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Research Article

α-Skew π-McCoy Rings

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As a generalization of α-skew McCoy rings, we introduce the concept of α-skew π-McCoy rings, and we study the relationships with another two new generalizations, α-skew π_1-McCoy rings and α-skew π_2-McCoy rings, observing the relations with α-skew McCoy rings, π-McCoy rings, α-skew Armendariz rings, π-regular rings, and other kinds of rings. Also, we investigate conditions such that α-skew π_1-McCoy rings imply α-skew π-McCoy rings and α-skew π_2-McCoy rings. We show that in the case where R is a nonreduced ring, if R is 2-primal, then R is an α-skew π-McCoy ring. And, let R be a weak (α, δ)-compatible ring; if R is an α-skew π_1-McCoy ring, then R is α-skew π_2-McCoy.

1. Introduction

Throughout this paper R is an associative ring with identity, unless otherwise stated, and α is an endomorphism of R. The polynomial ring over R with respect to α is denoted by $R[x;\alpha]$ (or simply, the skew polynomial ring) which elements are polynomials in x with coefficients in R, the addition is defined as usual and the multiplication depending on the relation $xr = \alpha(r)x$ for any $r \in R$. Most of the results in the polynomial rings have been done with the case where α is the identity. For a ring R, $P(R)$ is the prime radical (i.e., the intersection of all prime ideals of R), and $N(R)$ is the set of all nilpotent elements of R. A ring R is said to be an NI ring if $N(R)$ forms an ideal of R.

Following Nielsen [1], a ring R is said to be right McCoy; if two polynomials $f(x)$ and $g(x) \in R[x] \setminus \{0\}$ such that $f(x)g(x) = 0$, then there exists $r \in R \setminus \{0\}$ which satisfies $f(x)r = 0$. A left McCoy ring is defined similarly. If a ring is both right and left McCoy, then it is called a McCoy ring. Commutative rings are McCoy [2].

Başer et al. in [3] introduced the notion of α-skew McCoy ring with respect to an endomorphism α. Let α be an endomorphism of a ring R. The ring R is called α-skew McCoy; if two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ such that $f(x)g(x) = 0$, then $f(x)c = 0$ for some $c \in R \setminus \{0\}$. It is clear that a ring R is right McCoy if R is I_R-skew McCoy, where I_R is the identity endomorphism of R.

Jeon et al. in [4] studied a generalization of McCoy rings, which they have been called π-McCoy rings. A ring R is said to be π-McCoy; if whenever $f(x)g(x) \in N(R[x])$, then $f(x)c \in N(R[x])$ for some $c \in R \setminus \{0\}$, where $f(x)$ and $g(x)$ are in $R[x] \setminus \{0\}$. Thus the concept of π-McCoy rings is a generalization of the concept of McCoy rings, but the converse may not be true in general.

There are many relationships between McCoyness and other kinds of rings like Armendariz rings, regular rings, reduced rings (i.e., a ring without nonzero nilpotent elements), 2-primal rings (i.e., if $P(R) = N(R)$), and others. An Armendariz ring R is defined by Rege and Chhawchharia in [5]; if two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x] \setminus \{0\}$ such that $f(x)g(x) = 0$, then $a_i b_j = 0$ for all i and j. Also, it is proved in [5] that every Armendariz ring is McCoy, but the converse needs not to be true.

According to Hong et al. in [6, 7], the Armendariz property of a polynomial ring was extended to a skew polynomial ring. A ring R is called α-Armendariz (resp., α-skew Armendariz); if two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha]$ such that $f(x)g(x) = 0$, then $a_i b_j = 0$ (resp., $a_i \alpha^i b_j = 0$) for all $0 \le i \le n$ and $0 \le j \le m$. Also, it is proved in [7] that any α-Armendariz ring is α-skew

Armendariz, but the converse does not hold. Note that the notion of α-skew McCoy rings extends both McCoy rings and α-skew Armendariz rings [3].

Ouyang in [8] introduced the concept of a skew π-Armendariz ring and showed that this notion generalizes the concept of α-Armendariz ring defined by Hong et al. in [7]. Let R be a ring with an endomorphism α and an α-derivation δ. The ring R is called a skew π-Armendariz ring; if two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha,\delta]$ such that $f(x)g(x) \in N(R[x;\alpha,\delta])$, then $a_i b_j \in N(R)$ for each $0 \le i \le n$ and $0 \le j \le m$.

