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*Research Article*

# Some New Existence Results of Positive Solutions to an Even-Order Boundary Value Problem on Time Scales

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We consider a high-order three-point boundary value problem. Firstly, some new existence results of at least one positive solution for a noneigenvalue problem and an eigenvalue problem are established. Our approach is based on the application of three different fixed point theorems, which have extended and improved the famous Guo-Krasnosel'skii fixed point theorem at different aspects. Secondly, some examples are included to illustrate our results.

## 1. Introduction

We are concerned with the following even-order three-point boundary value problem on time scales $\mathbb{T}$:

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t) = f\left(t, y(\sigma(t))\right), \quad t \in [t_1, t_3] \subset \mathbb{T}, \\ y^{\Delta^{2i+1}}(t_1) = 0, \\ \alpha y^{\Delta^{2i}}(\sigma(t_3)) + \beta y^{\Delta^{2i+1}}(\sigma(t_3)) = y^{\Delta^{2i+1}}(t_2),\end{gathered} \tag{1}$$

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t) = \lambda a(t) g\left(y(\sigma(t))\right), \quad t \in [t_1, t_3] \subset \mathbb{T}, \\ y^{\Delta^{2i+1}}(t_1) = 0, \\ \alpha y^{\Delta^{2i}}(\sigma(t_3)) + \beta y^{\Delta^{2i+1}}(\sigma(t_3)) = y^{\Delta^{2i+1}}(t_2),\end{gathered} \tag{2}$$

for $0 \le i \le n-1$, where $\alpha > 0$ and $\beta > 1$ are given constants; $f : [t_1, \sigma(t_3)] \times [0, +\infty) \to [0, +\infty)$ is continuous and $\sigma(t_3)$ is right dense so that $\sigma^j(t_3) = \sigma(t_3)$ for $j \ge 1$; $\lambda > 0$ is a constant; $a : [t_1, \sigma(t_3)] \to [0, +\infty)$ is continuous and not identically zero on $[t_2, \sigma(t_3)]$; $g : [0, +\infty) \to [0, +\infty)$ is continuous. Throughout this paper, we suppose that $\mathbb{T}$ is any time scale and $[t_1, t_3]$ is a subset of $\mathbb{T}$ such that $[t_1, t_3] = \{t \in \mathbb{T} : t_1 \le t \le t_3\}$. Some preliminary definitions and theorems on time scales can be found in [1–4] which are excellent references for the calculus of time scales.

In recent years, the theory of time scales, which has received a lot of attention, was introduced by Hilger in his Ph.D. thesis [5] in 1988 in order to unify continuous and discrete analysis. In particular, the theory is also widely applied to stock market, biology, heat transfer, and epidemic models; for details, see [6–10] and the references therein. An important class of time scales are boundary value problems, and such investigations can provide accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. There is much attention paid to the existence of positive solution for higher-order two-point boundary value problems on time scales [11–15]. However, to the best of our knowledge, there are not many results concerning multipoint boundary value problems of higher-order on time scales; we refer the reader to [16–24] for some recent results.

We would like to mention some results of Anderson and Avery [16], Anderson and Karaca [17], Han and Liu [20], and Hu [21]. In [16], Anderson and Avery studied the following even-order BVP:

$$\begin{gathered}(-1)^n x^{(\Delta\nabla)^n}(t) = \lambda h(t) f(x(t)), \quad t \in [a, c] \subset \mathbb{T}, \\ x^{(\Delta\nabla)^i}(a) = 0, \quad x^{(\Delta\nabla)^i}(c) = \beta x^{(\Delta\nabla)^i}(b), \\ 0 \le i \le n-1.\end{gathered} \tag{3}$$

They have studied the existence of at least one positive solution to the BVP (3) using the functional-type cone expansion-compression fixed point theorem.

In [17], Anderson and Karaca were concerned with the dynamic three-point boundary value problem

$$\begin{gathered}(-1)^n y^{\Delta^{2n}}(t) = f(t, y^{\sigma}(t)), \quad t \in [a,b] \subset \mathbb{T}, \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \\ \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), \quad 0 \le i \le n-1.\end{gathered} \tag{4}$$

Existence results of bounded solutions of a noneigenvalue problem are first established as a result of the Schauder fixed point theorem. Second, the monotone method is discussed to ensure the existence of solutions of the BVP (4). Third, they established criteria for the existence of at least one positive solution of the eigenvalue problem by using the Krasnosel'skii fixed point theorem. Later, they investigated the existence of at least two positive solutions of the BVP (4) by using the Avery-Henderson fixed point theorem.

In [20], Han and Liu studied the existence and uniqueness of nontrivial solution for the following third-order $p$-Laplacian $m$-point eigenvalue problems on time scales:

$$\begin{gathered}\left(\phi_p\left(u^{\Delta\nabla}\right)\right)^{\nabla} + \lambda f\left(t, u(t), u^{\Delta}(t)\right) = 0, \quad t \in (0,T), \\ \alpha u(0) - \beta u^{\Delta}(0) = 0, \quad u(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u^{\Delta\nabla}(0) = 0,\end{gathered} \tag{5}$$

where $\phi_p(s)$ is $p$-Laplacian operator, that is, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\lambda > 0$ is a parameter, and $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$. They obtained several sufficient conditions of the existence and uniqueness of nontrivial solution of (5) when $\lambda$ is in some interval. Their approach is based on the Leray-Schauder nonlinear alternative.

