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*Research Article*

# The Asymptotic Stability of the Generalized 3D Navier-Stokes Equations

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We study the stability issue of the generalized 3D Navier-Stokes equations. It is shown that if the weak solution $u$ of the Navier-Stokes equations lies in the regular class $\nabla u \in L^p(0,\infty;B^0_{q,\infty}(\mathbb{R}^3))$, $(2\alpha/p)+(3/q)=2\alpha$, $2<q<\infty$, $0<\alpha<1$, then every weak solution $v(x,t)$ of the perturbed system converges asymptotically to $u(x,t)$ as $\|v(t)-u(t)\|_{L^2} \to 0$, $t\to\infty$.

## 1. Introduction and Main Result

In this study, we consider the Cauchy problem of the generalized 3D Navier-stokes equations:

$$
\begin{gathered}
u_t+(-\Delta)^{\alpha}u+(u\cdot\nabla)u+\nabla\pi=f, \quad (x,t)\in\mathbb{R}^3\times(0,\infty), \\
\nabla\cdot u=0, \\
u(x,0)=u_0.
\end{gathered}
\tag{1}
$$

Here, $0<\alpha<1$, and $u$ and $\pi$ denote unknown velocity and pressure, respectively. $f$ is the external force and $u_0$ is a given initial velocity.

It is well known that when $\alpha=1$, system (1) becomes the classic Navier-Stokes equations. For the Navier-Stokes equations, it is proved that it has a global weak solution

$$
u(x,t)\in L^{\infty}\left(0,T;L^2\right)\cap L^2\left(0,T;H^1\right), \quad \forall T>0 \tag{2}
$$

for given $u_0\in L^2(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$ [1]. However, the regularity of Leray weak solutions is still an open problem in mathematical fluid mechanics even if much effort has been made [2–4]. It is an interesting problem to investigate the stability properties of the Navier-Stokes equations and related fluid models [5–11]. As regard to the above system (1), the asymptotic stability of weak solution of the generalized 3D Navier-Stokes equation is described as follows. If $u$ is perturbed initially by $\omega_0$ without any smallness assumption, then the perturbed system $v$ is governed by the following equations:

$$
\begin{gathered}
v_t+(-\Delta)^{\alpha}v+(v\cdot\nabla)v+\nabla\pi=f, \\
\nabla\cdot v=0, \\
v(x,0)=u_0+\omega_0,
\end{gathered}
\tag{3}
$$

where $\omega_0$ is the initial perturbation. There is large literature on the stability issue of the classic Navier-Stokes equations and related fluid models [12–17]. The aim of this paper is to show the stability of weak solution in the framework of the homogeneous Besov space. More precisely, with the use of the Littlewood-Paley decomposition and the classic Fourier splitting technique, we can show that when the initial perturbation $\omega_0\in L^2(\mathbb{R}^3)$, then every weak solution $v(t)$ of the perturbed system (2) converges asymptotically to $u(t)$ as $\|v(t)-u(t)\|_{L^2}\to 0$, $t\to\infty$.

Now our result reads as follows.

**Theorem 1.** *Let $f\in L^2(0,T;H^{-\alpha}(\mathbb{R}^3))$, $\omega_0\in L^2(\mathbb{R}^3)$; Suppose that $u(x,t)$ is a weak solution of (1) and that $v(x,t)$ is a weak solution of the perturbed problem (2), respectively. Moreover, if $\nabla u$ also lies in the following regular class:*

$$
\nabla u\in L^p\left(0,\infty;B^0_{q,\infty}\left(\mathbb{R}^3\right)\right), \quad \frac{2\alpha}{p}+\frac{3}{q}=2\alpha, \quad 2<q<\infty, \tag{4}
$$

*then $\|v(t)-u(t)\|_{L^2}\to 0$ $(t\to\infty)$.*

The remainder of this paper is organized as follows. In the Section 2, we first recall the Littlewood-Paley decomposition and the Bony decomposition; then we give three key lemmas. And we prove asymptotic stability of the weak solution in the Section 3.

