \sum_{\substack{1 \le m \le N \\ (m,q)=1}} \frac{d(n) d(m)}{nm} \sum_{d|(q,2^k n + m)} \mu\left(\frac{q}{d}\right) \phi(d) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\
&= \frac{J(q)}{2^{k+1}} \sum_{\substack{1 \le n \le N/2^k \\ (n,q)=1}} \frac{d(n) d\left(2^k n\right)}{n^2} + \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \le n \le N \\ (n,q)=1 \\ 2^k n \equiv m (\bmod d) \\ 2^k n \ne m}} \sum_{\substack{1 \le m \le N \\ (m,q)=1}} \frac{d(n) d(m)}{nm} \\
&\quad - \frac{1}{2} \sum_{d|q} \mu\left(\frac{q}{d}\right) \phi(d) \sum_{\substack{1 \le n \le N \\ (n,q)=1 \\ 2^k n \equiv -m (\bmod d)}} \sum_{\substack{1 \le m \le N \\ (m,q)=1}} \frac{d(n) d(m)}{nm} + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\
&= \frac{J(q)}{2^{k+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d(n) d\left(2^k n\right)}{n^2} + O\left(\sum_{d|q} \phi(d) \sum_{1 \le n \le N} \sum_{(1-2^k n)/d \le l \le (N - 2^k n)/d} \frac{d(n) d\left(ld + 2^k n\right)}{n\left(ld + 2^k n\right)} \right) \\
&\quad + O\left(\sum_{d|q} \phi(d) \sum_{1 \le n \le N} \sum_{(1+2^k n)/d \le l \le (N + 2^k n)/d} \frac{d(n) d\left(ld - 2^k n\right)}{n\left(ld - 2^k n\right)} \right) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\
&= \frac{J(q)}{2^{k+1}} \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d(n) d\left(2^k n\right)}{n^2} + O\left(N^{\epsilon}\right) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right) \\
&= \frac{5 + 3k}{2^{k+1} \cdot 5} \cdot \frac{\zeta^4(2)}{\zeta(4)} J(q) \prod_{p|q} \frac{\left(1 - 1/p^2\right)^3}{1 + 1/p^2} + O\left(N^{\epsilon}\right) + O\left(\frac{q^{3/2} \log q \log^3 N}{N^{1/2}}\right).
\end{aligned} \tag{71}$$

Now taking $N = q^4$, we immediately get

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \chi\left(2^k\right)\left|L(1,\chi)\right|^4 = \frac{5+3k}{2^{k+1}\cdot 5}\cdot\frac{\zeta^4(2)}{\zeta(4)}J(q) \times \prod_{p\mid q}\frac{\left(1-1/p^2\right)^3}{1+1/p^2} + O\left(q^{\epsilon}\right). \quad (72)$$