Recently, in [21], Hu considered the following singular third-order three-point boundary value problem on time scales:

$$\begin{gathered}\left(u^{\Delta\Delta}(t)\right)^{\nabla} + w(t) f(t, u(t)) = 0, \quad t \in [a,b], \\ u(\rho(a)) - \beta u^{\Delta}(\rho(a)) = \alpha u(\eta), \\ \gamma u(\eta) = u(b), \qquad u^{\Delta\Delta}(\rho(a)) = 0,\end{gathered} \tag{6}$$

where $\eta \in (a,b)$, $\beta \ge 0$, $1 < \gamma < (b - \rho(a) + \beta)/(\eta - \rho(a) + \beta)$, and $0 \le \alpha < (b - \gamma\eta + (\gamma - 1)(\rho(a) - \beta))/(b - \eta)$. The functions $w(t) : (a,b) \to [0,+\infty)$ and $f : [a,b] \times (0,+\infty) \to [0,+\infty)$ are continuous. The nonlinearity $w$ may have singularity at $t = a$ and/or $t = b$ and $f$ may have singularity at $u = 0$. Some theorems on the existence of positive solutions of (6) were obtained by utilizing the fixed theorem of cone expansion and compression type.

We note that Yaslan [23] constructed the Green's function for the BVP (1) and obtained the following result.

**Theorem 1** (see [23]). *Assume that $\alpha > 0$, $\beta > 1$. In addition, there exist numbers $0 < r < R < \infty$ such that*

$$f(t,y) < \frac{1}{L^n} y, \quad \text{if } 0 \le y \le r, \tag{7}$$

$$f(t,y) > \frac{L^{n-1}}{k^{2n} M^{2n-1}} y, \quad \text{if } R \le y < \infty \tag{8}$$

*for $t \in [t_1, \sigma(t_3)]$, where $k, L,$ and $M$ are as in (16)–(21), respectively. Then the BVP (1) has at least one positive solution.*

One of the most frequently used tools for proving the existence of positive solutions to the integral equations is Krasnosel'skii's theorem on cone expansion and compression and its norm-type version due to Guo and Lakshmikantham (see [25]), and they have been applied extensively to all kinds of problems, such as ordinary differential equations, difference equations, and the general dynamic equations on time scales. In order to apply them easily, more and more authors have been dedicated to extending this theorem. In [26], the authors dealt with modifications of the classical Krasnosel'skii fixed point theorem concerning cone compression and expansion of norm type. For further abstract result, the reader is referred to the recent paper [27].

In [28], Zima proved the following fixed point theorem of Leggett-Williams type.

**Theorem 2** (see [28]). *Let $E$ be a real Banach space, $P$ a normal cone in $E$, and $\gamma$ the normal constant of $P$. For $u_0 \in P \setminus \{\theta\}$, let $P(u_0) = \{x \in P : \lambda u_0 \preceq x$ for some $\lambda > 0\}$. Assume that $\Omega_1$ and $\Omega_2$ are bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $F : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous and $u_0 \in P \setminus \{\theta\}$. If either*

- ($C_1$) *$\gamma\|x\| \le \|Fx\|$ for $x \in P(u_0) \cap \partial\Omega_1$ and $\|Fx\| \le \|x\|$ for $x \in P \cap \partial\Omega_2$ or*
- ($C_2$) *$\|Fx\| \le \|x\|$ for $x \in P \cap \partial\Omega_1$ and $\gamma\|x\| \le \|Fx\|$ for $x \in P(u_0) \cap \partial\Omega_2$*

*is satisfied, then $F$ has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.*

We would also like to mention the results of Zhang and Sun [29, 30], in [29, 30]. Zhang and Sun continued to extend this theorem by replacing the norm with some convex functional on cone $P$ (let $E$ be a real Banach space and $P$ a cone in $E$). $\rho : P \to \mathbb{R}$ is said to be a convex functional on $P$ if $\rho(tx + (1-t)y) \le t\rho(x) + (1-t)\rho(y)$ for all $x, y \in P$, and $t \in [0,1]$. They obtained the following main results.

**Theorem 3** (see [30]). *Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $E$ such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous and $\rho : P \to [0,+\infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(x) > 0$ for $x \ne \theta$. If one of the following two conditions is satisfied:*

- ($D_1$) *$\rho(Ax) \le \rho(x)$, for all $x \in P \cap \partial\Omega_1$, and $\inf_{x \in P \cap \partial\Omega_2} \rho(x) > 0$, $\rho(Ax) \ge \rho(x)$, for all $x \in P \cap \partial\Omega_2$,*

($D_2$) $\inf_{x\in P\cap\partial\Omega_1}\rho(x) > 0$, $\rho(Ax) \geq \rho(x)$, *for all* $x \in P \cap \partial\Omega_1$, *and* $\rho(Ax) \leq \rho(x)$, *for all* $x \in P \cap \partial\Omega_2$,

*then A has at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

*Remark 4* (see [30]). Obviously, $\rho(x) = \|x\|$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(x) > 0$ for $x \neq \theta$. Moreover, $\inf_{x\in P\cap\partial\Omega_1}\|x\| > 0$ and $\inf_{x\in P\cap\partial\Omega_2}\|x\| > 0$ since $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$.

In [31], Cid et al. combined the monotone iterative technique with the Guo-Krasnosel'skii fixed point theorem in cones. Their result is as follows.

**Theorem 5** (see [31]). *Let E be a real Banach space, K a normal cone with normal constant* $\gamma \geq 1$ *and nonempty interior, and* $A : K \to K$ *a completely continuous operator. Define* $S = \{x \in K : Ax \leq x\}$ *and suppose that*

(i) *there exist* $x_0 \in S$ *and* $\overline{R} > 0$ *such that* $B[x_0, \overline{R}] = \{x \in E : \|x - x_0\| \leq \overline{R}\} \subset K$, *that is,* $x_0 \in S \cap \mathrm{int}(K)$, *and that one of the two following conditions holds:*

(ii) *S is bounded;*

(iii) *there exists* $r > 0$ *such that* $S \cap B[\theta, r] = \emptyset$.