## 2. Some Auxiliary Lemmas

We recall some basic facts about the Littlewood-Paley decomposition (refer to [18]). Let $\mathcal{S}(\mathbb{R}^3)$ be Schwartz class of rapidly decreasing functions; supposing $f \in \mathcal{S}(\mathbb{R}^3)$, the Fourier transformation $\mathcal{F}$ is defined by

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,dx. \tag{5}$$

Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$, supported in $\mathcal{B} = \{\xi \in \mathbb{R}^3, |\xi| \le 4/3\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3, 3/4 \le |\xi| \le 8/3\}$, respectively, such that

$$\chi(\xi) + \sum_{j\ge 0} \varphi\left(2^{-j}\xi\right) = 1, \quad \xi \in \mathbb{R}^3. \tag{6}$$

Let $h = \mathcal{F}^{-1}\varphi$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$, we define the dyadic blocks as follows:

$$\begin{aligned} \Delta_j f &= \varphi\left(2^{-j}D\right) f \\ &= 2^{3j} \int_{\mathbb{R}^3} h\left(2^j y\right) f(x-y)\,dy, \quad \text{for } j \ge 0, \\ S_j f &= \chi\left(2^{-j}D\right) f = \sum_{-1\le k\le j-1} \Delta_k f \\ &= 2^{3j} \int_{\mathbb{R}^3} \widetilde{h}\left(2^j y\right) f(x-y)\,dy, \\ \Delta_{-1} f &= S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \le -2. \end{aligned} \tag{7}$$

We can easily verify that

$$\begin{aligned} \Delta_j \Delta_k f &= \varphi\left(2^{-j}\xi\right)\varphi\left(2^{-k}\xi\right)\widehat{f} = 0, \quad \text{if } |j-k| \ge 2, \\ \Delta_j\left(S_{k-1} f \Delta_k f\right) &= \varphi\left(2^{-j}\xi\right)\chi\left(2^{-(k-1)}\xi\right)\widehat{f} \\ &\quad \times \varphi\left(2^{-k}\xi\right)\widehat{f} = 0, \quad \text{if } |j-k| \ge 5. \end{aligned} \tag{8}$$

Especially for any $f \in L^2(\mathbb{R}^3)$, we have the Littlewood-Paley decomposition:

$$f = S_0(f) + \sum_{j\ge 0} \Delta_j f, \quad f \in \mathcal{S}'\left(\mathbb{R}^3\right). \tag{9}$$

Now we give the definition of the Besov space. Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$; the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^3)$ (see [18]) is defined by the full-dyadic decomposition, such as

$$B^s_{p,q}\left(\mathbb{R}^3\right) = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^3\right) : \|f\|_{B^s_{p,q}} < \infty \right\}, \tag{10}$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} \left( \sum_{j=-1}^{\infty} 2^{jsq} \left\|\Delta_j f\right\|^q_{L_p} \right)^{1/q}, & 1 \le q < \infty, \\ \sup_{j\ge -1} 2^{js} \left\|\Delta_j f\right\|_{L_p}, & q = \infty, \end{cases} \tag{11}$$

and $\mathcal{S}'(\mathbb{R}^3)$ is a dual space of $\mathcal{S}(\mathbb{R}^3)$.

The Bony decomposition (see [19]) will be frequently used; it is followed by

$$uv = T_u v + T_v u + R(u, v), \tag{12}$$

where

$$T_u v = \sum_j S_{j-1} u \Delta_j v, \qquad R(u, v) = \sum_{|j'-j|\le 1} \Delta_j u \Delta_{j'} v. \tag{13}$$

The following Bernstein inequality (see [18]) will be used in the next section.

**Lemma 2.** *Assume that $k, j \in Z$ and $1 \le p \le q \le \infty$, for $f \in \mathcal{S}(\mathbb{R}^3)$, one has*

$$\sup_{|\alpha|=k} \left\|\partial^\alpha \Delta_j f\right\|_{L^q(\mathbb{R}^3)} \le C 2^{jk+3j((1/p)-(1/q))} \left\|\Delta_j f\right\|_{L^p(\mathbb{R}^3)}, \tag{14}$$

*and the constant $C$ is independent of $j$ and $k$.*

In the following, we will introduce two lemmas, which will be employed in the proof of our theorem.