□

Lemma 18. *Let $q \geq 2$ be an odd number. For integers $k \geq 0$ and $l \geq 1$, one has*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \overline{\chi}\left(2^k\right)\left|L(2,\overline{\chi})\right|^4 = \frac{15k+17}{2^{2k+1}\cdot 17}\cdot\frac{\zeta^4(4)}{\zeta(8)}\cdot J(q)\prod_{p\mid q}\frac{\left(1-1/p^4\right)^3}{1+1/p^4} + O\left(q^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} L^2(2,\overline{\chi})L^2(1,\chi\chi_4) = \frac{1}{2\left(1-1/2^6\right)}\cdot\frac{L^4(3,\chi_4)}{\zeta(6)}\cdot J(q)\prod_{p\mid q}\frac{\left(1-\chi_4(p)/p^3\right)^3}{1+\chi_4(p)/p^3} + O\left(q^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \overline{\chi}\left(2^l\right)L^2(2,\overline{\chi})L^2(1,\chi\chi_4) \ll q^{\epsilon},$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \chi\left(2^k\right)L^2(2,\overline{\chi})L(1,\chi\chi_4)L(2,\chi) = \frac{((15/16)k+1)}{2^{2k+1}}\cdot\frac{\zeta^2(4)L^2(3,\chi_4)}{L(7,\chi_4)} \cdot J(q)\prod_{p\mid q}\frac{\left(1-\chi_4(p)/p^3\right)^2\left(1-1/p^4\right)^2}{1-\chi_4(p)/p^7} + O\left(q^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \overline{\chi}\left(2^k\right)L^2(2,\overline{\chi})L(1,\chi\chi_4)L(2,\chi) = \frac{1}{2^{k+1}}\cdot\frac{\zeta^2(4)L^2(3,\chi_4)}{L(7,\chi_4)} \cdot J(q)\prod_{p\mid q}\frac{\left(1-\chi_4(p)/p^3\right)^2\left(1-1/p^4\right)^2}{1-\chi_4(p)/p^7} + O\left(q^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \left|L(1,\overline{\chi}\chi_4)\right|^4 = \frac{27}{160}\cdot\frac{\zeta^4(2)}{\zeta(4)}\cdot J(q)\prod_{p\mid q}\frac{\left(1-1/p^2\right)^3}{1+1/p^2} + O\left(q^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \chi\left(2^l\right)L^2(1,\overline{\chi}\chi_4)L(1,\chi\chi_4)L(2,\chi) \ll q^{\epsilon},$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} L^2(1,\overline{\chi}\chi_4)L(1,\chi\chi_4)L(2,\chi) = \frac{9}{32}\cdot\frac{\zeta^2(2)L^2(3,\chi_4)}{L(5,\chi_4)} \cdot J(q)\prod_{p\mid q}\frac{\left(1-1/p^2\right)^2\left(1-\chi_4(p)/p^3\right)^2}{1-\chi_4(p)/p^5} + O\left(q^{\epsilon}\right),$$

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=1}}^{*} \overline{\chi}\left(2^k\right)\left|L(1,\overline{\chi}\chi_4)\right|^2\left|L(2,\overline{\chi})\right|^2 = \frac{2}{2^{2k}5}\cdot\zeta(4)L(3,\chi_4) \cdot J(q)\prod_{p\mid q}\left(1-\chi_4(p)/p^3\right)\left(1-1/p^4\right) \times\prod_{p\nmid q}\left(1+\frac{1}{p^2\left(1-\chi_4(p)/p\right)}\right) + O\left(q^{\epsilon}\right). \quad (73)$$

Proof. By Lemma 17 and the methods proving Lemma 18, we can get this lemma. □

5. Proof of Theorems 3 and 4

First we prove Theorem 3. By Theorem 11 and Lemma 18, we have

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \left|\sum_{1\leq a\leq q/4} a\chi(a)\right|^4 = \frac{q^6}{8^4\pi^4}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} \left|\overline{\chi}(2)L(1,\overline{\chi}) - \overline{\chi}(4)L(1,\overline{\chi}) + \frac{4}{\pi}L(2,\overline{\chi}\chi_4)\right|^4$$

$$= \frac{q^6}{8^4\pi^4}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\Big|\overline{\chi}(4)L^2(1,\overline{\chi})+\overline{\chi}(16)L^2(1,\overline{\chi})+\frac{16}{\pi^2}L^2(2,\overline{\chi}\chi_4)-2\overline{\chi}(8)L^2(1,\overline{\chi})+\frac{8}{\pi}\overline{\chi}(2)L(1,\overline{\chi})L(2,\overline{\chi}\chi_4)-\frac{8}{\pi}\overline{\chi}(4)L(1,\overline{\chi})L(2,\overline{\chi}\chi_4)\Big|^2$$

$$= \frac{3}{2048\pi^4}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}|L(1,\chi)|^4 - \frac{1}{512\pi^4}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\chi(2)|L(1,\chi)|^4 + \frac{1}{2048\pi^4}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\chi(4)|L(1,\chi)|^4 + \frac{1}{16\pi^8}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}|L(2,\chi\chi_4)|^4$$

$$- \frac{1}{256\pi^5}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}L^2(1,\overline{\chi})L(1,\chi)L(2,\chi\chi_4) + \frac{3}{256\pi^5}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(2)L^2(1,\overline{\chi})L(1,\chi)L(2,\chi\chi_4)$$