*If, moreover, A is nondecreasing in the set*

$$K_1 = \left\{x \in K : \frac{\overline{R}}{\gamma} \leq \|x\| \leq \gamma \|x_0\|\right\}, \tag{9}$$

*where* $\gamma \geq 1$ *is the normal constant of the cone, then there exists* $x \in K$, $x \neq \theta$, *such that* $x = Ax$.

In this paper, we will improve and extend Theorem 1 in two different directions. On the one hand, we will weaken the restriction on $f$ in (8) by means of Theorem 2 (see Theorem 10). On the other hand, by constructing appropriate convex functional shown in [29, 30], the properties of $f$ on bounded sets will be considered. Furthermore, the new existence result is obtained by Theorem 3 (see Theorem 11).

## 2. Preliminaries

Let $G(t, s)$ be Green's function for the following boundary value problem:

$$\begin{gathered} -y^{\Delta^2}(t) = 0, \quad t \in [t_1, t_3] \subset \mathbb{T}, \\ y^{\Delta}(t_1) = 0, \qquad \alpha y(\sigma(t_3)) + \beta y^{\Delta}(\sigma(t_3)) = y^{\Delta}(t_2). \end{gathered} \tag{10}$$

A direct calculation gives

$$G(t, s) = \begin{cases} H(t, s), & t_1 \leq s \leq t_2, \\ K(t, s), & t_2 < s \leq t_3, \end{cases} \tag{11}$$

where

$$\begin{gathered} H(t, s) = \begin{cases} \sigma(t_3) - t + \dfrac{\beta - 1}{\alpha}, & \sigma(s) \leq t, \\ \sigma(t_3) - \sigma(s) + \dfrac{\beta - 1}{\alpha}, & t \leq s, \end{cases} \\ K(t, s) = \begin{cases} \sigma(t_3) - t + \dfrac{\beta}{\alpha}, & \sigma(s) \leq t, \\ \sigma(t_3) - \sigma(s) + \dfrac{\beta}{\alpha}, & t \leq s. \end{cases} \end{gathered} \tag{12}$$

To state and prove the main results of this paper, we need the following lemmas.

**Lemma 6** (see [23]). *Let* $\alpha > 0$, $\beta > 1$. *Then the Green function* $G(t, s)$ *in* (11) *satisfies the following inequality:*

$$G(t, s) \geq \frac{t - t_1}{\sigma(t_3) - t_1} G(\sigma(t_3), s) \tag{13}$$

*for* $(t, s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

**Lemma 7** (see [23]). *Let* $\alpha > 0$, $\beta > 1$. *Then the Green function* $G(t, s)$ *in* (11) *satisfies*

$$0 < G(t, s) \leq G(\sigma(s), s) \tag{14}$$

*for* $(t, s) \in [t_1, \sigma(t_3)] \times [t_1, t_3]$.

**Lemma 8** (see [23]). *Let* $\alpha > 0$, $\beta > 1$ *and* $s \in [t_1, t_3]$. *Then the Green function* $G(t, s)$ *in* (11) *satisfies*

$$\min_{t\in[t_2,\sigma(t_3)]} G(t, s) \geq k \|G(\cdot, s)\|, \tag{15}$$

*where*

$$k = \frac{\beta - 1}{\alpha(\sigma(t_3) - \sigma(t_1)) + \beta - 1}, \tag{16}$$

*and* $\|\cdot\|$ *is defined by* $\|x\| = \max_{x\in[t_1,\sigma(t_3)]}|x(t)|$.

If we let $G_1(t, s) := G(t, s)$ for $G$ as in (11), then we can recursively define

$$G_j(t, s) = \int_{t_1}^{\sigma(t_3)} G_{j-1}(t, r) G(r, s) \Delta r \tag{17}$$

for $2 \leq j \leq n$ and $G_n(t, s)$ is Green's function for the homogeneous problem

$$\begin{gathered} (-1)^n y^{\Delta^{2n}}(t) = 0, \quad t \in [t_1, t_3] \subset \mathbb{T}, \\ y^{\Delta^{2i+1}}(t_1) = 0, \\ y^{\Delta^{2i}}(\sigma(t_3)) + \beta y^{\Delta^{2i+1}}(\sigma(t_3)) = y^{\Delta^{2i+1}}(t_2), \\ 0 \leq i \leq n - 1. \end{gathered} \tag{18}$$

**Lemma 9** (see [23]). *Let $\alpha > 0$, $\beta > 1$. Then the Green function $G_n(t,s)$ satisfies the following inequalities:*

$$0 \le G_n(t,s) \le L^{n-1}\|G(\cdot,s)\|, \quad (t,s) \in [t_1,\sigma(t_3)] \times [t_1,t_3],$$
$$G_n(t,s) \ge k^n M^{n-1}\|G(\cdot,s)\|, \quad t,s \in [t_2,\sigma(t_3)] \times [t_1,t_3], \tag{19}$$

*where $k$ is given in* (16),

$$L = \int_{t_1}^{\sigma(t_3)} \|G(\cdot,s)\| \Delta s > 0, \tag{20}$$

$$M = \int_{t_2}^{\sigma(t_3)} \|G(\cdot,s)\| \Delta s > 0. \tag{21}$$

Suppose that $\mathscr{B}$ denotes the Banach space $C[t_1,\sigma(t_3)]$ with the norm $\|y\| = \max_{t\in[t_1,\sigma(t_3)]}|y(t)|$.