**Lemma 3.** *Suppose that $u, w \in L^\infty(0, T; L^2) \cap L^2(0, T; H^\alpha)$, for all $T > 0$, $\nabla v \in L^p(0, \infty; B^0_{q,\infty})$, $(2\alpha/p) + (3/q) = 2\alpha$, $2 < q < \infty$.*

*Then the trilinear form*

$$F(u, v, w) = \int_0^T \int_{\mathbb{R}^3} (u \cdot \nabla v)\, w\, dx\, dt \tag{15}$$

*is continuous and*

$$\begin{aligned} |F(u, v, w)| \le{}& C\|u\|^{1/p}_{L^\infty(0,T;L^2)} \|u\|^{1-(1/p)}_{L^2(0,T;H^\alpha)} \|w\|^{1/p}_{L^\infty(0,T;L^2)} \\ & \times \|w\|^{1-(1/p)}_{L^2(0,T;H^\alpha)} \|\nabla v\|_{L^p(0,T;B^0_{q,\infty})}. \end{aligned} \tag{16}$$

*In particular, if $u = w$, then*

$$|F(w, v, w)| \le \frac{1}{2} \int_0^T \left\|\Lambda^\alpha w\right\|^2_{L^2} dt + C \int_0^T \|w\|^2_{L^2} \|\nabla v\|^p_{B^0_{q,\infty}}\, dt. \tag{17}$$

*Proof of Lemma 3.* We borrow the idea of [20] to prove this lemma. By using of the Littlewood-Paley decomposition and the Bony decomposition, we obtain

$$\begin{aligned}
F(u,v,w) &= \int_0^T \int_{\mathbb{R}^3} \left(u^i w\right) \partial_i v \, dx\, dt \\
&= \int_0^T \int_{\mathbb{R}^3} \left(T_{u^i} w + T_w u^i + R\left(u^i, w\right)\right) \\
&\qquad \times \left(\sum_j \Delta_j \partial_i v\right) dx\, dt \\
&= \sum_{|k-j|\le 4} \int_0^T \int_{\mathbb{R}^3} S_{k-1} u^i \Delta_k w \Delta_j \partial_i v \, dx\, dt \\
&\quad + \sum_{|k-j|\le 4} \int_0^T \int_{\mathbb{R}^3} \Delta_k u^i S_{k-1} w \Delta_j \partial_i v \, dx\, dt \\
&\quad + \sum_{|k-k'|\le 1} \sum_{k,k'\ge j-3} \int_0^T \int_{\mathbb{R}^3} \Delta_k u^i \Delta_{k'} w \Delta_j \partial_i v \, dx\, dt \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{18}$$

Then we estimate $I_1$, $I_2$, and $I_3$ one by one. Applying the Hölder inequality and the Bernstein inequality (40), we derive

$$\begin{aligned}
\left|I_1\right| &\le C \sum_{|k-j|\le 4} \sum_{k'\le k-2} \int_0^T \left\|\Delta_{k'} u^i\right\|_{L^{2q/(q-2)}} \left\|\Delta_k w\right\|_{L^2} \left\|\Delta_j \partial_i v\right\|_{L^q} dt \\
&\le C \sum_{|k-j|\le 4} \sum_{k'\le k-2} \int_0^T 2^{(3/q)k'} \left\|\Delta_{k'} u^i\right\|_{L^2} \left\|\Delta_k w\right\|_{L^2} \left\|\Delta_j \partial_i v\right\|_{L^q} dt \\
&\le C \sum_{|k-j|\le 4} \sum_{k'\le k-2} \int_0^T \left(2^{(\alpha/p')k'} \left\|\Delta_{k'} u\right\|_{L^2}\right) \\
&\qquad \times \left(2^{(\alpha/p')k} \left\|\Delta_k w\right\|_{L^2}\right) \\
&\qquad \times \left\|\Delta_j \nabla v\right\|_{L^q} 2^{((3/q)-(\alpha/p'))k'-(\alpha/p')k} dt,
\end{aligned} \tag{19}$$

where $(1/p) + (1/p') = 1$.

Since $|k-j| \le 4$, $k' < k$ and $(2\alpha/p) + (3/q) = 2\alpha$ with $2 < q < \infty$, then

$$\begin{aligned}
2^{((3/q)-(\alpha/p'))k'-(\alpha/p')k} &= 2^{((3/q)-\alpha+(\alpha/p))k'-(\alpha-(\alpha/p))k} \\
&= 2^{(3/2q)(k'-k)} \le C.
\end{aligned} \tag{20}$$

