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Research Article

Stability and Global Hopf Bifurcation Analysis on a Ratio-Dependent Predator-Prey Model with Two Time Delays

Huitao Zhao,[1,2] Yiping Lin,[1] and Yunxian Dai[1]

[1] *Department of Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650093, China*
[2] *Department of Mathematics and Information Science, Zhoukou Normal University, Zhoukou, Henan 466001, China*

Correspondence should be addressed to Yiping Lin; linyiping689@sohu.com

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A ratio-dependent predator-prey model with two time delays is studied. By means of an iteration technique, sufficient conditions are obtained for the global attractiveness of the positive equilibrium. By comparison arguments, the global stability of the semitrivial equilibrium is addressed. By using the theory of functional equation and Hopf bifurcation, the conditions on which positive equilibrium exists and the quality of Hopf bifurcation are given. Using a global Hopf bifurcation result of Wu (1998) for functional differential equations, the global existence of the periodic solutions is obtained. Finally, an example for numerical simulations is also included.

1. Introduction

The main purpose of this paper is to investigate the bifurcation phenomena from the delays for the following predator-prey system:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}x(t)\,y(t-\tau_2)}{my^2(t-\tau_2) + x^2(t)}\right], \\ \dot{y}(t) &= \frac{a_{21}x^2(t-\tau_1)\,y(t)}{my^2(t) + x^2(t-\tau_1)} - r_2 y(t), \end{aligned} \quad (1)$$

where $x(t)$ and $y(t)$ stand for the population (or density) of the prey and the predator at time t, respectively. From the biological sense, we assume that $x^2 + y^2 \neq 0$. r_1, r_2, a_{11}, a_{12}, a_{21}, and m are positive constants, in which r_1 denotes the intrinsic growth rate of the prey, a_{11} is the intraspecific competition rate of the prey, a_{12} is the capturing rate of the predator, a_{21}/a_{12} describes the efficiency of the predator in converting consumed prey into predator offspring, m is the interference coefficient of the predators, and r_2 is the predator mortality rate. The delay $\tau_1 \geq 0$ denotes the gestation period of the predator; $\tau_2 \geq 0$ is the hunting delay of the predator to prey.

This model is labeled "ratio-dependent," which means that the functional and numerical responses depend on the densities of both prey and predators, especially when predator has to search for food. Such a functional response is called a ratio-dependent response function (see [1] for more details). In system (1), the ratio-dependent response function is of the form $g(x/y) = c(x/y)^2/(m + (x/y)^2) = cx^2/(my^2 + x^2)$.

The ratio-dependent predator-prey model has been studied by several researchers recently and very rich dynamics have been observed [2–5]. For example, Xu et al. [4] studied a delayed ratio-dependent predator-prey model with the same ratio-dependent response function of system (1). By means of an iteration technique, they obtained the sufficient conditions for the global attractiveness of the positive equilibrium. By comparison arguments, they proved the global stability of the semitrivial equilibrium. Finally using the theory of functional equation and Hopf bifurcation, they gave the condition on which positive equilibrium exists and the formulae to determine the quality of Hopf bifurcation. But in their work, the global continuation of local Hopf bifurcation was not mentioned.

In general, periodic solutions through the Hopf bifurcation in delay differential equations are local for the values of parameters which are only in a small neighborhood of the critical values (see, e.g., [6, 7]). Therefore we would like to know if these nonconstant periodic solutions obtained through local bifurcation can continue for a large range of parameter values. Recently, a great deal of research has been devoted to the topics [8–12]. One of the methods used in them is the global Hopf bifurcation theorem by Wu [13]. For example, Song et al. [12] studied a predator-prey system with two delays, and using the methods in [13], they get the global existence of periodic solutions.

Motivated by [12], we will study the system (1); special attention is paid to the global continuation of local Hopf bifurcation. We suppose that the initial condition for system (1) takes the form

$$x(\theta) = \phi(\theta), \quad y(\theta) = \psi(\theta), \quad \phi(\theta) \geq 0, \quad \psi(\theta) \geq 0,$$
$$\theta \in [-\tau, 0] \left(\tau = \tau_1 + \tau_2\right), \quad \phi(0) > 0, \quad \psi(0) > 0, \tag{2}$$

where $(\phi(\theta), \psi(\theta)) \in \mathscr{C}([-\tau, 0], \mathbf{R}_{+0}^2)$, which is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}_{+0}^2$, where $\mathbf{R}_{+0}^2 = \{(x, y) \mid x \geq 0, y \geq 0\}$.

By the fundamental theory of functional differential equations [14], system (1) has a unique solution $(x(t), y(t))$ satisfying initial condition (2).

The rest of the paper is organized as follows. In Section 2, we show the positivity and the boundedness of solutions of system (1) with initial condition (2). In Section 3, we study the existence of Hopf bifurcation for system (1) at the positive equilibrium. In Section 4, using the normal form theory and the center manifold reduction, explicit formulae are derived to determine the direction of bifurcation and the stability and other properties of bifurcating periodic solutions. In Section 5, by means of an iteration technique, sufficient conditions are obtained for the global attractiveness of the positive equilibrium. In Section 6, we consider the global existence of bifurcating periodic solutions and give some numerical simulations. In Section 7, a brief discussion is given.

2. Positivity and Boundedness

In this section, we study the positivity and boundedness of solutions of system (1) with initial conditions (2).

Theorem 1. *Solutions of system* (1) *with initial condition* (2) *are positive for all* $t \geq 0$.

Proof. Assume $(x(t), y(t))$ to be a solution of system (1) with initial condition (2). Let us consider $y(t)$ for $t \geq 0$. It follows from the second equation of system (1) that

$$y(t) = y(0)\, e^{\int_0^t ((a_{21}x^2(s-\tau_1)/my^2(s)+x^2(s-\tau_1))-r_2)ds}; \tag{3}$$

then, from initial condition (2), we have $y(t) > 0$, for $t \geq 0$. We derive from the first equation of system (1) that

$$x(t) = x(0)\, e^{\int_0^t (r_1 - a_{11}x(s) - (a_{12}x(s)y(s-\tau_2)/my^2(s-\tau_2)+x^2(s)))ds}; \tag{4}$$

that is, $x(t) > 0$ for $t \geq 0$. This ends the proof. □

For the following discussion of boundedness, we first consider the following ordinary differential equation:

$$\dot{u} = \frac{a_{21}A_1^2 u(t)}{mu^2(t) + A_1^2} - r_2 u(t), \quad u(0) > 0, \tag{5}$$

where a_{21}, r_2, A_1, and m are positive constants. From Lemma 2.1 in [5], it is easy to verify the following result.

Lemma 2. *If* $a_{21} < r_2$, *the trivial equilibrium* $u^0 = 0$ *of* (5) *is globally stable. If* $a_{21} > r_2$, *then* (5) *admits a unique positive equilibrium* $u^* = \sqrt{(a_{21} - r_2)/mr_2}A_1$ *which is globally asymptotically stable in* $\Lambda = \{u \mid u \geq 0\}$.

Theorem 3. *Positive solutions of system* (1) *with initial condition* (2) *are ultimately bounded.*

Proof. Let $(x(t), y(t))$ be a positive solution of system (1) with initial condition (2). From the first equation of system (1), we have

$$\dot{x}(t) \leq x(t)\left[r_1 - a_{11}x(t)\right], \tag{6}$$

which yields

$$\limsup_{t\to+\infty} x(t) \leq \frac{r_1}{a_{11}}; \tag{7}$$

hence, for $\epsilon > 0$ sufficiently small, there is a $T_1 > 0$ such that if $t > T_1$, $x(t) < (r_1/a_{11}) + \epsilon$.

We now consider the boundedness of $y(t)$. If $a_{21} \leq r_2$, we derive from the second equation of system (1) that

$$\dot{y}(t) \leq \left(a_{21} - r_2\right) y(t) \leq 0; \tag{8}$$

from monotone bounded theorem, it is easy to show that $\lim_{t\to+\infty} y(t) \leq y(0)$.

Therefore, we assume below that $a_{21} > r_2$. We derive from the second equation of system (1) that, for $t > T_1 + \tau$,

$$\dot{y}(t) \leq \frac{a_{21}\left(r_1/a_{11} + \epsilon\right)^2 y(t)}{my^2(t) + \left(r_1/a_{11} + \epsilon\right)^2} - r_2 y(t); \tag{9}$$

noting that $a_{21} > r_2$, by Lemma 2, a comparison argument shows that

$$\limsup_{t\to+\infty} y(t) \leq \sqrt{\frac{a_{21} - r_2}{mr_2}}\left(\frac{r_1}{a_{11}} + \epsilon\right). \tag{10}$$

This completes the proof. □

3. Local Stability and Hopf Bifurcation

In this section, we discuss the local stability of the positive equilibrium and the semitrivial equilibrium of system (1) and establish the existence of Hopf bifurcation at the positive equilibrium.

It is easy to show that system (1) always has a semitrivial equilibrium $E_1(r_1/a_{11}, 0)$. Further, if the following condition holds:

(H1) $r_1^2 a_{21}^2 m > a_{12}^2 r_2 (a_{21} - r_2) > 0$,

then system (1) has a unique positive equilibrium $E^*(x^*, y^*)$, where

$$x^* = \frac{r_1 a_{21} - r_2 a_{12} h}{a_{11} a_{21}}, \qquad y^* = h x^*, \tag{11}$$

where

$$h = \sqrt{\frac{a_{21} - r_2}{m r_2}}. \tag{12}$$

For convenience, let us introduce new variables $X(t) = x(t - \tau_1), Y(t) = y(t), \tau = \tau_1 + \tau_2$, rewriting $X(t), Y(t)$ as $x(t), y(t)$, so that system (1) can be written as the following system with a single delay:

$$\begin{aligned} \dot{x}(t) &= x(t)\left[r_1 - a_{11} x(t) - \frac{a_{12} x(t) y(t-\tau)}{m y^2(t-\tau) + x^2(t)}\right], \\ \dot{y}(t) &= \frac{a_{21} x^2(t) y(t)}{m y^2(t) + x^2(t)} - r_2 y(t). \end{aligned} \tag{13}$$

Clearly, system (13) has the same equilibrium as system (1).

The characteristic equation of system (13) at the semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ is of the form

$$(\lambda + r_1)(\lambda + r_2 - a_{21}) = 0. \tag{14}$$

Clearly, (14) always has a root $\lambda = -r_1$, and if $a_{21} < r_2$, the other root of (14) is negative; if $a_{21} > r_2$, the other root of (14) is positive. Hence the semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ is locally asymptotically stable (unstable) if $a_{21} < r_2$ $(a_{21} > r_2)$.