Motivated by all of the previous, in this paper, we introduced in Section 2 the concept of α-skew π-McCoy rings by considering the skew polynomial ring $R[x;\alpha]$ instead of the ring $R[x]$ in the condition of π-McCoy ring. Consequently, some results of π-McCoy rings would be considered as special cases of α-skew π-McCoy rings. We showed that the notion of α-skew π-McCoy rings generalizes the notion of α-skew McCoy rings introduced by Başer et al. [3]. We proved that, in case of commutative rings, if R is a π-regular ring but not regular, then R is an α-skew π-McCoy ring. Also, if R is a local, one-sided Artinian, nonreduced ring with an automorphism α, then $R[x;\alpha]$ is an α-skew π-McCoy ring. Moreover, let R be a nonreduced, right Noetherian ring. If R is an Abelian π-regular ring, then R is an α-skew π-McCoy ring. In Section 3 we studied the relationships between α-skew π-McCoy rings and new two different concepts (α-skew π_1-McCoy ring and α-skew π_2-McCoy ring) that related and are close to the notion of α-skew π-McCoy rings and depending on various visions of α-skew McCoy rings and α-skew Armendariz rings. We proved that if S is any ring, then $R = T_n(S)$ is an α-skew π_1-McCoy ring for all $n \ge 2$. Also, let α be an endomorphism of a ring R, and Let R be a semicommutative ring satisfies the α-condition. If R is an α-skew π_1-McCoy ring, then R is α-skew π-McCoy. Furthermore, let R be a weak (α,δ)-compatible ring. If R is an α-skew π_1-McCoy ring, then R is α-skew π_2-McCoy.

Finally, we mention that skew polynomial rings play an important role and have applications in several domains like coding theory, Galois representations theory in positive characteristic, cryptography, control theory, and solving ordinary differential equations.

2. α-Skew π-McCoy Rings

Motivating by [3, 4, 8], we introduced the following concept.

Definition 1. Let α be an endomorphism of a ring R. A ring R is called α-skew π-McCoy; if two polynomials $f(x)$ and $g(x) \in R[x,\alpha] \setminus \{0\}$ such that $f(x)g(x) \in N(R[x;\alpha])$, then $f(x)c \in N(R[x;\alpha])$, for some $c \in R \setminus \{0\}$.

It is clear that every π-McCoy ring is I_R-skew π-McCoy ring, where I_R is the identity endomorphism of R.

Proposition 2. *Every α-skew McCoy ring is an α-skew π-McCoy ring.*

Proof. Suppose that R is an α-skew McCoy ring and $f(x)g(x) \in N(R[x;\alpha])$ for $f(x)$ and $g(x) \in R[x,\alpha] \setminus \{0\}$. Then there exists a positive integer n such that $(f(x)g(x))^n = 0$ and $(f(x)g(x))^{n-1} \neq 0$, and there exists a positive integer m such that $(g(x)f(x))^m = 0$ and $(g(x)f(x))^{m-1} \neq 0$. Now, if $f(x)g(x) = 0$ and $g(x)f(x) = 0$ ($f(x)g(x) = 0$ and $g(x)f(x) \neq 0$, $f(x)g(x) \neq 0$, and $g(x)f(x) = 0$, $f(x)g(x) \neq 0$, and $g(x)f(x) \neq 0$), then there exist $a, b \in R \setminus \{0\}$ such that $f(x)a = 0$ and $g(x)b = 0$ ($f(x)a = 0$ and $bg(x) = 0$, $af(x) = 0$ and $g(x)b = 0$, and $f(x)a = 0$ and $g(x)b = 0$, resp.,) because R is α-skew McCoy, which implies that $f(x)a \in N(R[x;\alpha])$ or $bg(x) \in N(R[x;\alpha])$ for some $a, b \in R\setminus\{0\}$. On the other hand, let $f(x)g(x) \in N(R[x;\alpha])$, then $g(x)f(x) \in N(R[x;\alpha])$. Since R is α-skew McCoy ring, hence $cg(x) \in N(R[x;\alpha])$ and so R is an α-skew π-McCoy ring. □

The converse of Proposition 2 may not be true in general.