Thanks to the Sobolev embedding $B^{\alpha/p'}_{2,\infty}(\mathbb{R}^3) \hookrightarrow B^{\alpha/p'}_{2,2}(\mathbb{R}^3) = H^{\alpha/p'}(\mathbb{R}^3)$, we have the following estimate:

$$\left|I_1\right| \le C \int_0^T \|u\|_{H^{\alpha/p'}} \|w\|_{H^{\alpha/p'}} \|\nabla v\|_{B^0_{q,\infty}} dt. \tag{21}$$

Similarly, for $I_2$, we also have

$$\left|I_2\right| \le C \int_0^T \|u\|_{H^{\alpha/p'}} \|w\|_{H^{\alpha/p'}} \|\nabla v\|_{B^0_{q,\infty}} dt. \tag{22}$$

To estimate the last term $I_3$, by using the Hölder inequality and the Bernstein inequality we obtain

$$\begin{aligned}
\left|I_3\right| &\le C \sum_{|k-k'|\le 1} \sum_{k,k'\ge j-3} \int_0^T \left\|\Delta_k u^i\right\|_{L^2} \left\|\Delta_{k'} w\right\|_{L^2} \left\|\Delta_j \partial_i v\right\|_{L^\infty} dt \\
&\le C \sum_{|k-k'|\le 1} \sum_{k,k'\ge j-3} \int_0^T \left\|\Delta_k u^i\right\|_{L^2} \left\|\Delta_{k'} w\right\|_{L^2} \\
&\qquad \times \left(2^{(3/q)j} \left\|\Delta_j \partial_i v\right\|_{L^q}\right) dt \\
&\le C \sum_{|k-k'|\le 1} \sum_{k,k'\ge j-3} \int_0^T \left(2^{(\alpha/p')k} \left\|\Delta_k u\right\|_{L^2}\right) \\
&\qquad \times \left(2^{(\alpha/p')k'} \left\|\Delta_{k'} w\right\|_{L^2}\right) \\
&\qquad \times \left\|\Delta_j \nabla v\right\|_{L^q} 2^{-(3/q)j \; -(\alpha/p')(k+k')} dt.
\end{aligned} \tag{23}$$

Since $|k-k'| \le 1$, $k, k' \ge j-3$ and $(2\alpha/p) + (3/q) = 2\alpha$, $2 < q < \infty$, we have

$$\begin{aligned}
&2^{-(3/q)j-(\alpha/p')(k+k')} = 2^{-(3/q)j-(3/2)(k+k')(1/q)} \le 2^{9/q} \le C, \\
&\left|I_3\right| \le C \int_0^T \|u\|_{H^{\alpha/p'}} \|w\|_{H^{\alpha/p'}} \|\nabla v\|_{B^0_{q,\infty}} dt.
\end{aligned} \tag{24}$$

So, we can derive

$$\begin{aligned}
&|F(u,v,w)| \\
&\quad \le C \int_0^T \|u\|_{H^{\alpha/p'}} \|w\|_{H^{\alpha/p'}} \|\nabla v\|_{B^0_{q,\infty}} dt \\
&\quad \le C \left(\int_0^T \|u\|^{2p'}_{H^{\alpha/p'}} dt\right)^{1/2p'} \left(\int_0^T \|w\|^{2p'}_{H^{\alpha/p'}} dt\right)^{1/2p'} \\
&\qquad \times \left(\int_0^T \|\nabla v\|^{p}_{B^0_{q,\infty}} dt\right)^{1/p} \\
&\quad \le C \|u\|_{L^{2p'}(0,T;H^{\alpha/p'})} \|w\|_{L^{2p'}(0,T;H^{\alpha/p'})} \|\nabla v\|_{L^p(0,T;B^0_{q,\infty})}.
\end{aligned} \tag{25}$$

Applying the interpolation inequality, we have

$$\begin{aligned}
\|u\|_{L^{2p'}(0,T;H^{\alpha/p'})} &\le C \|u\|^{1-(1/p')}_{L^\infty(0,T;L^2)} \cdot \|u\|^{1/p'}_{L^2(0,T;H^\alpha)} \\
&\le C \|u\|^{1/p}_{L^\infty(0,T;L^2)} \cdot \|u\|^{1-(1/p)}_{L^2(0,T;H^\alpha)}.
\end{aligned} \tag{26}$$