## 3. Existence Theorem of Positive Solutions

In this section, we apply Theorems 2 and 3 to establish two existence criteria for the problem (1).

**Theorem 10.** *Assume that the following conditions hold.*

$(H_1)$ *There exists $r_1 > 0$ such that*

$$f(t,y) \le \frac{r_1}{L^n} \tag{22}$$

*for $t \in [t_1,\sigma(t_3)]$ and $y \in [0,r_1]$.*

$(H_2)$ *There exist $t_0 \in (t_2,\sigma(t_3)]$, $a > 0$, $r_2 > 0$, $r_2 \ne r_1$, and the continuous functions $\varphi : [t_2,\sigma(t_3)] \to [0,+\infty)$, $h : (0,r_2] \to [0,+\infty)$ such that*

$$f(t,y) \ge \varphi(t)h(y) \tag{23}$$

*for $t \in [t_2,\sigma(t_3)]$ and $y \in [0,r_2]$, $h(y)/y^a$ is nonincreasing on $(0,r_2]$, and*

$$h(r_2)\left(\frac{k^n M^{n-1}}{L^{n-1}}\right)^a \int_{t_2}^{\sigma(t_3)} G_n(t_0,s)\varphi(s)\Delta s \ge r_2, \tag{24}$$

*where $k$, $L$, and $M$ are defined in* (16)–(21)*, respectively. Then the problem* (1) *has at least one positive solution on $[t_1,\sigma(t_3)]$.*

**Theorem 11.** *If there exist constants $a$ and $b$ with $0 < b < a$ satisfying one of the following two conditions:*

$(H_3)$ *$a \ge (L^{n-1}/k^n M^{n-1})^2 b$, $f(t,y) \ge a/k^n M^n$, $(t,y) \in [t_2,\sigma(t_3)] \times [a(k^n M^{n-1}/L^{n-1}), a]$ and $f(t,y) \le b/L^n$, $(t,y) \in [t_1,\sigma(t_3)] \times [0,(L^{n-1}/k^n M^{n-1})b]$;*

$(H_4)$ *$a/b \ge L^n/k^n M^n$, $f(t,y) \le a/L^n$, $(t,y) \in [t_1,\sigma(t_3)] \times [0,(L^{n-1}/k^n M^{n-1})a]$ and $f(t,y) \ge b/k^n M^n$, $(t,y) \in [t_2,\sigma(t_3)] \times [b(k^n M^{n-1}/L^{n-1}), b]$,*

*where $k$, $L$, and $M$ are defined in* (16)–(21)*, respectively. Then the problem* (1) *has at least one positive solution.*

*Proof of Theorem 10.* Consider the Banach space $\mathscr{B}$ and set

$$P = \left\{ y \in \mathscr{B},\ y(t) \ge 0,\ t \in [t_1,\sigma(t_3)],\ \min_{t\in[t_2,\sigma(t_3)]} y(t) \ge \frac{k^n M^{n-1}}{L^{n-1}}\|y\| \right\}. \tag{25}$$

It is easy to show that $P$ is a normal cone in $\mathscr{B}$ (with $\gamma = 1$). Obviously, for $x,\ y \in \mathscr{B}, x \preceq y$ if and only if $x(t) \le y(t)$ for every $t \in [t_1,\sigma(t_3)]$. Fix $u_0 \equiv 1$ on $[t_1,\sigma(t_3)]$. Then

$$P(u_0) = \left\{ y \in \mathscr{B} : y(t) > 0,\ t \in [t_1,\sigma(t_3)],\ \min_{t\in[t_2,\sigma(t_3)]} y(t) \ge \frac{k^n M^{n-1}}{L^{n-1}}\|y\| \right\}. \tag{26}$$

For $y \in P$ and $t \in [t_1,\sigma(t_3)]$, we define the integral operator

$$Ay(t) = \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(s,y(\sigma(s)))\Delta s \tag{27}$$

clearly, and every fixed point of $A$ is a solution of the problem (1). From the proof of Theorem 3 of [23], we can know that $A : P \to P$ is completely continuous.

Let $\Omega_1 = \{y \in \mathscr{B} : \|y\| < r_1\}$, $\Omega_2 = \{y \in \mathscr{B} : \|y\| < r_2\}$, and then we can assume that $r_1 < r_2$. By $(H_1)$, for $y \in P \cap \partial\Omega_1$ and $t \in [t_1,\sigma(t_3)]$, we get

$$Ay(t) \le L^{n-1}\int_{t_1}^{\sigma(t_3)} \|G(\cdot,s)\| \frac{r_1}{L^n}\Delta s = r_1. \tag{28}$$

Hence, $\|Ay\| \le \|y\|$ for $y \in P \cap \partial\Omega_1$.

Let $y \in P(u_0) \cap \partial\Omega_2$, and then $\|y\| = r_2$ and $y(t) \in (0,r_2]$ for $t \in [t_1,\sigma(t_3)]$. By $(H_2)$, we obtain

$$\begin{aligned}
Ay(t_0) &= \int_{t_1}^{\sigma(t_3)} G_n(t_0,s) f(s,y(\sigma(s)))\Delta s \\
&\ge \int_{t_2}^{\sigma(t_3)} G_n(t_0,s)\varphi(s)h(y(\sigma(s)))\Delta s \\
&= \int_{t_2}^{\sigma(t_3)} G_n(t_0,s)\varphi(s)\frac{h(y(\sigma(s)))}{y^a(\sigma(s))}y^a(\sigma(s))\Delta s \\
&\ge \frac{h(r_2)}{r_2^a}\int_{t_2}^{\sigma(t_3)} G_n(t_0,s)\varphi(s)y^a(\sigma(s))\Delta s \\
&\ge \frac{h(r_2)}{r_2^a}\int_{t_2}^{\sigma(t_3)} G_n(t_0,s)\varphi(s)\left(\frac{k^n M^{n-1}}{L^{n-1}}\|y\|\right)^a\Delta s \\
&= h(r_2)\left(\frac{k^n M^{n-1}}{L^{n-1}}\right)^a \int_{t_2}^{\sigma(t_3)} G_n(t_0,s)\varphi(s)\Delta s \\
&\ge r_2.
\end{aligned} \tag{29}$$