The characteristic equation of system (13) at the positive equilibrium $E^*(x^*, y^*)$ is of the form

$$\lambda^2 + p_0 \lambda + p_1 + p_2 e^{-\lambda\tau} = 0, \tag{15}$$

where

$$\begin{aligned} p_0 &= r_1 - \frac{2a_{12} r_2^2 h}{a_{21}^2} + \frac{2r_2(a_{21} - r_2)}{a_{21}}, \\ p_1 &= \left(r_1 - \frac{2a_{12} r_2^2 h}{a_{21}^2}\right)\frac{2r_2(a_{21} - r_2)}{a_{21}}, \\ p_2 &= \frac{2a_{12} r_2^2 h (2r_2 - a_{21})(a_{21} - r_2)}{a_{21}^3}, \end{aligned} \tag{16}$$

where h is defined as (12).

When $\tau = 0$, (15) becomes

$$\lambda^2 + p_0 \lambda + p_1 + p_2 = 0. \tag{17}$$

It is easy to show that

$$p_1 + p_2 = \frac{2r_2(a_{21} - r_2)(r_1 a_{21} - a_{12} r_2 h)}{a_{21}^2}. \tag{18}$$

Obviously, if (H1) holds, then $p_1 + p_2 > 0$. Hence, the positive equilibrium $E^*(x^*, y^*)$ of system (13) is locally stable when $\tau = 0$ if

$$r_1 > \frac{2a_{12} r_2^2 h}{a_{21}^2} - \frac{2r_2(a_{21} - r_2)}{a_{21}}, \tag{19}$$

and it is unstable when $\tau = 0$ if

$$r_1 < \frac{2a_{12} r_2^2 h}{a_{21}^2} - \frac{2r_2(a_{21} - r_2)}{a_{21}}. \tag{20}$$

We assume that $\lambda = i\omega(\omega > 0)$ is a root of (15); this is the case if and only if ω satisfies the following equation:

$$-\omega^2 + p_0 \omega i + p_1 + p_2 e^{-i\omega\tau} = 0. \tag{21}$$

Separating the real and imaginary parts, we obtain the following system for ω:

$$\begin{aligned} p_2 \cos \omega\tau &= \omega^2 - p_1, \\ p_2 \sin \omega\tau &= p_0 \omega. \end{aligned} \tag{22}$$

It follows that

$$\omega^4 + \left(p_0^2 - 2p_1\right)\omega^2 + p_1^2 - p_2^2 = 0. \tag{23}$$

Letting $z = \omega^2$, (42) becomes

$$z^2 + \left(p_0^2 - 2p_1\right) z + p_1^2 - p_2^2 = 0. \tag{24}$$

By a direct calculation, it follows that

$$\begin{aligned} p_0^2 - 2p_1 &= \left(r_1 - \frac{2a_{12} r_2^2 h}{a_{21}^2}\right)^2 + \left(\frac{2r_2(a_{21} - r_2)}{a_{21}}\right)^2 > 0, \\ p_1 - p_2 &= \frac{2r_2(a_{21} - r_2)}{a_{21}}\left(r_1 - \frac{4a_{12} r_2^2 h + a_{12} a_{21} r_2 h}{a_{21}^2}\right). \end{aligned} \tag{25}$$

Note that if (H1) holds, then $p_1 + p_2 > 0$. Hence if (H1) and $p_1 - p_2 > 0$ hold, (24) has no positive roots. Accordingly, if (H1) and $p_1 - p_2 > 0$ hold, the positive equilibrium E^* of system (13) exists and is locally asymptotically stable for all $\tau \geq 0$. If (H1) and $p_1 - p_2 < 0$ hold, then (24) has a unique positive root ω_0, where

$$\omega_0^2 = \frac{1}{2}\left(2p_1 - p_0^2 + \sqrt{p_0^4 - 4p_0^2 p_1 + 4p_2^2}\right). \tag{26}$$

Then, we can get

$$\tau_n = \frac{1}{\omega_0} \arccos \frac{\omega_0^2 - p_1}{p_2} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \ldots, \tag{27}$$

at which (15) admits a pair of purely imaginary roots of the form $\pm\omega_0$.

Let $p_1 - p_2 < 0$ and τ_0 be defined above. Denote

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau) \tag{28}$$

the root of (15) satisfying

$$\alpha(\tau_n) = 0, \qquad \omega(\tau_n) = \omega_0. \tag{29}$$

It is not difficult to verify that the following result holds.

Lemma 4. *If (H1) and $p_1 - p_2 < 0$ hold, the transversal condition $(d(\operatorname{Re}\lambda)/d\tau)|_{\tau=\tau_n} > 0$ holds.*

Proof. Differentiating (15) with respect τ, we obtain that

$$2\lambda\frac{d\lambda}{d\tau} + p_0\frac{d\lambda}{d\tau} - p_2\tau e^{-\lambda\tau}\frac{d\lambda}{d\tau} = p_2\lambda e^{-\lambda\tau}; \tag{30}$$

it follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p_0}{-\lambda p_2 e^{-\lambda\tau}} - \frac{\tau}{\lambda}; \tag{31}$$

from (15) and (31), we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + p_0}{-\lambda\left(\lambda^2 + p_0\lambda + p_1\right)} - \frac{\tau}{\lambda}. \tag{32}$$

We therefore derive that

$$\begin{aligned}
&\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\bigg|_{\tau=\tau_n}\right\} \\
&= \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_n}\right\} \\
&= \operatorname{sign}\left\{\operatorname{Re}\left[\frac{2\lambda + p_0}{-\lambda\left(\lambda^2 + p_0\lambda + p_1\right)}\right]_{\tau=\tau_n}\right\} \\
&= \operatorname{sign}\left\{\frac{\omega_0^2\left(p_0^2 - 2p_1 + \omega_0^2\right)}{\omega_0^4 p_0^2 + \left(\omega_0 p_1 - \omega_0^3\right)^2}\right\}.
\end{aligned} \tag{33}$$

Noting that $p_0^2 - 2p_1 > 0$, hence, if (H1) and $p_1 - p_2 < 0$ hold, we have $(d(\operatorname{Re}\lambda)/d\tau)|_{\tau=\tau_n} > 0$. Accordingly, the transversal condition holds and a Hopf bifurcation occurs at $\tau = \tau_n$. □

By Lemma B in [5], we have the following results.

Theorem 5. *Suppose (H1) holds and let h be defined in* (12), *for system* (13), *one has the following.*

(i) *If $r_1 > (2a_{12}r_2^2h/a_{21}^2) - (2r_2(a_{21} - r_2)/a_{21})$ and $r_1 > (4a_{12}r_2^2h + a_{12}a_{21}r_2h)/a_{21}^2$, then the positive equilibrium E^* is locally asymptotically stable for all $\tau \geq 0$.*

(ii) *If $r_1 > (2a_{12}r_2^2h/a_{21}^2) - (2r_2(a_{21} - r_2)/a_{21})$ and $r_1 < (4a_{12}r_2^2h + a_{12}a_{21}r_2h)/a_{21}^2$, then there exists a positive number τ_0 such that the positive equilibrium E^* is locally asymptotically stable if $\tau \in [0, \tau_0)$ and is unstable if $\tau > \tau_0$. Further, system* (13) *undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$.*

4. Direction and Stability of Hopf Bifurcations

In Section 3, we have shown that system (13) admits a periodic solution bifurcated from the positive equilibrium E^* at the critical value τ_0. In this section, we derive explicit formulae to determine the direction of Hopf bifurcations and stability of periodic solutions bifurcated from the positive equilibrium E^* at critical value τ_0 by using the normal form theory and the center manifold reduction (see, e.g., [15, 16]).

Set $\tau = \tau_0 + \mu$; then $\mu = 0$ is a Hopf bifurcation value of system (13). Thus we can consider the problem above in the phase space $\mathscr{C} = \mathscr{C}([-\tau, 0], \mathbf{R}^2)$.

Let $u_1(t) = x(t) - x^*$, $u_2(t) = y(t) - y^*$. System (13) is transformed into

$$\begin{aligned}
\dot{u}_1(t) &= c_1u_1(t) + c_4u_2(t-\tau) \\
&\quad + \sum_{i+j\geq 2}\frac{1}{i!j!}f_{ij}^{(1)}u_1^i(t)u_2^j(t-\tau), \\
\dot{u}_2(t) &= c_2u_1(t) + c_3u_2(t) + \sum_{i+j\geq 2}\frac{1}{i!j!}f_{ij}^{(2)}u_1^i(t)u_2^j(t),
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
c_1 &= -r_1 + \frac{2a_{12}r_2^2h}{a_{21}^2}, \qquad c_2 = \frac{2r_2h\left(a_{21} - r_2\right)}{a_{21}}, \\
c_3 &= -r_2 + \frac{r_2\left(2r_2 - a_{21}\right)}{a_{21}}, \qquad c_4 = -\frac{a_{12}r_2\left(2r_2 - a_{21}\right)}{a_{21}^2}, \\
f^{(1)} &= x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}x(t)y(t-\tau)}{my^2(t-\tau) + x^2(t)}\right], \\
f^{(2)} &= \frac{a_{21}x^2(t)y(t)}{my^2(t) + x^2(t)} - r_2y(t), \\
f_{ij}^{(1)} &= \frac{\partial^{i+j}f^{(1)}}{\partial x^i\partial y(t-\tau)^j}\bigg|_{(x^*,y^*)}, \\
f_{ij}^{(2)} &= \frac{\partial^{i+j}f^{(2)}}{\partial x^i\partial y^j}\bigg|_{(x^*,y^*)}, \quad i, j \geq 0.
\end{aligned} \tag{35}$$

For the simplicity of notations, we rewrite (34) as

$$\dot{u}(t) = L_\mu u_t + f(\mu, u_t), \tag{36}$$

where $u(t) = (u_1(t), u_1(t))^T \in \mathbf{R}^2$, $u_t(\theta) \in \mathscr{C}$ is defined by $u_t(\theta) = u(t + \theta)$, and $L_\mu : \mathscr{C} \to \mathbf{R}, f : \mathbf{R} \times \mathscr{C} \to \mathbf{R}$ are given, respectively, by

$$L_\mu\phi = \begin{bmatrix} c_1 & 0 \\ c_2 & c_3 \end{bmatrix}\phi(0) + \begin{bmatrix} 0 & c_4 \\ 0 & 0 \end{bmatrix}\phi(-\tau), \tag{37}$$

$$f(\mu, \phi) = \begin{bmatrix} \sum_{i+j\geq 2}\frac{1}{i!j!}f_{ij}^{(1)}\phi_1^i(t)\phi_2^j(t-\tau) \\ \sum_{i+j\geq 2}\frac{1}{i!j!}f_{ij}^{(2)}\phi_1^i(t)\phi_2^j(t) \end{bmatrix}. \tag{38}$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-\tau, 0]$ such that

$$L_\mu\phi = \int_{-\tau}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \text{for } \phi \in \mathscr{C}. \tag{39}$$

In fact, we can choose

$$\eta(\theta,\mu)=\begin{bmatrix}c_1 & 0\\ c_2 & c_3\end{bmatrix}\delta(\theta)+\begin{bmatrix}0 & c_4\\ 0 & 0\end{bmatrix}\delta(\theta+\tau), \quad (40)$$

where δ is the Dirac delta function. For $\phi\in\mathscr{C}^1([-\tau,0],\mathbf{R}^2)$, define

$$A(\mu)\phi=\begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & \theta\in[-\tau,0),\\ \displaystyle\int_{-\tau}^{0}d\eta(s,\mu)\phi(s), & \theta=0,\end{cases}$$
$$R(\mu)\phi=\begin{cases}0, & \theta\in[-\tau,0),\\ f(\mu,\phi), & \theta=0.\end{cases} \quad (41)$$

Then when $\theta=0$, system (36) is equivalent to

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t, \quad (42)$$

where $u_t(\theta)=u(t+\theta)$ for $\theta\in[-\tau,0]$.