Then

$$\begin{aligned}
|F(u,v,w)| &\le C \|u\|^{1/p}_{L^\infty(0,T;L^2)} \|u\|^{1-(1/p)}_{L^2(0,T;H^\alpha)} \|w\|^{1/p}_{L^\infty(0,T;L^2)} \\
&\quad \times \|w\|^{1-(1/p)}_{L^2(0,T;H^\alpha)} \|\nabla v\|_{L^p(0,T;B^0_{q,\infty})}.
\end{aligned} \tag{27}$$

Especially if $u = w$, by using the interpolation inequality, we get

$$\begin{aligned}|F(u,v,w)| &\le C\int_0^T \|w\|_{H^{\alpha/p'}}^2 \|\nabla v\|_{B^0_{q,\infty}} dt \\ &\le C\int_0^T \|w\|_{L^2}^{2(1-(1/p'))} \|\Lambda^\alpha w\|_{L^2}^{2/p'} \|\nabla v\|_{B^0_{q,\infty}} dt \\ &\le \frac{1}{2}\int_0^T \|\Lambda^\alpha w\|_{L^2}^2 dt + C\int_0^T \|w\|_{L^2}^2 \|\nabla v\|_{B^0_{q,\infty}}^p dt.\end{aligned} \tag{28}$$

Hence, the proof of the lemma is complete. □

Let $w(x,t) = v(x,t) - u(x,t)$ denote the difference of $v(x,t)$ and $u(x,t)$, where $u(x,t)$ is a weak solution of (1) and $v(x,t)$ is a weak solution of the perturbed problem (2). Thus $w(x,t)$ satisfies the following equations:

$$\begin{gathered} w_t + (-\Delta)^\alpha w + (v\cdot\nabla) w + (w\cdot\nabla) u + \nabla\pi = 0, \\ (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \nabla\cdot w = 0, \\ w(x,0) = w_0. \end{gathered} \tag{29}$$

**Lemma 4.** *Let $w(x,t)$ be the solution of the above problem. Then*

$$|\widehat{w}(\xi,t)| \le e^{-|\xi|^{2\alpha}t} |\widehat{w}_0(\xi)| + C|\xi| t. \tag{30}$$

*Proof of Lemma 4.* Taking the Fourier transformation of the first equation of (38), we get

$$\widehat{w}_t + |\xi|^{2\alpha}\widehat{w} = F[-(v\cdot\nabla)w - (w\cdot\nabla)u - \nabla\pi] =: G(\xi,t). \tag{31}$$

We can easily obtain

$$\begin{aligned}|F[-(v\cdot\nabla)w]| &\le \sum_{i,j}\int_{\mathbb{R}^3} |v_i w_j| |\xi_j| dx \le |\xi| \|v\|_{L^2}\|w\|_{L^2}, \\ |F[-(w\cdot\nabla)u]| &\le \sum_{i,j}\int_{\mathbb{R}^3} |w_i u_j| |\xi_j| dx \le |\xi| \|w\|_{L^2}\|u\|_{L^2}.\end{aligned} \tag{32}$$

Applying the operator $\nabla\,\text{div}$ to the first equation of (38), we have

$$\Delta\pi = \sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(-v_i w_j - w_i u_j), \tag{33}$$

and taking the Fourier transformation, we get

$$|\xi|^2 F[\pi] = \sum_{i,j}\xi_i\xi_j F[-v_i w_j - w_i u_j]; \tag{34}$$

thus

$$|F[\nabla\pi]| \le |\xi| |F[\pi]| \le |\xi| \|w\|_{L^2}(\|u\|_{L^2} + \|v\|_{L^2}). \tag{35}$$

Then we have

$$|G(\xi,t)| \le |\xi| \|w\|_{L^2}(\|u\|_{L^2} + \|v\|_{L^2}). \tag{36}$$

Thus solving the ordinary differential equation (31) and using (36) gives

$$\begin{aligned}|\widehat{w}(\xi,t)| &= \left|\widehat{w}_0(\xi)e^{-|\xi|^{2\alpha}t} + \int_0^t e^{-|\xi|^{2\alpha}(t-s)} G(\xi,s) ds\right| \\ &\le |\widehat{w}_0(\xi)| e^{-|\xi|^{2\alpha}t} + C|\xi|\int_0^t \|w\|_{L^2}(\|u\|_{L^2} + \|v\|_{L^2}) ds \\ &\le e^{-|\xi|^{2\alpha}t}|\widehat{w}_0(\xi)| + C|\xi| t,\end{aligned} \tag{37}$$

which is the desired assertion of Lemma 4. □

## 3. Proof of Theorem 1

The following argument is follows the classic Fourier splitting methods which is first used by Schonbek [21] (see also [22]).