This gives $\|Ay\| \ge \|y\|$ for $y \in P(u_0) \cap \partial\Omega_2$. By Theorem 2, $A$ has a fixed point $y^* \in P$ such that $r_1 \le \|y^*\| \le r_2$. This ends the proof of Theorem 10. □

*Proof of Theorem 11.* Set

$$\Omega_1 = \{y \in \mathscr{B} \mid \rho(y) < b\},$$
$$\Omega_2 = \{y \in \mathscr{B} \mid \rho(y) < a\}. \tag{30}$$

It is clear that $\Omega_1$ and $\Omega_2$ are open sets with $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$.

We define $\rho(y) = \max_{t\in[t_2,\sigma(t_3)]} y(t)$. Then $\rho : P \to [0,+\infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(u) > 0$ for $u \neq 0$, where $P$ is defined in (25).

If $y \in P \cap \partial\Omega_1$, then $\rho(y) = b$ and for $t \in [t_1, \sigma(t_3)]$, we have

$$0 \le y(t) \le \|y\| \le \frac{L^{n-1}}{k^n M^{n-1}} b. \tag{31}$$

Therefor,

$$\begin{aligned}\rho(Ay) &= \max_{t\in[t_1,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(s, y(\sigma(s)))\Delta s\\ &\le L^{n-1}\int_{t_1}^{\sigma(t_3)} \|G(\cdot,s)\| f(s,y(\sigma(s)))\Delta s\\ &\le L^{n-1}\frac{b}{L^n}\int_{t_1}^{\sigma(t_3)}\|G(\cdot,s)\|\Delta s\\ &= b = \rho(y).\end{aligned} \tag{32}$$

If $y \in P \cap \partial\Omega_2$, then $\rho(y) = a$ and for $t \in [t_2, \sigma(t_3)]$, we have

$$a\frac{k^n M^{n-1}}{L^{n-1}} \le \|y\| \frac{k^n M^{n-1}}{L^{n-1}} \le y(t) \le \rho(y) = a. \tag{33}$$

Therefore,

$$\begin{aligned}\rho(Ay) &\ge k^n M^{n-1}\int_{t_2}^{\sigma(t_3)} \|G(\cdot,s)\| f(s,y(\sigma(s)))\Delta s\\ &\ge k^n M^{n-1}\frac{a}{k^n M^n}\int_{t_2}^{\sigma(t_3)}\|G(\cdot,s)\|\Delta s\\ &= a = \rho(y).\end{aligned} \tag{34}$$

Thus, condition $(D_1)$ in Theorem 3 is satisfied, and the problem (1) has at least one positive solution.

In the following, we check condition $(D_2)$ in Theorem 3.

If $y \in P \cap \partial\Omega_1$, then $\rho(y) = b$ and for $t \in [t_1, \sigma(t_3)]$, we have

$$b\frac{k^n M^{n-1}}{L^{n-1}} \le \|y\| \frac{k^n M^{n-1}}{L^{n-1}} \le y(t) \le \rho(y) = b. \tag{35}$$

Therefore,

$$\begin{aligned}\rho(Ay) &\ge k^n M^{n-1}\int_{t_2}^{\sigma(t_3)} \|G(\cdot,s)\| f(s,y(\sigma(s)))\Delta s\\ &\ge k^n M^{n-1}\frac{b}{k^n M^n}\int_{t_2}^{\sigma(t_3)}\|G(\cdot,s)\|\Delta s\\ &= b = \rho(y).\end{aligned} \tag{36}$$

If $y \in P \cap \partial\Omega_2$, then $\rho(y) = a$ and for $t \in [t_1, \sigma(t_3)]$, we have

$$0 \le y(t) \le \|y\| \le \frac{L^{n-1}}{k^n M^{n-1}} a. \tag{37}$$

Therefore,

$$\begin{aligned}\rho(Ay) &= \max_{t\in[t_1,\sigma(t_3)]} \int_{t_1}^{\sigma(t_3)} G_n(t,s) f(s, y(\sigma(s)))\Delta s\\ &\le L^{n-1}\int_{t_1}^{\sigma(t_3)} \|G(\cdot,s)\| f(s,y(\sigma(s)))\Delta s\\ &\le L^{n-1}\frac{a}{L^n}\int_{t_1}^{\sigma(t_3)}\|G(\cdot,s)\|\Delta s\\ &= a = \rho(y).\end{aligned} \tag{38}$$

Thus, condition $(D_2)$ in Theorem 3 is satisfied, and the problem (1) has at least one positive solution. □

*Remark 12.* In Theorem 10, we substitute condition $(H_2)$ for condition (8) in Theorem 1, and thus the more general and comprehensive functions are incorporated. In Theorem 11, we substitute condition $(H_3)$ or $(H_4)$ for conditions (7) and (8) in Theorem 1, where we only require that the "heights" of the nonlinear term are appropriate on some bounded sets. Moreover, the existence is independent of the growth of $f$ outside these bounded sets.