For $\psi\in\mathscr{C}^1([0,\tau],(\mathbf{R}^2)^*)$, define

$$A^*\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,\tau],\\ \displaystyle\int_{-\tau}^{0}d\eta^T(t,0)\psi(-t), & s=0,\end{cases} \quad (43)$$

and a bilinear inner product,

$$\langle\psi(s),\phi(\theta)\rangle=\overline{\psi}(0)\phi(0)-\int_{-\tau}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi, \quad (44)$$

where $\eta(\theta)=\eta(\theta,0)$ and $\overline{(\cdot)}$ denotes the conjugate complex of $(\cdot)$. Then $A(0)$ and A^* are adjoint operators. By the discussion in Section 3, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively.

Suppose that $q(\theta)=(1,\rho)^Te^{i\omega_0\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0$. Then $A(0)q(\theta)=i\omega_0 q(\theta)$. From the definition of $A(0)$, it is easy to get $\rho=(i\omega_0-c_3)/c_2$.

Similarly, let $q^*(s)=D(1,\rho^*)e^{-i\omega_0 s}$ be the eigenvector of A^* corresponding to $-i\omega_0$. By the definition of A^*, we can compute $\rho^*=(-i\omega_0-c_1)/c_2$.

In order to assure $\langle q^*(s),q(\theta)\rangle=1$, we need to determine the value of D. From (44) and the definitions of q and q^*, we have $D=1/(1+\overline{\rho}^*\rho+c_4\rho\tau_0e^{i\tau_0\omega_0})$ such that $\langle q^*(s),q(\theta)\rangle=1$ and $\langle q^*(s),\overline{q}(\theta)\rangle=0$.

In the following, we first compute the coordinates to describe the center manifold C_0 at $\mu=0$. Define

$$z(t)=\langle q^*,u_t\rangle,\quad W(t,\theta)=u_t(\theta)-2\operatorname{Re}\{z(t)q(\theta)\}. \quad (45)$$

On the center manifold C_0, we have

$$\begin{aligned}W(t,\theta)&=W(z(t),\overline{z}(t),\theta)\\&=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}\\&\quad+W_{30}(\theta)\frac{z^3}{6}+\cdots,\end{aligned} \quad (46)$$

where z and $\overline{z}$ are local coordinates for center manifold C_0 in the directions of q and $\overline{q}$. Note that W is real if u_t is real. We consider only real solutions. For the solution $u_t\in C_0$, since $\mu=0$, we have

$$\begin{aligned}\dot{z}&=\omega_0z+i\langle q^*(\theta),f(0,W(z(t),\overline{z}(t),\theta)\\&\qquad+2\operatorname{Re}\{z(t)q(\theta)\})\rangle\\&=i\omega_0z+\overline{q}^*(0)f(0,W(z(t),\overline{z}(t),0)\\&\qquad+2\operatorname{Re}\{z(t)q(0)\})\\&\triangleq i\omega_0z+\overline{q}^*(0)f_0(z,\overline{z})=i\omega_0z+g(z,\overline{z}),\end{aligned} \quad (47)$$

where

$$\begin{aligned}g(z,\overline{z})&=\overline{q}^*(0)f_0(z,\overline{z})\\&=g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z}^2}{2}+g_{21}\frac{z^2\overline{z}}{2}+\cdots.\end{aligned} \quad (48)$$

By (45), we have

$$u_t(\theta)=(u_{1t}(\theta),u_{2t}(\theta))^T=W(t,\theta)+zq(\theta)+\overline{z}\overline{q}(\theta). \quad (49)$$

It follows from (38) and (48) that

$$\begin{aligned}g_{20}&=2\overline{D}\Big[\frac{1}{2}f_{20}^{(1)}\rho^2+f_{11}^{(1)}\rho e^{-i\tau_0\omega_0}+\frac{1}{2}f_{02}^{(1)}e^{-2i\tau_0\omega_0}\\&\qquad+\overline{\rho}^*\Big(\frac{1}{2}f_{20}^{(2)}\rho^2+f_{11}^{(2)}\rho+\frac{1}{2}f_{02}^{(2)}\Big)\Big],\\
g_{11}&=\overline{D}\Big[f_{20}^{(1)}\rho\overline{\rho}+f_{11}^{(1)}\big(\rho e^{i\tau_0\omega_0}+\overline{\rho}e^{-i\tau_0\omega_0}\big)\\&\qquad+f_{02}^{(1)}+\overline{\rho}^*\big(f_{20}^{(2)}\rho\overline{\rho}+f_{11}^{(2)}(\rho+\overline{\rho})+f_{02}^{(2)}\big)\Big],\\
g_{02}&=2\overline{D}\Big[\frac{1}{2}f_{20}^{(1)}\overline{\rho}^2+f_{11}^{(1)}\rho e^{i\tau_0\omega_0}\\&\qquad+\frac{1}{2}f_{02}^{(1)}e^{2i\tau_0\omega_0}+\overline{\rho}^*\Big(\frac{1}{2}f_{20}^{(2)}\overline{\rho}^2\\&\qquad+f_{11}^{(2)}\overline{\rho}+\frac{1}{2}f_{02}^{(2)}\Big)\Big],\\
g_{21}&=2\overline{D}\Big[\frac{1}{2}f_{20}^{(1)}\big(2\rho W_{11}^{(1)}(0)+\overline{\rho}W_{20}^{(1)}(0)\big)\\&\qquad+f_{11}^{(1)}\Big(\rho W_{11}^{(2)}(-\tau_0)+\frac{1}{2}\overline{\rho}W_{20}^{(2)}(-\tau_0)\\&\qquad+\frac{1}{2}W_{20}^{(1)}(0)e^{i\tau_0\omega_0}+W_{11}^{(1)}(0)e^{-i\tau_0\omega_0}\Big)\\&\qquad+\frac{1}{2}f_{02}^{(1)}\big(2W_{11}^{(2)}(-\tau_0)e^{-i\tau_0\omega_0}+W_{20}^{(2)}(-\tau_0)e^{i\tau_0\omega_0}\big)\end{aligned}$$

$$+\frac{1}{2}f_{21}^{(1)}\left(\rho^2 e^{i\tau_0\omega_0}+2\rho\overline{\rho}e^{-i\tau_0\omega_0}\right)$$
$$+\frac{1}{2}f_{12}^{(1)}\left(\overline{\rho}e^{-2i\tau_0\omega_0}+2\rho\right)$$
$$+\frac{1}{2}f_{30}^{(1)}\rho^2\overline{\rho}+\frac{1}{2}f_{03}^{(1)}e^{-i\tau_0\omega_0}\Big]$$
$$+2\overline{D}\overline{\rho}^*\Big[\frac{1}{2}f_{20}^{(2)}\left(2\rho W_{11}^{(1)}(0)+\overline{\rho}W_{20}^{(1)}(0)\right)$$
$$+f_{11}^{(2)}\Big(\rho W_{11}^{(2)}(0)+\frac{1}{2}\overline{\rho}W_{20}^{(2)}(0)$$
$$+\frac{1}{2}W_{20}^{(1)}(0)+W_{11}^{(1)}(0)\Big)$$
$$+\frac{1}{2}f_{02}^{(2)}\left(2W_{11}^{(2)}(0)+W_{20}^{(2)}(0)\right)$$
$$+\frac{1}{2}f_{21}^{(2)}\left(\rho^2+2\rho\overline{\rho}\right)$$
$$+\frac{1}{2}f_{12}^{(2)}\left(\overline{\rho}+2\rho\right)+\frac{1}{2}f_{30}^{(2)}\rho^2\overline{\rho}+\frac{1}{2}f_{03}^{(2)}\Big]. \tag{50}$$

In order to assure the value of g_{21}, we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. By (42) and (45), we have

$$\begin{aligned}\dot{W}&=\dot{u}_t-\dot{z}q-\dot{\overline{z}}\,\overline{q}\\&=\begin{cases}AW-2\operatorname{Re}\{\overline{q}^*(0)f_0q(\theta)\}, & \theta\in[-\tau_0,0),\\AW-2\operatorname{Re}\{\overline{q}^*(0)f_0q(\theta)\}+f_0, & \theta=0,\end{cases}\\&\triangleq AW+H(z,\overline{z},\theta),\end{aligned} \tag{51}$$

where

$$H(z,\overline{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\overline{z}+H_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots. \tag{52}$$

Notice that near the origin on the center manifold C_0, we have

$$\dot{W}=W_z\dot{z}+W_{\overline{z}}\dot{\overline{z}}; \tag{53}$$

thus, we have

$$\begin{gathered}\left(A-2i\omega_k\tau_kI\right)W_{20}(\theta)=-H_{20}(\theta),\\AW_{11}(\theta)=-H_{11}(\theta).\end{gathered} \tag{54}$$

By (51), for $\theta\in[-\tau_0,0)$, we have

$$\begin{aligned}H(z,\overline{z},\theta)&=-\overline{q}^*(0)f_0q(\theta)-q^*(0)\overline{f}_0\overline{q}(\theta)\\&=-gq(\theta)-\overline{gq}(\theta).\end{aligned} \tag{55}$$

Comparing the coefficients with (51) gives that

$$\begin{aligned}H_{20}(\theta)&=-g_{20}q(\theta)-\overline{g}_{02}\overline{q}(\theta),\\H_{11}(\theta)&=-g_{11}q(\theta)-\overline{g}_{11}\overline{q}(\theta).\end{aligned} \tag{56}$$

From (56), (54), and the definition of $A(0)$, we can get

$$\dot{W}_{20}(\theta)=2i\omega_0W_{20}(\theta)+g_{20}q(\theta)+\overline{g}_{02}\overline{q}(\theta). \tag{57}$$

Notice that $q(\theta)=q(0)e^{i\omega_0\theta}$; we have

$$W_{20}(\theta)=\frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta}+\frac{i\overline{g}_{02}}{3\omega_0}\overline{q}(0)e^{-i\omega_0\theta}+E_1e^{2i\omega_0\theta}, \tag{58}$$

where $E_1=(E_1^{(1)},E_1^{(2)})\in\mathbf{R}^2$ is a constant vector. In the same way, we can also obtain

$$W_{11}(\theta)=-\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta}+\frac{i\overline{g}_{11}}{\omega_0}\overline{q}(0)e^{-i\omega_0\theta}+E_2, \tag{59}$$

where $E_2=(E_2^{(1)},E_2^{(2)})\in\mathbf{R}^2$ is also a constant vector. In what follows, we will compute E_1 and E_2. From the definition of $A(0)$ and (54), we have

$$\int_{-\tau_0}^{0}d\eta(\theta)W_{20}(\theta)=2i\omega_0W_{20}(0)-H_{20}(0), \tag{60}$$

$$\int_{-\tau_0}^{0}d\eta(\theta)W_{11}(\theta)=-H_{11}(0), \tag{61}$$

where $\eta(\theta)=\eta(0,\theta)$.