$$- \frac{3}{256\pi^5}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(4)L^2(1,\overline{\chi})L(1,\chi)L(2,\chi\chi_4) + \frac{1}{256\pi^5}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(8)L^2(1,\overline{\chi})L(1,\chi)L(2,\chi\chi_4)$$

$$+ \frac{1}{128\pi^6}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(4)L^2(1,\overline{\chi})L^2(2,\chi\chi_4) - \frac{1}{64\pi^6}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(8)L^2(1,\overline{\chi})L^2(2,\chi\chi_4) + \frac{1}{128\pi^6}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(16)L^2(1,\overline{\chi})L^2(2,\chi\chi_4)$$

$$+ \frac{1}{16\pi^7}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(2)L(1,\overline{\chi})L(2,\overline{\chi}\chi_4)L^2(2,\chi\chi_4) - \frac{1}{16\pi^7}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\overline{\chi}(4)L(1,\overline{\chi})L(2,\overline{\chi}\chi_4)L^2(2,\chi\chi_4)$$

$$+ \frac{1}{32\pi^6}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}|L(1,\chi)|^2|L(2,\chi\chi_4)|^2 - \frac{1}{32\pi^6}q^6\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*}\chi(2)|L(1,\chi)|^2|L(2,\chi\chi_4)|^2$$

$$= \frac{7}{2^{17}\cdot 3^2}q^6 J(q)\prod_{p|q}\frac{(1-1/p^2)^3}{1+1/p^2} + \frac{35}{2^{16}\cdot 3^2\cdot 17}q^6 J(q)\prod_{p|q}\frac{(1-1/p^4)^3}{1+1/p^4} - \frac{L^2(3,\chi_4)}{2^{14}\cdot 3^2\cdot\pi\cdot L(5,\chi_4)} \times q^6 J(q)\prod_{p|q}\frac{(1-1/p^2)^2(1-\chi_4(p)/p^3)^2}{1-\chi_4(p)/p^5}$$

$$+ \frac{L(3,\chi_4)}{2^8\cdot 5\cdot 3^2\cdot\pi^2}q^6 J(q) \times \prod_{p|q}\left(1-\frac{\chi_4(p)}{p^3}\right)\left(1-\frac{1}{p^4}\right) \times \prod_{p\nmid q}\left(1+\frac{1}{p^2(1-\chi_4(p)/p)}\right) + O\left(q^{6+\epsilon}\right). \tag{74}$$

This proves Theorem 3.

On the other hand, by Theorem 14 and Lemma 18, we have

$$\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}^{*}\left|\sum_{1\le a\le q/4}a\chi(a)\right|^4 = \frac{q^6}{16^4\pi^8}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}^{*}\left|(\overline{\chi}(4)-2\overline{\chi}(2)-8)L(2,\overline{\chi})+4\pi L(1,\overline{\chi}\chi_4)\right|^4$$

$$= \frac{q^6}{16^4\pi^8}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}}^{*}\Big|(\overline{\chi}(4)-2\overline{\chi}(2)-8)^2L^2(2,\overline{\chi})$$

$$+ 16\pi^2 L^2(1,\overline{\chi}\chi_4) + 8\pi(\overline{\chi}(4) - 2\overline{\chi}(2) - 8) \times L(1,\overline{\chi}\chi_4) L(2,\overline{\chi})\Big|^2$$

$$= \frac{5281}{65536\pi^8} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} |L(2,\overline{\chi})|^4 + \frac{427}{8192\pi^8} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(2)|L(2,\overline{\chi})|^4 - \frac{227}{8192\pi^8} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(4)|L(2,\overline{\chi})|^4 - \frac{7}{1024\pi^8} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(8)|L(2,\overline{\chi})|^4$$

$$+ \frac{1}{512\pi^8} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(16)|L(2,\overline{\chi})|^4 + \frac{1}{32\pi^6} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} L^2(2,\overline{\chi}) L^2(1,\chi\chi_4) + \frac{1}{64\pi^6} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(2) L^2(2,\overline{\chi}) L^2(1,\chi\chi_4)$$

$$- \frac{3}{512\pi^6} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(4) L^2(2,\overline{\chi}) L^2(1,\chi\chi_4) - \frac{1}{512\pi^6} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(8) L^2(2,\overline{\chi}) L^2(1,\chi\chi_4) + \frac{1}{2048\pi^6} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(16) L^2(2,\overline{\chi}) L^2(1,\chi\chi_4)$$