## 4. Solvability of the Problem (2)

In this section, we apply Theorem 5 to obtain sufficient conditions for the existence of solutions for the problem (2). We note that the conspicuous advantage of Theorem 5 is that the conditions over the set $S$ of the lower solution are deleted.

We define the cone

$$\begin{aligned}K = \{y \in \mathscr{B} : y(t) \ge 0,\ t \in [t_1, \sigma(t_3)],\\ y(t) \ge \omega\|y\| \text{ on } [t_2, \sigma(t_3)]\}\end{aligned} \tag{39}$$

for some $0 < \omega \le k^n M^{n-1}/L^{n-1}$ ($\omega$ will be fixed later).

Since the problem (2) is equivalent to the following integral equation:

$$y(t) = \int_{t_1}^{\sigma(t_3)} G_n(t,s)\lambda a(s) g(y(\sigma(s)))\Delta s, \quad t \in [t_1, \sigma(t_3)], \tag{40}$$

we still define the operator $A : K \to \mathscr{B}$ as follows:

$$Ay(t) = \int_{t_1}^{\sigma(t_3)} G_n(t,s)\lambda a(s) g(y(\sigma(s)))\Delta s, \quad t \in [t_1, \sigma(t_3)]. \tag{41}$$

By the proof of Section 3, we can know that $A : K \to K$ is completely continuous. On the other hand, it is easy to check that $K$ is normal with normal constant $\gamma = 1$ and with nonempty interior.

Define

$$e = L^{n-1} \int_{t_1}^{\sigma(t_3)} \|G(\cdot, s)\| a(s) \Delta s. \tag{42}$$

The main result is as follows.

**Theorem 13.** *Assume that* $\lim_{y\to\infty}(g(y)/y) = +\infty$ *and there exists* $C \in [0, +\infty)$ *such that* $g$ *is nondecreasing on* $[0, C)$. *If*

$$0 < \lambda < \sup_{y\in(0,C)} \frac{y}{eg(y)}, \tag{43}$$

*then the problem* (2) *has at least a positive solution.*

*Proof.* Since $\lambda < \sup_{y\in(0,C)}(y/eg(y))$, take $R_0 \in (0, C)$ and $0 < \omega \le k^n M^{n-1}/L^{n-1}$ such that

$$R_0 - \lambda eg(R_0) > \omega R_0. \tag{44}$$

First, we will prove that the function $y_0(t) = R_0$, $t \in [t_1, \sigma(t_3)]$ satisfies condition (i) in Theorem 5. It is easy to check that $y_0 \in \operatorname{int}(K)$. At the same time, for each $t \in [t_1, \sigma(t_3)]$, we have

$$\begin{aligned} Ay_0(t) &= \int_{t_1}^{\sigma(t_3)} G_n(t,s)\lambda a(s) g(y_0(\sigma(s)))\Delta s \\ &\le \lambda eg(R_0) < R_0 = y_0(t). \end{aligned} \tag{45}$$

In addition, since $\|y_0 - Ay_0\| = R_0$, we have that for $t \in [t_2, \sigma(t_3)]$

$$\begin{aligned} y_0(t) - Ay_0(t) &\ge R_0 - \lambda eg(R_0) > \omega R_0 \\ &= \omega \|y_0 - Ay_0\|. \end{aligned} \tag{46}$$

In consequence, $y_0 \in S$ and $y_0$ satisfies condition (i).

Condition (ii) in Theorem 5 holds because of the assumed asymptotic behavior of $g$ at infinity. In fact, for fixed $\lambda > 0$ choose $D > 0$ large enough such that

$$\lambda D\omega \min_{t\in[t_2,\sigma(t_3)]} \int_{t_2}^{\sigma(t_3)} G_n(t,s) a(s)\Delta s > 1 \tag{47}$$

and $Y > 0$ satisfying $g(y) \ge Dy$ provided that $y \ge Y$. Now suppose that $y \in K$ with $\|y\| \ge Y/\omega$. Then $y(t) \ge Y$ for all $t \in [t_2, \sigma(t_3)]$ and

$$\begin{aligned} Ay(t) &\ge \int_{t_2}^{\sigma(t_3)} G_n(t,s)\lambda a(s) g(y(\sigma(s)))\Delta s \\ &\ge \lambda D\omega \|y\| \min_{t\in[t_2,\sigma(t_3)]} \int_{t_2}^{\sigma(t_3)} G_n(t,s) a(s)\Delta s > \|y\| \end{aligned} \tag{48}$$

for all $t \in [t_2, \sigma(t_3)]$, which implies that $Ay \nleq y$. Therefore, the set $S$ is contained in the ball centered at the origin and radius $Y/\omega$.

Finally, since $g$ is nondecreasing in $[0, R_0]$, if we put $y_1, y_2 \in K$ with $y_1(t) \le y_2(t) \le R_0$, $t \in [t_2, \sigma(t_3)]$ we have

$$\begin{aligned} &Ay_2(t) - Ay_1(t) \\ &\quad= \int_{t_1}^{\sigma(t_3)} G_n(t,s)\lambda a(s) g(y_2(\sigma(s)))\Delta s \\ &\qquad- \int_{t_1}^{\sigma(t_3)} G_n(t,s)\lambda a(s) g(y_1(\sigma(s)))\Delta s \ge 0 \end{aligned} \tag{49}$$