From (51), (58), and (60) and noting that

$$\left[i\omega_0I-\int_{-\tau_0}^{0}e^{i\omega_0\theta}d\eta(\theta)\right]q(0)=0, \tag{62}$$

we have

$$E_1^{(1)}=\frac{1}{A_1}\begin{vmatrix}e_1 & -c_4e^{-2i\omega_0\tau_0}\\e_2 & 2i\omega_0-c_3\end{vmatrix},\qquad E_1^{(2)}=\frac{1}{A_1}\begin{vmatrix}2i\omega_0-c_1 & e_1\\-c_2 & e_2\end{vmatrix}, \tag{63}$$

where

$$\begin{aligned}A_1&=\left(2i\omega_0-c_1\right)\left(2i\omega_0-c_3\right)-c_2c_4e^{-2i\omega_0\tau_0},\\e_1&=f_{20}^{(1)}\rho^2+2f_{11}^{(1)}\rho e^{-i\tau_0\omega_0}+f_{02}^{(1)}e^{-2i\tau_0\omega_0},\\e_2&=f_{20}^{(2)}\rho^2+2f_{11}^{(2)}\rho+f_{02}^{(2)}.\end{aligned} \tag{64}$$

From (52), (59), and (61) and noting that

$$\left[-i\omega_0I-\int_{-\tau_0}^{0}e^{-i\omega_0\theta}d\eta(\theta)\right]\overline{q}(0)=0, \tag{65}$$

we have

$$E_2^{(1)}=\frac{1}{A_2}\begin{vmatrix}e_3 & -c_4\\e_4 & -c_3\end{vmatrix},\qquad E_2^{(2)}=\frac{1}{A_2}\begin{vmatrix}-c_1 & e_3\\-c_2 & e_4\end{vmatrix}, \tag{66}$$

where

$$\begin{aligned}A_2&=c_1c_3-c_2c_4,\\e_3&=f_{20}^{(1)}\rho\overline{\rho}+f_{11}^{(1)}\left(\rho e^{i\tau_0\omega_0}+\overline{\rho}e^{-i\tau_0\omega_0}\right)+f_{02}^{(1)},\\e_4&=f_{20}^{(2)}\rho\overline{\rho}+f_{11}^{(2)}\left(\rho+\overline{\rho}\right)+f_{02}^{(2)}.\end{aligned} \tag{67}$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (58) and (59). Furthermore, we can determine each g_{ij}. Therefore, each g_{ij} is determined by the parameters and delay in (13). Thus, we can compute the following values [15]:

$$\begin{gathered} c_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \\ T_2 = -\frac{\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\omega_0\tau_0}, \\ \beta_2 = 2\operatorname{Re}\{c_1(0)\}, \end{gathered} \tag{68}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value τ_k; that is, μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0\,(< 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcation exists for $\tau > \tau_0\,(< \tau_0)$; β_2 determines the stability of the bifurcation periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0\,(> 0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0\,(< 0)$.

5. Global Attractiveness

In this section, following Chaplygin [17], taking into account the upper and lower solution technique and using monotone iterative methods [18, 19], we discuss the global attractiveness of the positive equilibrium $E^*(x^*, y^*)$ and the global stability of the semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ of system (1), respectively.

Theorem 6. *Suppose (H1) holds and let h be defined above, then the positive equilibrium $E^*(x^*, y^*)$ of system (1) is globally attractive provided that the following holds:*

(H2) $r_1 > \max\{a_{12}/2\sqrt{m}, (3a_{12}/m) + (2a_{12}r_2/a_{21})h\}$,

Proof. Let $(x(t), y(t))$ be any positive solution of system (1) with initial conditions (2).

Let

$$\begin{aligned} U_1 &= \limsup_{t\to+\infty} x(t), & V_1 &= \liminf_{t\to+\infty} x(t), \\ U_2 &= \limsup_{t\to+\infty} y(t), & V_2 &= \liminf_{t\to+\infty} y(t). \end{aligned} \tag{69}$$

Using iteration method, we will proof that $U_1 = V_1 = x^*$, $U_2 = V_2 = y^*$.

From the first equation of system (1), we have

$$\dot{x}(t) \le x(t)\left[r_1 - a_{11}x(t)\right]; \tag{70}$$

by comparison, it follows that

$$U_1 = \limsup_{t\to+\infty} x(t) \le \frac{r_1}{a_{11}} := M_1^x; \tag{71}$$

hence, for $\epsilon > 0$ sufficiently small, there exists a $T_1 > 0$ such that if $t > T_1$, $x(t) \le M_1^x + \epsilon$.

From the second equation of system (1), we have, for $t > T_1 + \tau$,

$$\dot{y}(t) \le \frac{a_{21}(M_1^x + \epsilon)^2 y(t)}{my^2(t) + (M_1^x + \epsilon)^2} - r_2 y(t). \tag{72}$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \frac{a_{21}(M_1^x + \epsilon)^2 u(t)}{mu^2(t) + (M_1^x + \epsilon)^2} - r_2 u(t). \tag{73}$$

Since (H1) holds, by Lemma 2, it follows from (73) that

$$\lim_{t\to+\infty} u(t) = \left(M_1^x + \epsilon\right)h, \tag{74}$$

where h is defined in (12). By comparison, we obtain that

$$U_2 = \limsup_{t\to+\infty} y(t) \le \left(M_1^x + \epsilon\right)h; \tag{75}$$

since this inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, it follows that $U_2 \le M_1^y$, where

$$M_1^y = M_1^x h. \tag{76}$$

Hence, for $\epsilon > 0$ sufficiently small, there is a $T_2 > T_1 + \tau$ such that if $t > T_2$, $y(t) \le M_1^y + \epsilon$.

For $\epsilon > 0$ sufficiently small, noting that $my^2(t-\tau_2) + x^2 \ge 2\sqrt{m}xy(t-\tau_2)$, we derive from the first equation of system (1) that, for $t > T_2$,

$$\dot{x}(t) \ge x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}}{2\sqrt{m}}\right]; \tag{77}$$

by comparison, it follows that

$$V_1 = \liminf_{t\to+\infty} x(t) \ge \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}}{2\sqrt{m}}\right) := N_1^x; \tag{78}$$

hence, for $\epsilon > 0$ sufficiently small, there is a $T_3 > T_2 + \tau$, such that if $t > T_3$, $x(t) \ge N_1^x - \varepsilon$.

For $\epsilon > 0$ sufficiently small, we derive from the second equation of system (1) that, for $t > T_3 + \tau$,

$$\dot{y}(t) \ge \frac{a_{21}(N_1^x - \epsilon)^2 y(t)}{my^2(t) + (N_1^x - \epsilon)^2} - r_2 y(t). \tag{79}$$

Consider the following auxiliary equation:

$$\dot{u}(t) = \frac{a_{21}(N_1^x - \epsilon)^2 u(t)}{mu^2(t) + (N_1^x - \epsilon)^2} - r_2 u(t). \tag{80}$$

Since (H1) holds, by Lemma (5), it follows from (80) that

$$\lim_{t\to+\infty} u(t) = \left(N_1^x - \epsilon\right)h; \tag{81}$$

by comparison we derive that

$$V_2 = \liminf_{t\to+\infty} y(t) \ge \left(N_1^x - \epsilon\right)h. \tag{82}$$

Since this inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, we conclude that $V_2 \geq N_1^y$, where

$$N_1^y = N_1^x h. \tag{83}$$

Therefore, for $\epsilon > 0$ sufficiently small, there is a $T_4 > T_3 + \tau$ such that if $t > T_4$, $y(t) \geq N_1^y - \epsilon$.

Again, for $\epsilon > 0$ sufficiently small, it follows from the first equation of system (1) that, for $t > T_4$,

$$\dot{x}(t) \leq x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}(N_1^x - \epsilon)(N_1^y - \epsilon)}{m(M_1^y + \epsilon)^2 + (M_1^x + \epsilon)^2}\right]; \tag{84}$$

by comparison we derive that

$$U_1 = \limsup_{t\to+\infty} x(t) \leq \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}(N_1^x - \epsilon)(N_1^y - \epsilon)}{m(M_1^y + \epsilon)^2 + (M_1^x + \epsilon)^2}\right). \tag{85}$$

Since the above inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, it follows that $U \leq M_2^x$, where

$$M_2^x = \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}N_1^x N_1^y}{m(M_1^y)^2 + (M_1^x)^2}\right); \tag{86}$$

hence, for $\epsilon > 0$ sufficiently small, there is a $T_5 > T_4 + \tau$ such that if $t > T_5$, $x(t) \leq M_2^x + \epsilon$.

It follows from the second equation of system (1) that, for $t > T_5$,

$$\dot{y}(t) \leq \frac{a_{21}(M_2^x + \epsilon)^2 y(t)}{my^2(t) + (M_2^x + \epsilon)^2} - r_2 y(t). \tag{87}$$

By Lemma 2 and a comparison argument we derive from (87) that

$$U_2 = \limsup_{t\to+\infty} y(t) \leq (M_2^x + \epsilon)h; \tag{88}$$

since this inequality holds true for $\epsilon > 0$ sufficiently small, we get $U_2 \leq M_2^y$, where

$$M_2^y = M_2^x h; \tag{89}$$

hence, for $\epsilon > 0$ sufficiently small, there is a $T_6 > T_5 + \tau$ such that if $t > T_6$, $y(t) \leq M_2^y + \epsilon$.