$$+ \frac{1}{64\pi^7} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \chi(4) L^2(2,\overline{\chi}) L(1,\chi\chi_4) L(2,\chi) - \frac{3}{128\pi^7} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \chi(2) L^2(2,\overline{\chi}) L(1,\chi\chi_4) L(2,\chi) - \frac{147}{1024\pi^7} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} L^2(2,\overline{\chi}) L(1,\chi\chi_4) L(2,\chi)$$

$$- \frac{59}{1024\pi^7} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(2) L^2(2,\overline{\chi}) L(1,\chi\chi_4) L(2,\chi) + \frac{105}{4096\pi^7} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(4) L^2(2,\overline{\chi}) L(1,\chi\chi_4) L(2,\chi) + \frac{15}{2048\pi^7} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(8) L^2(2,\overline{\chi}) L(1,\chi\chi_4) L(2,\chi)$$

$$- \frac{1}{512\pi^7} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(16) L^2(2,\overline{\chi}) L(1,\chi\chi_4) L(2,\chi) + \frac{1}{256\pi^4} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} |L(1,\overline{\chi}\chi_4)|^4 + \frac{1}{256\pi^5} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \chi(4) L^2(1,\overline{\chi}\chi_4) L(1,\chi\chi_4) L(2,\chi)$$

$$- \frac{1}{128\pi^5} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \chi(2) L^2(1,\overline{\chi}\chi_4) L(1,\chi\chi_4) L(2,\chi) - \frac{1}{32\pi^5} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} L^2(1,\overline{\chi}\chi_4) L(1,\chi\chi_4) L(2,\chi)$$

$$+ \frac{69}{1024\pi^6} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} |L(1,\overline{\chi}\chi_4)|^2 |L(2,\overline{\chi})|^2 + \frac{7}{256\pi^6} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(2) |L(1,\overline{\chi}\chi_4)|^2 |L(2,\overline{\chi})|^2 - \frac{1}{64\pi^6} q^6 \sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=1}} \overline{\chi}(4) |L(1,\overline{\chi}\chi_4)|^2 |L(2,\overline{\chi})|^2$$

$$= \frac{385}{2^{20}\cdot 51} \cdot q^6 J(q) \prod_{p\mid q} \frac{(1 - 1/p^4)^3}{1 + 1/p^4} + \frac{15 L^4(3,\chi_4)}{\pi^{12}} \cdot q^6 J(q) \prod_{p\mid q} \frac{(1 - \chi_4(p)/p^3)^3}{1 + \chi_4(p)/p^3}$$

$$- \frac{1423\pi L^2(3,\chi_4)}{2^{16}\cdot 5^2 \cdot 3^4 \cdot L(7,\chi_4)} \cdot q^6 J(q) \prod_{p\mid q} \frac{(1 - \chi_4(p)/p^3)^2 (1 - 1/p^4)^2}{1 - \chi_4(p)/p^7}$$

$$+\frac{3}{2^{16}}\cdot q^{6}J(q)\prod_{p\mid q}\frac{\left(1-1/p^{2}\right)^{3}}{1+1/p^{2}}$$
$$-\frac{L^{2}(3,\chi_4)}{2^{12}\cdot\pi\cdot L(5,\chi_4)}$$
$$\cdot q^{6}J(q)\prod_{p\mid q}\frac{\left(1-1/p^{2}\right)^{2}\left(1-\chi_4(p)/p^{3}\right)^{2}}{1-\chi_4(p)/p^{5}}$$
$$+\frac{L(3,\chi_4)}{2^{10}\cdot 3\cdot\pi^{2}}\cdot q^{6}J(q)\prod_{p\mid q}\left(1-\chi_4(p)/p^{3}\right)\left(1-1/p^{4}\right)$$
$$\times\prod_{p\nmid q}\left(1+\frac{1}{p^{2}\left(1-\chi_4(p)/p\right)}\right)$$
$$+O\left(q^{6+\epsilon}\right). \quad (75)$$

This completes the proof of Theorem 4.

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