Taking the inner product of the first equation in (38) with $w$ together with the divergence-free condition of $v, w$ we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \int_{\mathbb{R}^3} |\Lambda^\alpha w|^2 dx = -\int_{\mathbb{R}^3} (w\cdot\nabla) u \cdot w\, dx. \tag{38}$$

Applying Plancherel's theorem to (38) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\widehat{w}(\xi,t)|^2 d\xi + \int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widehat{w}(\xi,t)|^2 d\xi \\ = -\int_{\mathbb{R}^3}(w\cdot\nabla)u\cdot w\,dx.\end{aligned} \tag{39}$$

Let $f(t)$ be a continuous function of $t$ with $f(0) = 1$, $f(t) > 0$ and $f'(t) > 0$, we can derive the following:

$$\begin{aligned}&\frac{d}{dt}\left(f(t)\int_{\mathbb{R}^3}|\widehat{w}(\xi,t)|^2 d\xi\right) \\ &\quad + 2f(t)\int_{\mathbb{R}^3}|\xi|^{2\alpha}|\widehat{w}(\xi,t)|^2 d\xi \\ &= -2f(t)\int_{\mathbb{R}^3}(w\cdot\nabla)u\cdot w\,dx \\ &\quad + f'(t)\int_{\mathbb{R}^3}|\widehat{w}(\xi,t)|^2 d\xi.\end{aligned} \tag{40}$$

By integrating in time from 0 to $t$ for (40), we have

$$\begin{aligned}&f(t)\int_{\mathbb{R}^3}|\widehat{w}(\xi,t)|^2 d\xi \\ &\quad + 2\int_0^t f(s)\int_{\mathbb{R}^3}|\xi|^{2\alpha}|\widehat{w}(\xi,s)|^2 d\xi\, ds \\ &= \int_{\mathbb{R}^3}|\widehat{w}_0|^2 d\xi - 2\int_0^t f(s)\int_{\mathbb{R}^3}(w\cdot\nabla)u\cdot w\,dx\,ds \\ &\quad + \int_0^t f'(s)\int_{\mathbb{R}^3}|\widehat{w}(\xi,s)|^2 d\xi\, ds.\end{aligned} \tag{41}$$

Noting that $f(t)$ is a scalar function and applying Lemma 3, we get

$$\begin{aligned}&\left|\int_0^t f(s)\int_{\mathbb{R}^3}(w\cdot\nabla)u\cdot w\,dx\,ds\right|\\&\quad\le\frac{1}{2}\int_0^t f(s)\left\|\Lambda^\alpha w\right\|_{L^2}^2 ds+C\int_0^t f(s)\|w\|_{L^2}^2\|\nabla u\|_{B^0_{q,\infty}}^p ds\\&\quad\le\frac{1}{2}\int_0^t f(s)\int_{\mathbb{R}^3}|\xi|^{2\alpha}\left|\widehat{w}(\xi,s)\right|^2 d\xi\,dt\\&\qquad+C\int_0^t f(s)\|w\|_{L^2}^2\|\nabla u\|_{B^0_{q,\infty}}^p ds.\end{aligned}\tag{42}$$

Then,

$$\begin{aligned}&f(t)\int_{\mathbb{R}^3}\left|\widehat{w}(\xi,t)\right|^2 d\xi\\&\qquad+\int_0^t f(s)\int_{\mathbb{R}^3}|\xi|^{2\alpha}\left|\widehat{w}(\xi,s)\right|^2 d\xi\,ds\\&\quad\le\int_{\mathbb{R}^3}\left|\widehat{w}_0\right|^2 d\xi+\int_0^t f'(s)\int_{\mathbb{R}^3}\left|\widehat{w}(\xi,s)\right|^2 d\xi\,ds\\&\qquad+C\int_0^t f(s)\|w\|_{L^2}^2\|\nabla u\|_{B^0_{q,\infty}}^p ds.\end{aligned}\tag{43}$$