For $\epsilon > 0$ sufficiently small, it follows from the first equation of system (1) that, for $t > T_6$,

$$\dot{x}(t) \geq x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}(M_2^x + \epsilon)(M_2^y + \epsilon)}{m(N_1^y - \epsilon)^2 + (N_1^x - \epsilon)^2}\right]; \tag{90}$$

by comparison, we can obtain that

$$V_1 = \liminf_{t\to+\infty} x(t) \geq \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}(M_2^x + \epsilon)(M_2^y + \epsilon)}{m(N_1^y - \epsilon)^2 + (N_1^x - \epsilon)^2}\right). \tag{91}$$

Since the above inequality holds true for arbitrary $\epsilon > 0$ sufficiently small, it follows that $V \geq N_2^x$, where

$$N_2^x = \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}M_2^x M_2^y}{m(N_1^y)^2 + (N_1^x)^2}\right); \tag{92}$$

therefore, for $\epsilon > 0$ sufficiently small, there is a $T_7 > T_6 + \tau$ such that if $t > T_7$, $x(t) \geq N_2^x - \epsilon$.

For $\epsilon > 0$ sufficiently small, we derive from the second equation of system (1) that, for $t > T_7 + \tau$,

$$\dot{y}(t) \geq \frac{a_{21}(N_2^x - \epsilon)^2 y(t)}{my^2(t) + (N_2^x - \epsilon)^2} - r_2 y(t). \tag{93}$$

Since (H1) holds, by Lemma 2 and a comparison argument, it follows (93) that

$$V_2 = \liminf_{t\to+\infty} y(t) \geq (N_2^x - \epsilon)h; \tag{94}$$

since, for arbitrary $\epsilon > 0$ sufficiently small, this inequality holds true, we conclude that $V_2 \geq N_2^y$, where

$$N_2^y = N_2^x h. \tag{95}$$

Continuing this process, we obtain four sequences M_n^x, M_n^y, V_n^x, and V_n^y $(n = 1, 2, \ldots)$ such that, for $n \geq 2$,

$$\begin{aligned} M_n^x &= \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}N_{n-1}^x N_{n-1}^y}{m(M_{n-1}^y)^2 + (M_{n-1}^x)^2}\right), \\ N_n^x &= \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}M_n^x M_n^y}{m(N_{n-1}^y)^2 + (N_{n-1}^x)^2}\right), \\ M_n^y &= M_n^x h, \qquad N_n^y = N_n^x h, \end{aligned} \tag{96}$$

where h is defined in (12). It is readily seen that

$$N_n^x \leq V_1 \leq U_1 \leq M_n^x, \qquad N_n^y \leq V_2 \leq U_2 \leq M_n^y. \tag{97}$$

It is easy to know that the sequences M_n^x, M_n^y are not increasing and the sequences N_n^x, N_n^y are not decreasing; from accumulation point theorem, the limit of each sequence in M_n^x, M_n^y, N_n^x, and N_n^y exists, Denote

$$\begin{aligned} \overline{x} &= \lim_{t\to+\infty} M_n^x, \qquad \underline{x} = \lim_{t\to+\infty} N_n^x, \\ \overline{y} &= \lim_{t\to+\infty} M_n^y, \qquad \underline{y} = \lim_{t\to+\infty} N_n^y. \end{aligned} \tag{98}$$

We therefore obtain from (96) and (98) that

$$\begin{aligned} \overline{x} &= \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}\underline{x}\,\underline{y}}{m\overline{y}^2 + \overline{x}^2}\right), \\ \underline{x} &= \frac{1}{a_{11}}\left(r_1 - \frac{a_{12}\overline{x}\,\overline{y}}{m\underline{y}^2 + \underline{x}^2}\right), \\ \overline{y} &= \overline{x}h, \qquad \underline{y} = \underline{x}h. \end{aligned} \tag{99}$$

To complete the proof, it is sufficient to prove that $\overline{x} = \underline{x}, \overline{y} = \underline{y}$. It follows from (99) that

$$a_{11}\left(1+mh^2\right)\overline{x}^3 = r_1\left(1+mh^2\right)\overline{x}^2 - a_{12}h\underline{x}^2, \tag{100}$$

$$a_{11}\left(1+mh^2\right)\underline{x}^3 = r_1\left(1+mh^2\right)\underline{x}^2 - a_{12}h\overline{x}^2. \tag{101}$$

Letting (100) minus (101), we have

$$\begin{aligned} a_{11}\left(1+mh^2\right)\left(\overline{x}-\underline{x}\right)\left(\overline{x}^2+\overline{x}\underline{x}+\underline{x}^2\right) \\ = \left[r_1\left(1+mh^2\right)+a_{12}h\right]\left(\overline{x}-\underline{x}\right)\left(\overline{x}+\underline{x}\right). \end{aligned} \tag{102}$$

If $\overline{x} \neq \underline{x}$, we derive from (102) that

$$\begin{aligned} a_{11}\left(1+mh^2\right)\left(\overline{x}^2+\overline{x}\underline{x}+\underline{x}^2\right) \\ = \left[r_1\left(1+mh^2\right)+a_{12}h\right]\left(\overline{x}+\underline{x}\right). \end{aligned} \tag{103}$$

Letting $A = a_{11}(1+mh^2)$, $B = r_1(1+mh^2) + a_{12}h$, we derive from (103) that

$$\overline{x}\underline{x} = \left(\overline{x}+\underline{x}\right)^2 - \frac{B}{A}\left(\overline{x}+\underline{x}\right). \tag{104}$$

It follows from (104) that

$$\begin{aligned} \left(\overline{x}+\underline{x}\right)^2 - 4\overline{x}\underline{x} &= \left(\overline{x}+\underline{x}\right)^2 - 4\left[\left(\overline{x}+\underline{x}\right)^2 - \frac{B}{A}\left(\overline{x}+\underline{x}\right)\right] \\ &= \left(\overline{x}+\underline{x}\right)\left[\frac{4B}{A} - 3\left(\overline{x}+\underline{x}\right)\right]; \end{aligned} \tag{105}$$

noting that $\overline{x} \geq N_1^x$, $\underline{x} \geq N_1^x$, we derive from (105) that

$$\left(\overline{x}+\underline{x}\right)^2 - 4\overline{x}\underline{x} \leq 2\left(\overline{x}+\underline{x}\right)\left[\frac{2B}{A} - 3N_1^x\right]. \tag{106}$$

Substituting (78) into (106), it follows that

$$\left(\overline{x}+\underline{x}\right)^2 - 4\overline{x}\underline{x} \leq -\frac{2\left(\overline{x}+\underline{x}\right)}{a_{11}}\left[r_1 - \frac{3a_{12}}{m} - \frac{2a_{12}h}{1+mh^2}\right]. \tag{107}$$

Hence, if (H2) holds, we have $(\overline{x}+\underline{x})^2 - 4\overline{x}\underline{x} < 0$; this is a contradiction. Accordingly, we have $\overline{x} = \underline{x}$. Therefore, from (99), we have $\overline{y} = \underline{y}$. Hence, the positive equilibrium E^* is globally attractive. The proof is complete. □

Using the same methods in [4, 20], we can also get a similar result.

Theorem 7. *If $r_1 > a_{12}/2\sqrt{m}$ and $a_{21} < r_2$, the semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ of system (1) is globally asymptotically stable.*

6. Global Continuation of Local Hopf Bifurcations

In this section, we study the global continuation of periodic solutions bifurcating from the positive equilibrium E^* of system (13). Throughout this section, we follow closely the notations in [13]. For simplification of notations, setting $z(t) = (z_1(t), z_2(t))^T = (x(t), y(t))^T$, we may rewrite system (13) as the following functional differential equation:

$$\dot{z}(t) = \mathscr{F}\left(z_t, \tau, p\right), \tag{108}$$

where $z_t(\theta) = (z_{1t}(\theta), z_{2t}(\theta))^T = (z_1(t+\theta), \text{and } z_2(t+\theta))^T \in \mathscr{C}([-\tau, 0], \mathbf{R}^2)$. It is obvious that if (H1) holds, then system (13) has a semitrivial equilibrium $E_1(r_1/a_{11}, 0)$ and a positive equilibrium $E^*(x^*, y^*)$. Following the work of [13], we need to define

$$\begin{aligned} \mathbf{X} &= \mathscr{C}\left(\left[-\tau, 0\right], \mathbf{R}^2\right), \\ \Gamma &= \operatorname{Cl}\{(z, \tau, p) \in \mathbf{X} \times \mathbf{R} \times \mathbf{R}^+;\ z \text{ is a nonconstant} \\ &\qquad \text{periodic solution of (108)}\}, \\ \mathscr{N} &= \left\{(\overline{z}, \overline{\tau}, \overline{p}); \mathscr{F}\left(\overline{z}, \overline{\tau}, \overline{p}\right) = 0\right\}. \end{aligned} \tag{109}$$

Let $\ell_{(E^*, \tau_j, 2\pi/\omega_0)}$ denote the connected component passing through $(E^*, \tau_j, 2\pi/\omega_0)$ in Γ, where τ_j is defined by (26). From Theorem 5, we know that $\ell_{(E^*, \tau_j, 2\pi/\omega_0)}$ is nonempty.

We first state the global Hopf bifurcation theory due to Wu [13] for functional differential equations.

Lemma 8. *Assume that (z_*, τ, p) is an isolated center satisfying the hypotheses (A_1)–(A_4) in [13]. Denote by $\ell_{(z_*, \tau, p)}$ the connected component of (z_*, τ, p) in Γ. Then either*

(i) *$\ell_{(z_*, \tau, p)}$ is unbounded or*

(ii) *$\ell_{(z_*, \tau, p)}$ is bounded; $\ell_{(z_*, \tau, p)} \cap \Gamma$ is finite and*

$$\sum_{(z, \tau, p) \in \ell_{(z_*, \tau, p)} \cap \mathscr{N}} \gamma_m\left(z_*, \tau, p\right) = 0, \tag{110}$$

for all $m = 1, 2, \ldots$, where $\gamma_m(z_*, \tau, p)$ is the mth crossing number of (z_*, τ, p) if $m \in J(z_*, \tau, p)$ or it is zero if otherwise.

Clearly, if (ii) in Lemma 8 is not true, then $\ell_{(z_*, \tau, p)}$ is unbounded. Thus, if the projections of $\ell_{(z_*, \tau, p)}$ onto z-space and onto p-space are bounded, then the projection onto τ-space is unbounded. Further, if we can show that the projection of $\ell_{(z_*, \tau, p)}$ onto τ-space is away from zero, then the projection of $\ell_{(z_*, \tau, p)}$ onto τ-space must include interval $[\tau, +\infty)$. Following this ideal, we can prove our results on the global continuation of local Hopf bifurcation.