and for $t \in [t_2, \sigma(t_3)]$ and $r \in [t_1, \sigma(t_3)]$

$$\begin{aligned} Ay_2(t) - Ay_1(t) &= \int_{t_1}^{\sigma(t_3)} G_n(t,s)\lambda a(s) \\ &\quad\times [g(y_2(\sigma(s))) - g(y_1(\sigma(s)))]\Delta s \\ &\ge \int_{t_1}^{\sigma(t_3)} k^n M^{n-1} \|G(\cdot,s)\| \lambda a(s) \\ &\quad\times [g(y_2(\sigma(s))) - g(y_1(\sigma(s)))]\Delta s \\ &\ge \frac{k^n M^{n-1}}{L^{n-1}} \\ &\quad\times \int_{t_1}^{\sigma(t_3)} G_n(r,s)\lambda a(s) \\ &\qquad\times [g(y_2(\sigma(s))) \\ &\qquad\quad - g(y_1(\sigma(s)))]\Delta s \\ &\ge \omega \int_{t_1}^{\sigma(t_3)} G_n(r,s)\lambda a(s) \\ &\quad\times [g(y_2(\sigma(s))) - g(y_1(\sigma(s)))]\Delta s \\ &= \omega [Ay_2(r) - Ay_1(r)], \end{aligned} \tag{50}$$

and therefore $\min_{t\in[t_2,\sigma(t_3)]}[Ay_2(t) - Ay_1(t)] \ge \omega\|Ay_2 - Ay_1\|$ and $A$ is nondecreasing. Thus, all conditions in Theorem 5 are satisfied. We can assert that the problem (2) has at least one positive solution. □

*Remark 14.* In general, the previous results ensure the solvability of the problem (2) only for "enough small" $\lambda$, that is, for $0 < \lambda < \lambda_1$, where $\lambda_1$ is a constant explicitly established. However, Theorem 13 implies that the problem (2) is solvable for all $\lambda > 0$ even if $g(y) = 0$ for some $y \in (0, C]$. We should point out that the spectrum structure of the corresponding linear problem of (2) is still unknown so far; therefore, we can not directly apply the results shown in [32] due to Webb and Infante and in [33] due to Webb and Lan to the problem (2).

## 5. Some Examples

In this section, we present three examples to explain our results.

*Example 1.* Let $\mathbb{T} = \{(2/5)^n : n \in \mathbb{N}_0\} \cup [2,3]$, where $\mathbb{N}_0$ denotes the set of all nonnegative integers. Consider the following BVP:

$$\begin{aligned} -y^{\Delta^2}(t) &= f(t, y(\sigma(t))), \quad t \in \left[\frac{2}{5}, 3\right] \subset \mathbb{T}, \\ y^{\Delta}\left(\frac{2}{5}\right) &= 0, \qquad y(\sigma(3)) + 3y^{\Delta}(\sigma(3)) = y^{\Delta}(1), \end{aligned} \tag{51}$$

where

$$f(t,y) = \begin{cases} \dfrac{1}{20}(t+1)y^2(y-5)^2, & (t,y) \in \left[\dfrac{2}{5}, 3\right] \times [0,5], \\ (t+1)^6(y-5)^7, & (t,y) \in \left[\dfrac{2}{5}, 3\right] \times (5, +\infty). \end{cases} \tag{52}$$

It is easy to check that $f : [(2/5), 3] \times [0, +\infty) \to [0, +\infty)$ is continuous. By computation, we can get that

$$\begin{aligned} k &= \frac{1}{2}, \\ G(t,s) &= \begin{cases} H(t,s), & \dfrac{2}{5} \le s \le 1, \\ K(t,s), & 1 < s \le 3, \end{cases} \end{aligned} \tag{53}$$

where

$$\begin{aligned} H(t,s) &= \begin{cases} 5-t, & \sigma(s) \le t, \\ 5-\sigma(s), & t \le s, \end{cases} \\ K(t,s) &= \begin{cases} 6-t, & \sigma(s) \le t, \\ 6-\sigma(s), & t \le s. \end{cases} \end{aligned} \tag{54}$$

Therefore, we can obtain that

$$\begin{aligned} L &= \int_{2/5}^{3} \|G(\cdot,s)\| \Delta s \\ &= \int_{2/5}^{1} (5-\sigma(s)) \Delta s + \int_{1}^{2} (6-\sigma(s)) \Delta s + \int_{2}^{3} (6-\sigma(s)) \Delta s \\ &= \frac{99}{10}. \end{aligned} \tag{55}$$

Choose $r_1 = 1/50$, $r_2 = 2$, $t_0 = 2$, $a = 2$, $\varphi(t) = (1/20)(t+1)$, and $h(y) = y^2(y-5)^2$, and it is very easy to check that

$$\begin{aligned} f(t,y) &\le \frac{1}{20} \times (3+1) \frac{1}{50^2} \times \left(\frac{1}{50} - 5\right)^2 \\ &< \frac{1}{495} = \frac{1/50}{99/10} = \frac{r_1}{L}, \\ &\text{for } t \in \left[\frac{2}{5}, 3\right], y \in \left[0, \frac{1}{50}\right]. \\ \frac{1}{20}&(t+1)y^2(y-5)^2 = \varphi(t)h(y), \\ &\text{for } t \in [1,3], y \in (0,2], \end{aligned} \tag{56}$$

$h(y)/y^a = y^2(y-5)^2/y^2 = (y-5)^2$ is nonincreasing on $(0,2]$, and

$$\begin{aligned} h(2)&\left(\frac{kM^0}{L^0}\right)^2 \int_1^3 G(2,s) \frac{1}{20}(s+1) \Delta s \\ &= \frac{9}{20}\left[\int_1^2 (6-2)(s+1)\Delta s + \int_2^3 (6-\sigma(s))(s+1)\Delta s\right] \\ &= \frac{9}{20} \times \frac{121}{6} \\ &> 2 = r_2. \end{aligned} \tag{57}$$

The assumptions of Theorem 10 are satisfied; therefore, the problem (51) has a positive solution $y^*$ such that $1/50 \le \|y^*\| \le 2$.