Let $B(t)=\{\xi\in\mathbb{R}^3: f(t)|\xi|^{2\alpha}<f'(t)\}$, we have

$$\begin{aligned}f(s)\int_{\mathbb{R}^3}|\xi|^{2\alpha}\left|\widehat{w}(\xi,s)\right|^2 d\xi&\ge f'(s)\int_{\mathbb{R}^3}\left|\widehat{w}(\xi,s)\right|^2 d\xi\\&\quad-f'(s)\int_{B(s)}\left|\widehat{w}(\xi,s)\right|^2 d\xi.\end{aligned}\tag{44}$$

Then,

$$\begin{aligned}&f(t)\int_{\mathbb{R}^3}\left|\widehat{w}(\xi,t)\right|^2 d\xi\\&\quad\le\int_{\mathbb{R}^3}\left|\widehat{w}_0(\xi)\right|^2 d\xi+C\int_0^t f(s)\|w\|_{L^2}^2\|\nabla u\|_{B^0_{q,\infty}}^p ds\\&\qquad+\int_0^t f'(s)\int_{B(s)}\left|\widehat{w}(\xi,s)\right|^2 d\xi\,ds.\end{aligned}\tag{45}$$

In addition,

$$\begin{aligned}&\int_0^t f'(s)\int_{B(s)}\left|\widehat{w}(\xi,s)\right|^2 d\xi\,ds\\&\quad\le C\int_0^t f'(s)\int_{B(s)}\left(e^{-2|\xi|^{2\alpha}s}\left|\widehat{w}_0(\xi)\right|^2+|\xi|^2 s^2\right)d\xi\,ds\\&\quad\le C\int_0^t f'(s)\left(\int_{\mathbb{R}^3}e^{-2|\xi|^{2\alpha}s}\left|\widehat{w}_0(\xi)\right|^2 d\xi\right)ds\\&\qquad+C\int_0^t f'(s)\,s^2\left(\frac{f'(s)}{f(s)}\right)^{5/2\alpha}ds.\end{aligned}\tag{46}$$

Choose $f(t)=(1+t)^2$, then

$$\begin{aligned}&(1+t)^2\int_{\mathbb{R}^3}\left|\widehat{w}(\xi,t)\right|^2 d\xi\\&\quad\le C+C\int_0^t(1+s)^2\|w\|_{L^2}^2\|\nabla u\|_{B^0_{q,\infty}}^p ds\\&\qquad+C\int_0^t(1+s)\int_{\mathbb{R}^3}e^{-2|\xi|^{2\alpha}s}\left|\widehat{w}_0(\xi,s)\right|^2 d\xi\,ds\\&\qquad+C(1+t)^{4-(5/2\alpha)},\\&(1+t)^2\|w\|_{L^2}^2\\&\quad\le C\int_0^t(1+s)\int_{\mathbb{R}^3}e^{-2|\xi|^{2\alpha}s}\left|\widehat{w}_0(\xi)\right|^2 d\xi\,ds\\&\qquad+C\int_0^t(1+s)^2\|w\|_{L^2}^2\|\nabla u\|_{B^0_{q,\infty}}^p ds\\&\qquad+C(1+t)^{4-(5/2\alpha)}.\end{aligned}\tag{47}$$

By using the Gronwall inequality, it follows that

$$\begin{aligned}&(1+t)^2\|w\|_{L^2}^2\\&\le\left\{C\int_0^t(1+s)\int_{\mathbb{R}^3}e^{-2|\xi|^{2\alpha}s}\left|\widehat{w}_0(\xi)\right|^2 d\xi\,ds+C(1+t)^{4-(5/2\alpha)}\right\}\\&\quad\times\exp\left(\int_0^t\|\nabla u\|_{B^0_{q,\infty}}^p ds\right).\end{aligned}\tag{48}$$

Since

$$\int_{\mathbb{R}^3}e^{-2|\xi|^{2\alpha}t}\left|\widehat{w}_0(\xi)\right|^2 d\xi\le C(1+t)^{-3/2\alpha}\longrightarrow 0,\quad t\longrightarrow\infty,\tag{49}$$

we derive

$$\begin{aligned}\|w\|_{L^2}&\le C(1+t)^{-2}\int_0^t(1+s)\int_{\mathbb{R}^3}e^{-2|\xi|^{2\alpha}s}\left|\widehat{w}_0(\xi)\right|^2 d\xi\,ds\\&\quad+C(1+t)^{2-(5/2\alpha)}\longrightarrow 0,\quad t\longrightarrow\infty,\end{aligned}\tag{50}$$

which completes the proof of Theorem 1.


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