Lemma 9. *If condition (H1) holds, then all nonconstant periodic solutions of* (13) *with initial conditions,*

$$\begin{aligned} &x(\theta) = \phi(\theta), \quad y(\theta) = \psi(\theta), \quad \phi(\theta) \geq 0, \quad \psi(\theta) \geq 0, \\ &\theta \in [-\tau, 0]\left(\tau = \tau_1 + \tau_2\right), \quad \phi(0) > 0, \quad \psi(0) > 0, \end{aligned} \tag{111}$$

are uniformly bounded.

Proof. Suppose that $x = x(t), y = y(t)$ are nonconstant periodic solutions of system (13) and define

$$x(\xi_1) = \min\{x(t)\}, \quad x(\eta_1) = \max\{x(t)\},$$
$$y(\xi_2) = \min\{y(t)\}, \quad y(\eta_2) = \max\{y(t)\}. \tag{112}$$

It follows from system (13) that

$$x(t) = x(0)\exp\left\{\int_0^t \left(r_1 - a_{11}x(s) - \frac{a_{12}x(s)\,y(s-\tau)}{my^2(s-\tau)+x^2(s)}\right)ds\right\},$$
$$y(t) = y(0)\exp\left\{\int_0^t \left(-r_2 + \frac{a_{21}x^2(s)}{my^2(s)+x^2(s)}\right)ds\right\}, \tag{113}$$

which implies that the solutions of system (13) cannot cross the x-axis and y-axis. Thus the nonconstant periodic orbits must be located in the interior of each quadrant. It follows from initial conditions of system (13) that $(t) > 0, y(t) > 0$. From system (13), we can get

$$0 = r_1 - a_{11}x(\eta_1) - \frac{a_{12}x(\eta_1)\,y(\eta_1-\tau)}{my^2(\eta_1-\tau)+x^2(\eta_1)},$$
$$0 = -r_2 + \frac{a_{21}x^2(\eta_2)}{my^2(\eta_2)+x^2(\eta_2)}. \tag{114}$$

Since $x(t) > 0, y(t) > 0$, it follows from the first equation of (114) that

$$0 < x(\eta_1) \le \frac{r_1}{a_{11}}; \tag{115}$$

on the other hand, by the second equation of (114) and (115), we have

$$0 < y(\eta_2) \le h\frac{r_1}{a_{11}}, \tag{116}$$

where h is defined in (12). From the discussion above, the lemma follows immediately. □

Lemma 10. *If conditions (H1) and (H2) hold, then system* (13) *has no nonconstant periodic solution with period* τ.

Proof. Suppose for a contradiction that system (13) has nonconstant periodic solution with period τ. Then the following system (117) of ordinary differential equations has nonconstant periodic solution:

$$\dot{x}(t) = x(t)\left[r_1 - a_{11}x(t) - \frac{a_{12}x(t)\,y(t)}{my^2(t)+x^2(t)}\right],$$
$$\dot{y}(t) = \frac{a_{21}x^2(t)\,y(t)}{my^2(t)+x^2(t)} - r_2 y(t), \tag{117}$$

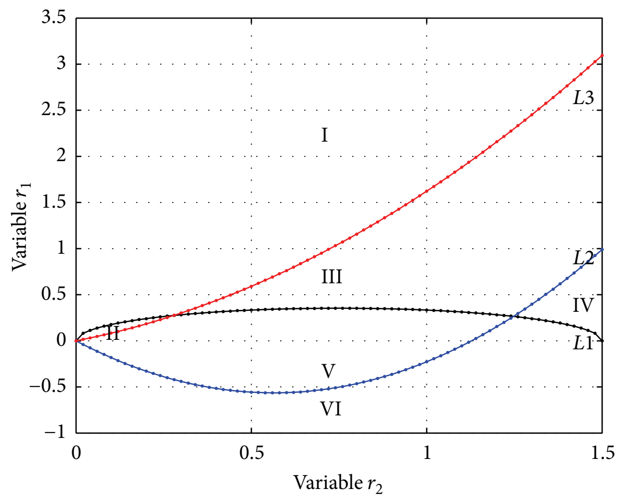

FIGURE 1: The bifurcation diagram of system (1) with $a_{11} = 0.1$, $a_{12} = 1$, $a_{21} = 3/2$, and $m = 2$, where $L1 : r_1 = -(2/9)r_2^2 + r_2/3, L2 : r_1 = (8/9)r_2^2\sqrt{((3/2)-r_2)/(2r_2)} - (4/3)r_2((3/2)-r_2)$, and $L3 : r_1 = ((16/9)r_2 + (1/6))\sqrt{((3/2)-r_2)/(2r_2)}$.

which has the same equilibria as system (13), that is, $E_1(r_1/a_{11}, 0)$ and a positive equilibrium $E^*(x^*, y^*)$. Note that x-axis and y-axis are the invariable manifold of system (13) and the orbits of system (13) do not intersect each other. Thus, there is no solution crossing the coordinate axis. On the other hand, note the fact that if system (117) has a periodic solution, then there must be the equilibrium in its interior and E_1 are located on the coordinate axis. Thus, we conclude that the periodic orbit of system (117) must lie in the first quadrant. From the proof of Theorem 6, we known that if (H1) and (H2) hold, the positive equilibrium is asymptotically stable and globally attractive; thus, there is no periodic orbit in the first quadrant. This ends the proof. □

Theorem 11. *Suppose the conditions (H1) and (H2) hold; let* ω_0 *and* $\tau_j(j = 0, 1, \ldots)$ *be defined in* (26). *If* $(2a_{12}r_2^2h/a_{21}^2) - (2r_2(a_{21} - r_2)/a_{21}) < r_1 < ((4a_{12}r_2^2h + a_{12}a_{21}r_2h)/a_{21}^2)$, *then system* (13) *has at least* $j - 1$ *periodic solutions for every* $\tau > \tau_j, (j = 1, 2, \ldots)$.

Proof. It is sufficient to prove that the projection of $\ell_{(E^*,\tau_j,2\pi/\omega_0)}$ onto τ-space is $[\overline{\tau}, +\infty)$ for each $j > 0$, where $\overline{\tau} \le \tau_j$.

The characteristic matrix of (108) at an equilibrium $\overline{z} = (\overline{z}^{(1)}, \overline{z}^{(2)}) \in \mathbf{R}^2$ takes the following form:

$$\Delta(\overline{z}, \tau, p)(\lambda) = \lambda\mathrm{Id} - D\mathscr{F}(\overline{z}, \overline{\tau}, \overline{p})\left(e^{\lambda}\mathrm{Id}\right). \tag{118}$$

$(\overline{z}, \overline{\tau}, \overline{p})$ is called a center if $\mathscr{F}(\overline{z}, \overline{\tau}, \overline{p}) = 0$ and $\det(\Delta(\overline{z}, \overline{\tau}, \overline{p})((2\pi/p)i)) = 0$. A center is said to be isolated if it is the only center in some neighborhood of $(\overline{z}, \overline{\tau}, \overline{p})$. It follows from (118) that

$$\det(\Delta(E_1, \tau, p)(\lambda)) = (\lambda + r_1)(\lambda + r_2 - a_{21}) = 0, \tag{119}$$

$$\det(\Delta(E^*, \tau, p)(\lambda)) = \lambda^2 + p_0\lambda + p_1 + p_2e^{-\lambda\tau} = 0, \tag{120}$$

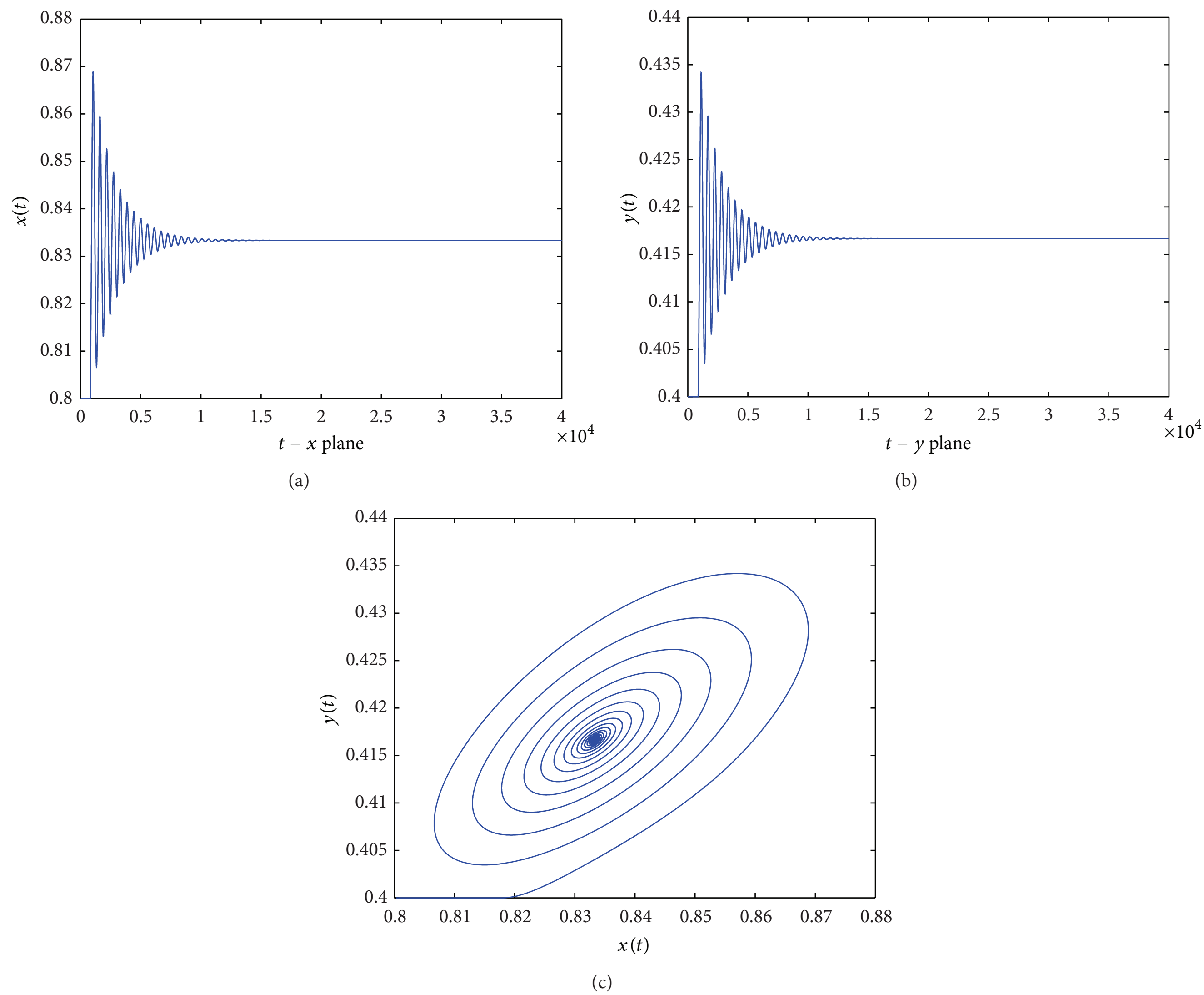

FIGURE 2: The trajectories and phase graphs of system (1) with $\tau = \tau_1 + \tau_2 = 6 + 4 = 10$.