*Example 2.* Let $\mathbb{T} = \{1,2,3,4,5\}$, and consider the following BVP:

$$\begin{aligned} y^{\Delta^2}(t) &= f(t, y(t+1)), \quad t \in [1,4] \subset \mathbb{T}, \\ y^{\Delta^{2i+1}}(1) &= 0, \qquad y^{\Delta^{2i}}(5) + 2y^{\Delta^{2i+1}}(5) = y^{\Delta^{2i+1}}(3), \end{aligned} \tag{58}$$

where $0 \le i \le 1$, $n = 2$, $t_1 = 1$, $t_2 = 3$, $t_3 = 4$, $\beta = 2$, $\alpha = 1$ and

$$f(t,y) = \begin{cases} \dfrac{1}{121} \cdot \dfrac{4}{275} y & y \le \dfrac{275}{4}, \\ \dfrac{3909 \times 3901 \times 275}{484 \times 559 \times 9} y & \\ \quad - \dfrac{1075011 \times 1072739}{44^2 \times 559 \times 9}, & y \ge \dfrac{275}{4}. \end{cases} \tag{59}$$

It is easy to check that $f : [1,5] \times [0,+\infty) \to [0,+\infty)$ is continuous. We can directly obtain that

$$G(t,s) = \begin{cases} H(t,s), & 1 \le s \le 3, \\ K(t,s), & 3 < s \le 4, \end{cases} \tag{60}$$

where

$$\begin{aligned} H(t,s) &= \begin{cases} 6-t, & s+1 \le t, \\ 5-s, & t \le s, \end{cases} \\ K(t,s) &= \begin{cases} 7-t, & s+1 \le t, \\ 6-s, & t \le s. \end{cases} \end{aligned} \tag{61}$$

Therefore, we can get that

$$\begin{aligned} M &= \int_3^{\sigma(4)} G(\sigma(s), s) \Delta s = 4, \\ L &= \int_1^{\sigma(4)} G(\sigma(s), s) \Delta s = 11, \\ k &= \frac{1}{5}, \qquad \frac{k^n M^{n-1}}{L^{n-1}} = \frac{k^2 M}{L} = \frac{4}{275}. \end{aligned} \tag{62}$$

Choose $a = 71^2$, $b = 1$, and then we can get that $f$ satisfies

$$\begin{aligned} f(t,y) &\geq \frac{a}{k^n M^n} = \frac{a}{k^2 M^2} = \frac{25}{16}\cdot 71^2, \\ &\text{for } (t,y) \in [3,5] \times \left[a\frac{4}{275}, a\right]; \\ f(t,y) &\leq \frac{b}{L^n} = \frac{b}{L^2} = \frac{1}{121}, \\ &\text{for } (t,y) \in [1,5] \times \left[0, \frac{275}{4}b\right]. \end{aligned} \tag{63}$$

Therefore, condition $(H_3)$ in Theorem 11 is satisfied, and the BVP (58) has at least one positive solution.

If we take

$$f(t,y) = \begin{cases} \dfrac{1}{20}|\sin t| + 27\cdot\dfrac{275}{64}y, & y \leq \dfrac{64}{275}, \\ \dfrac{1}{20}|\sin t| + 27\times\dfrac{275}{64}\times\dfrac{64}{275}, & y \geq \dfrac{64}{275}, \end{cases} \tag{64}$$

and choose $a = 3630$, $b = 16$, then

$$\frac{a}{b} = \frac{3630}{16} \geq \frac{L^2}{k^2 M^2} = 121 \times \frac{25}{16}. \tag{65}$$

By computation, we can get that

$$\begin{aligned} f(t,y) &\leq \frac{a}{L^n} = \frac{3630}{11^2} \\ &= 30, \quad \text{for } (t,y) \in [1,5] \times \left[0, \frac{275}{4}\times 3630\right], \\ f(t,y) &\geq \frac{b}{k^n M^n} = \frac{16}{1/25 \times 16} \\ &= 25, \quad \text{for } (t,y) \in [3,5] \times \left[\frac{16\times 4}{275}, 16\right]. \end{aligned} \tag{66}$$

Therefore, condition $(H_4)$ in Theorem 11 is satisfied, and the BVP (58) has at least one positive solution.

*Example 3.* Let $\mathbb{T}$ be defined in Example 1, and we consider the following BVP:

$$\begin{aligned} &-y^{\Delta^2}(t) = \lambda a(t)\, g(y(\sigma(t))), \quad t \in \left[\frac{2}{5}, 3\right] \subset \mathbb{T}, \\ &y^{\Delta}\left(\frac{2}{5}\right) = 0, \quad y(\sigma(3)) + 3y^{\Delta}(\sigma(3)) = y^{\Delta}(1), \end{aligned} \tag{67}$$

where $a(t) = 1$, $g(y) = 2y^3 - 9y^2 + 12y + \sqrt{y}$.

By computation, we can get that

$$e = \int_{2/5}^{1} G(\sigma(s), s)\,\Delta s = \frac{12}{5}. \tag{68}$$

Function $g(y)$ is not monotone in $[0, +\infty)$; however, from Example 2 in [31], we can know that $g(y)$ is nondecreasing on $[0, 1.087577)$.

It is easy to check that $\lim_{y\to\infty}(g(y)/y) = \lim_{y\to 0}(g(y)/y) = +\infty$. Then Theorem 13 implies the existence of a positive solution of the BVP (67) for $0 < \lambda < 0.07526$.

*Remark 15.* From the expressions of nonlinear terms which we defined in Examples 1 and 2, we can see that Theorem 1 cannot be directly applied to Examples 1 and 2.

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