where p_0, p_1, and p_2 are defined as in Section 3. From the discussion in Section 3, each of (119) and (120) has no purely imaginary root provided that $r_1 > (4a_{12}r_2^2h + a_{12}a_{21}r_2h)/a_{21}^2$. Thus, we conclude that (108) has no the center of the form as (E_1, τ, p) and (E^*, τ, p). On the other hand, from the discussion in Section 3 about the local Hopf bifurcation, it is easy to verify that $(E^*, \tau_j, 2\pi/\omega_0)$ is an isolated center, and there exist $\epsilon > 0$, $\delta > 0$, and a smooth curve $\lambda : (\tau_j - \delta, \tau_j + \delta) \to \mathscr{C}$ such that $\det(\Delta(\lambda(\tau))) = 0$, $|\lambda(\tau) - \omega_0| < \epsilon$ for all $\tau \in [\tau_j - \delta, \tau_j + \delta]$ and

$$\lambda\left(\tau_j\right) = \omega_0 i, \qquad \left.\frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau}\right|_{\tau=\tau_j} > 0. \tag{121}$$

Let

$$\Omega_{\epsilon,(2\pi/\omega_0)} = \left\{(\eta, p); 0 < \eta < \epsilon, \left|p - \frac{2\pi}{\omega_0}\right| < \epsilon\right\}. \tag{122}$$

It is easy to verify that, on $[\tau_j - \delta, \tau_j + \delta] \times \partial\Omega_{\epsilon, 2\pi/\omega_0}$,

$$\det\left(\Delta\left(E^*, \tau, p\right)\left(\eta + \frac{2\pi}{p}i\right)\right) = 0 \quad \text{if and only if } \eta = 0,\ \tau = \tau_j,\ p = \frac{2\pi}{\omega_0}. \tag{123}$$

Therefore, the hypotheses (A_1)–(A_4) in [13] are satisfied. Moreover, if we define

$$H^{\pm}\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right)(\eta, p) = \det\left(\Delta\left(E^*, \tau_j \pm \delta, p\right)\left(\eta + \frac{2\pi}{p}i\right)\right), \tag{124}$$

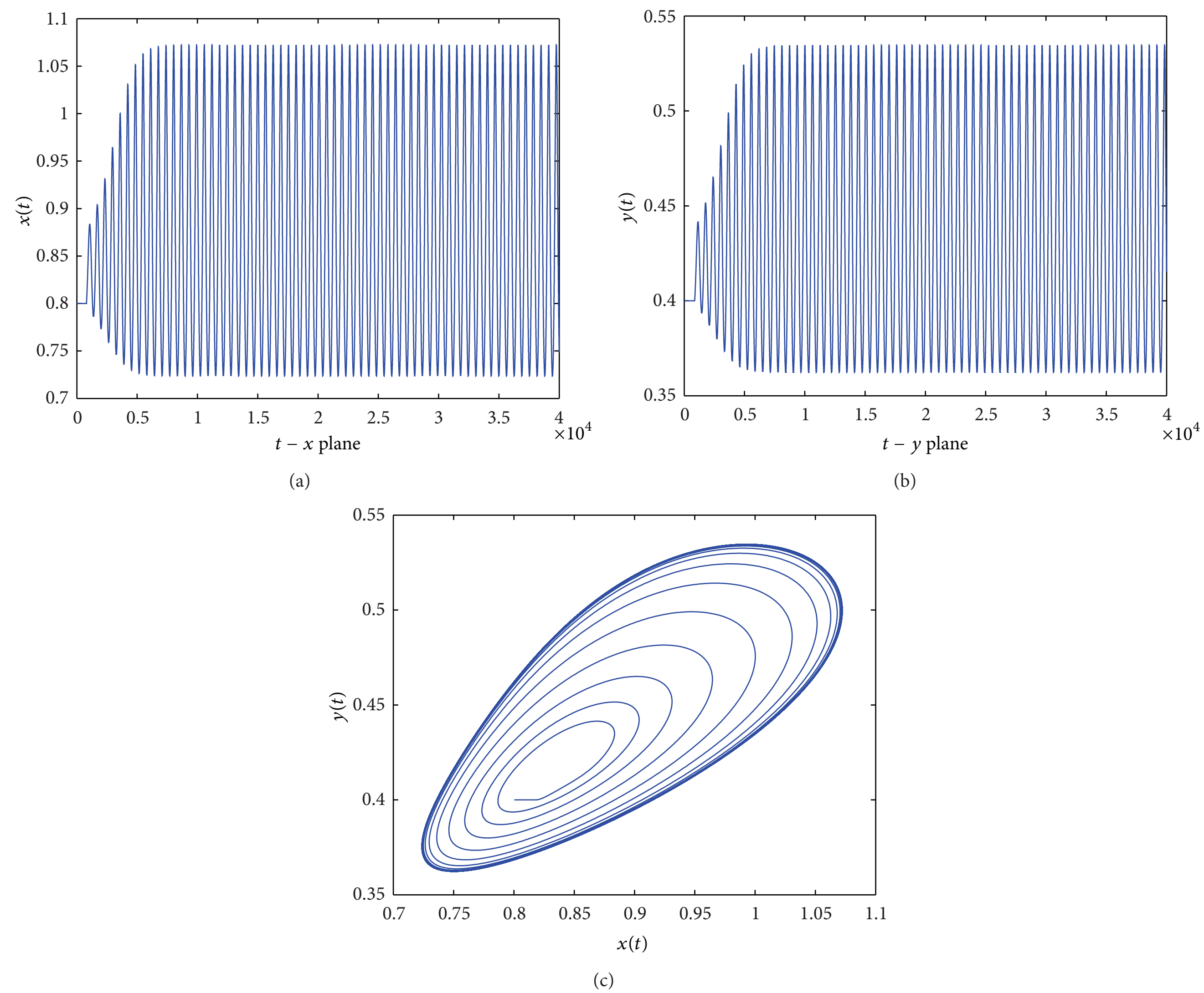

FIGURE 3: The trajectories and phase graphs of system (1) with $\tau = \tau_1 + \tau_2 = 6 + 6 = 12$.

then we have the crossing number of isolated center $(E^*, \tau_j, (2\pi/\omega_0))$ as follows:

$$\begin{aligned}\gamma\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right) &= \deg_B\left(H^-\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right), \Omega_{\epsilon,2\pi/\omega_0}\right) \\ &\quad - \deg_B\left(H^+\left(E^*, \tau_j, \frac{2\pi}{\omega_0}\right), \Omega_{\epsilon,2\pi/\omega_0}\right) \\ &= -1.\end{aligned} \tag{125}$$

Thus, we have

$$\sum_{(\overline{z},\overline{\tau},\overline{p})\in\mathscr{C}_{(E^*,\tau_j,2\pi/\omega_0)}} \gamma\left(\overline{z}, \overline{\tau}, \overline{p}\right) < 0, \tag{126}$$

where $(\overline{z}, \overline{\tau}, \overline{p})$ has all or parts of the form $(E^*, \tau_k, 2\pi/\omega_0)(k = 0, 1, \ldots)$. It follows from Lemma 8 that the connected component $\ell_{(E^*,\tau_j,2\pi/\omega_0)}$ through $(E^*, \tau_j, 2\pi/\omega_0)$ in Γ is unbounded. From (26), we can know that if (H1) holds, for $j \geq 1$,

$$\tau_j = \frac{1}{\omega_0}\arccos\frac{\omega_0^2 - p_1}{p_2} + \frac{2j\pi}{\omega_0} > \frac{2\pi}{\omega_0}. \tag{127}$$

Now we prove that the projection of $\ell_{(E^*,\tau_j,2\pi/\omega_0)}$ onto τ-space is $[\overline{\tau}, +\infty)$, where $\overline{\tau} \leq \tau_j$. Clearly, it follows from the proof of Lemma 10 that system (13) with $\tau = 0$ has no nontrivial periodic solution. Hence, the projection of $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ onto τ-space is away from zero.

For a contradiction, we suppose that the projection of $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ onto τ-space is bounded; this means that the projection of $\ell_{(E^*,\tau_j,(2\pi/\omega_0))}$ onto τ-space is included in an interval $(0, \tau^*)$. Noticing $(2\pi/\omega_0) < \tau_j$ and applying Lemma 10 we have $0 < p < \tau^*$ for $(z(t), \tau, p)$ belonging

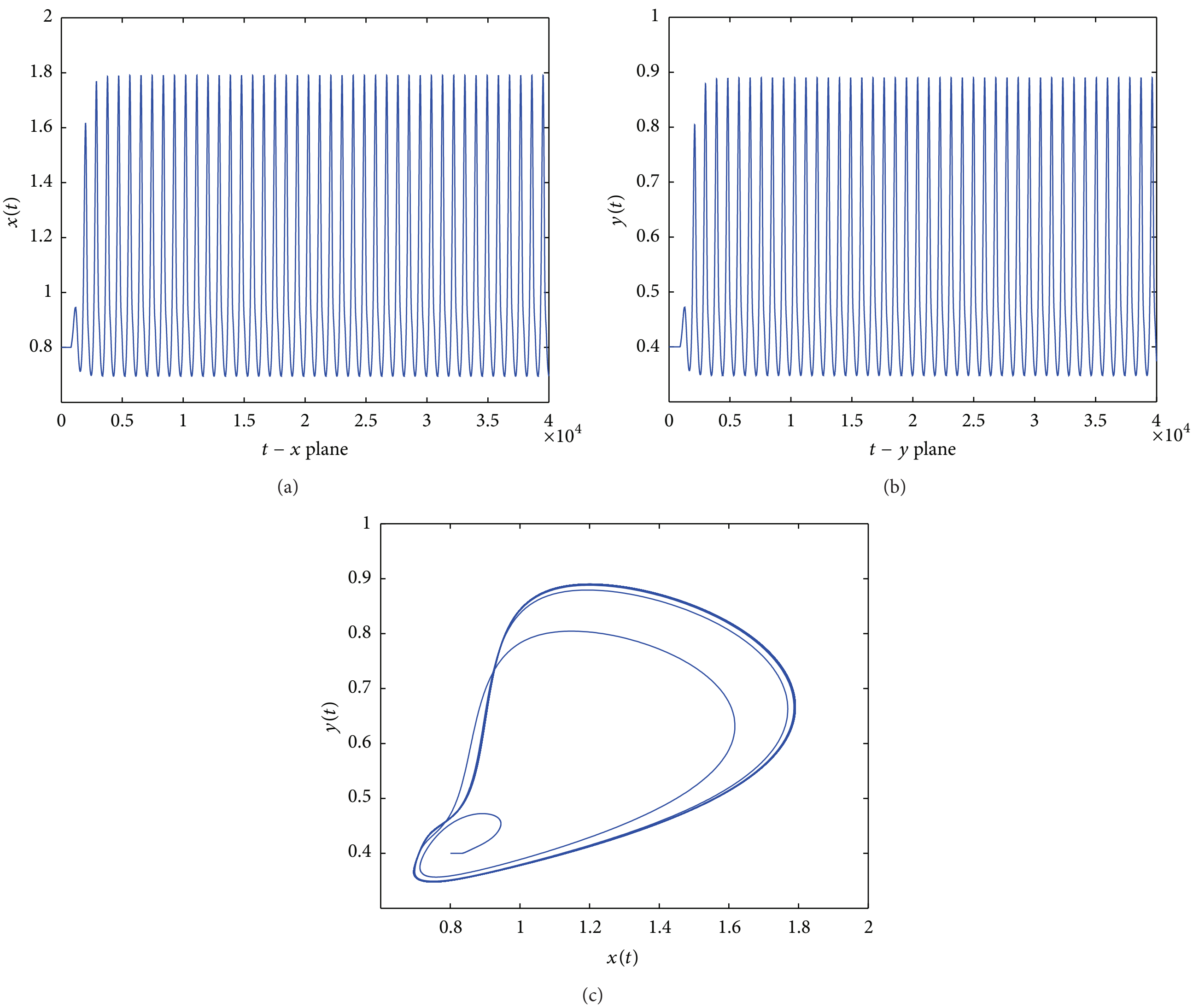

FIGURE 4: The trajectories and phase graphs of system (1) with $\tau = \tau_1 + \tau_2 = 10 + 8 = 18$.

to $\ell_{(E^*, \tau_j, (2\pi/\omega_0))}$. Applying Lemma 9, we know that the projection of $\ell_{(E^*, \tau_j, (2\pi/\omega_0))}$ onto z-space is bounded. So the component of $\ell_{(E^*, \tau_j, (2\pi/\omega_0))}$ is bounded. This contradicts our conclusion that $\ell_{(E^*, \tau_j, (2\pi/\omega_0))}$ is unbounded. The contradiction implies that the projection of $\ell_{(E^* \tau_j, (2\pi/\omega_0))}$ onto τ-space is unbounded above.

Hence, system (13) has at least $j - 1$ periodic solution for every $\tau > \tau_j$, $(j = 1, 2, \ldots)$. This completes the proof. □

Example 12. In system (1), we first choose $a_{11} = 0.1$, $a_{12} = 1$, $a_{21} = 3/2$, and $m = 2$. As depicted in Figure 1, a bifurcation diagram is given for system (1) with respect to the parameters r_1 and r_2. By the discussion in Section 3, system (1) always has a semitrivial equilibrium E_1, and if $r_2 > a_{21}$, E_1 is asymptotically stable; otherwise, E_1 is unstable. So if we choose $0 < r_2 < a_{21} = 3/2$, as depicted in Figure 1, E_1 is always unstable. In domains II, V, and VI, the positive equilibrium is not feasible. In domains I, III, and IV, system (1) has a unique positive equilibrium; it is locally asymptotically stable in domain I and is unstable in domain IV. In domain III, system (1) undergoes a Hopf bifurcation at the positive equilibrium at some τ_0. Further, we choose $r_1 = 5/12$, $r_2 = 1$, $a_{11} = 0.1, a_{12} = 1$, $a_{21} = 3/2$, and $m = 2$. In this case, system (1) has a positive equilibrium $E^* = (5/6, 5/12)$. By computation, we have $\omega_0 \approx 0.1063$, $\tau_0 \approx 10.8795$, and $\tau_1 \approx 69.9876$. From Theorem 5, E^* is stable when $\tau < \tau_0$ as illustrated by numerical simulations (see Figure 2). When τ passes through the critical value τ_0, the equilibrium E^* loses its stability and a Hopf bifurcation occurs; that is, a family of periodic solution bifurcates from E^*. By the algorithm derived in Section 3 and Section 4, we have $\lambda'(\tau_0) = 0.0053 - 0.0058i, c_1(0) = -0.4357 + 0.0265i$, which implies that $\mu_2 > 0$, $\beta_2 < 0$, and $T_2 > 0$. Thus, by the discussion in Section 4, the Hopf bifurcation is supercritical for $\tau > \tau_0$, the bifurcating periodic solutions from E^* at τ_0 are asymptotically stable, and the period of these periodic solutions is increasing with the increasing of τ, which are depicted in Figures 3, 4, and 5.

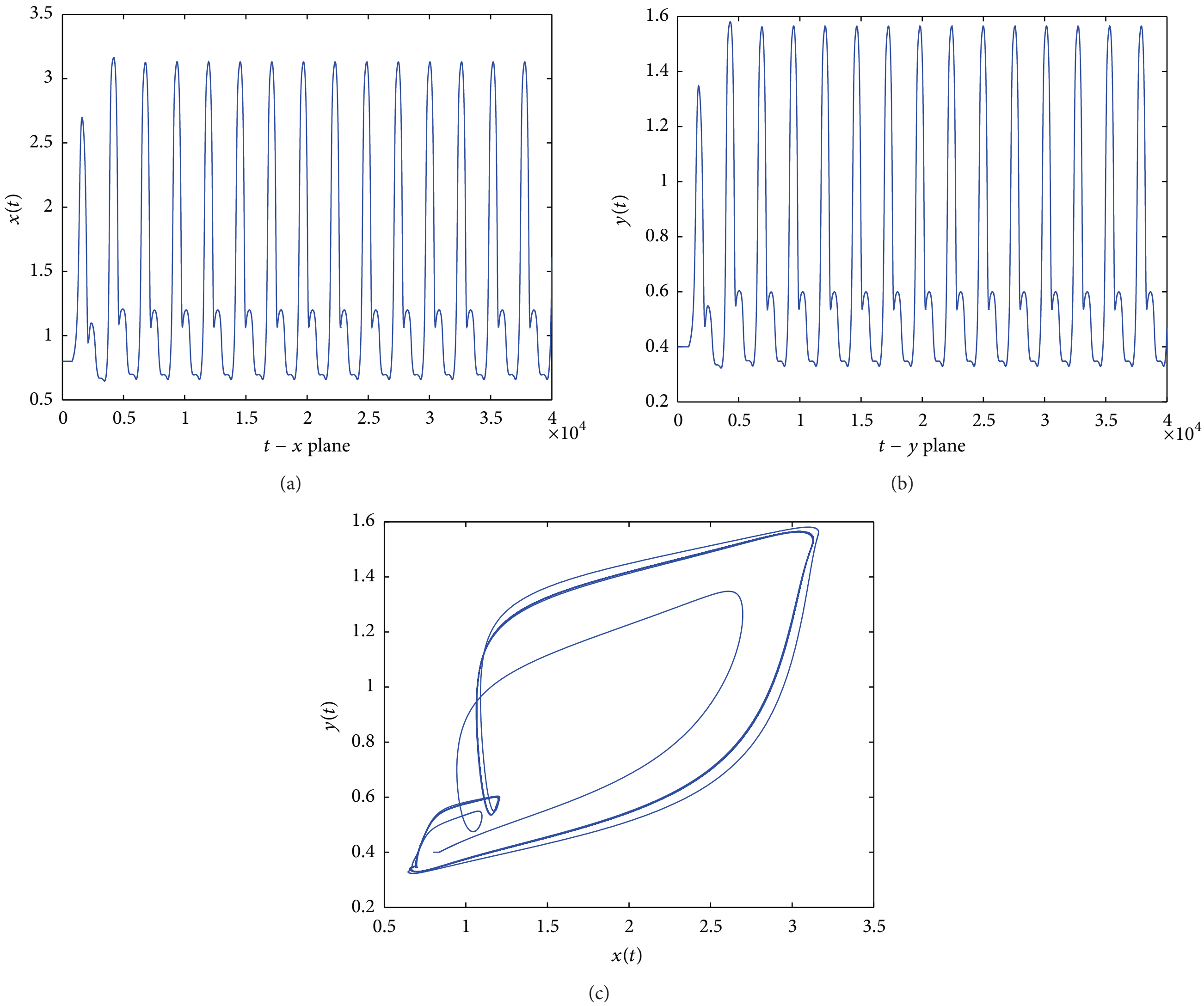

Figure 5: The trajectories and phase graphs of system (1) with $\tau = \tau_1 + \tau_2 = 10 + 60 = 70$.

Furthermore, Figure 5 shows that the local Hopf bifurcation implies the global Hopf bifurcation after the second critical value of $\tau_1 = 69.9876$.

7. Discussion

In this paper, we have studied a ratio-dependent predator-prey model with two time delays. By analyzing the corresponding characteristic equation, the local stability of the positive equilibrium and the semitrivial equilibrium of system (1) was discussed. We have obtained the estimated length of gestation delay which would not affect the stable coexistence of both prey and predator species at their equilibrium values. The existence of Hopf bifurcation for system (1) at the positive equilibrium was also established. From theoretical analysis it was shown that the larger values of gestation time delay cause fluctuation in individual population density and hence the system becomes unstable. As the estimated length of delay to preserve stability and the critical length of time delay for Hopf bifurcation are dependent upon the parameters of system, it is possible to impose some control, which will prevent the possible abnormal oscillation in population density. The global attractiveness result in Theorem 6 implied that system (1) is permanent if the intrinsic growth rate of the prey and the conversion rate and the interference rate of the predator are high, and the death rate of the predator is low. From Theorem 7 we see that if the death rate of the predator is greater than the conversion rate of the predator, the predator population become extinct for any gestation delay. In particular, the results about boundedness and attractiveness are similar to the results of [4]. From the discussion in Sections 3 and 4, we see that if the values of r_1, r_2, a_{11}, a_{12}, a_{21}, and m are given, we can get the Hopf bifurcation value of τ, and further we may determine the direction of Hopf bifurcation and the stability of periodic solutions bifurcating from the positive equilibrium E^* at the critical point τ_0.

Furthermore, we show that the local Hopf bifurcation implies the global Hopf bifurcation after the second critical value of delay.

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