Example 3. Let Z_3 be the ring of integers modulo 3. Consider the 2×2 matrix ring $R = \text{Mat}_2(Z_3)$ over Z_3 and an endomorphism $\alpha : R \to R$ defined by

$$\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \tag{1}$$

For

$$f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x,$$
$$g(x) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad x \in R[x;\alpha], \tag{2}$$

we have $f(x)g(x) = 0 \in N(R[x;\alpha])$, but for any $c \in R \setminus \{0\}$, $f(x)c \neq 0$, thus R is not α-skew McCoy [3]. On the other hand R is not π-McCoy [4]. Furthermore, this shows that the 2×2 upper triangular matrix ring

$$R' = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Z_3\right\} \tag{3}$$

over Z_3 is not α-skew McCoy [3], but R' is a π-McCoy ring [4]. In addition R' is an α-skew π-McCoy ring by Lemma 7(d) below.

The idea of the following example appears in [4].

Example 4. Let S be the 3×3 full matrix ring over the power series ring $F[[t]]$ over a field F. Let

$$K = \{M = (m_{ij}) \in S \mid m_{ij} \in tF[[t]]$$
$$\text{for } 1 \le i, j \le 2, m_{ij} = 0 \ \text{ for } i = 3 \text{ or } j = 3\}, \tag{4}$$
$$G = \{M = (m_{ij}) \in S \mid m_{ij} \in F,\ m_{ij} = 0 \ \text{ for } i \neq j\}.$$

Let R be the subring of S generated by K and G. Let $F = Z_2$. Note that every element of R is of the form

$$\begin{pmatrix} a + f_1 & f_2 & 0 \\ f_3 & a + f_4 & 0 \\ 0 & 0 & a \end{pmatrix} \tag{5}$$

for some $a \in F$ and $f_i \in F[[t]]$ $(i = 1, 2, 3, 4)$. Let $\alpha : R \to R$ be an endomorphism of R defined by

$$\alpha\left(\begin{pmatrix} a+f_1 & f_2 & 0 \\ f_3 & a+f_4 & 0 \\ 0 & 0 & a \end{pmatrix}\right) = \begin{pmatrix} a+f_1 & -f_2 & 0 \\ -f_3 & a+f_4 & 0 \\ 0 & 0 & a \end{pmatrix}. \quad (6)$$

Consider two polynomials over R,

$$f(x) = \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 0 \end{pmatrix} x^3, \quad (7)$$

$$g(x) = \begin{pmatrix} 0 & 0 & 0 \\ t & -t & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} t & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x \in R[x;\alpha].$$

$f(x)g(x) = 0 \in N(R[x;\alpha])$, but for every $r \in R$, $f(x)r \notin N(R[x;\alpha])$, hence R is not α-skew π-McCoy. On the other hand R is not π-McCoy [4].

Example 5. Let Z_4 be the ring of integers modulo 4. Consider the ring

$$R = \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in Z_4\right\}. \quad (8)$$

Let $\alpha : R \to R$ be an endomorphism defined by

$$\alpha\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}. \quad (9)$$

R is an α-skew McCoy ring [3], hence R is an α-skew π-McCoy ring.

Example 6. Consider the ring

$$R = \left\{\begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \mid a \in Z,\ t \in Q\right\}, \quad (10)$$

where Z and Q are the set of all integers and all rational numbers, respectively. Let $\alpha : R \to R$ be an automorphism of R defined by

$$\alpha\left(\begin{pmatrix} a & t \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & \frac{t}{2} \\ 0 & a \end{pmatrix}. \quad (11)$$

R is an α-skew Armendariz ring [6], hence R is α-skew McCoy [3], and then R is an α-skew π-McCoy ring.

Lemma 7. *(a) Let R be a ring, and Let α be an endomorphism of R. If there exists a nonzero ideal I of R such that $I[x;\alpha] \subseteq N(R[x;\alpha])$, then R is α-skew π-McCoy.*

(b) Every nonsemiprime ring is an α-skew π-McCoy ring.

(c) Let R be a ring with at least one nonzero nilpotent ideal. Then $\mathrm{Mat}_n(R)$ $(n \geq 1)$ is an α-skew π-McCoy ring.

(d) Let R be any ring. $U_n(R)$ and $L_n(R)$ are α-skew π-McCoy for $n \geq 2$.

(e) Let R be a ring, and Let n be any positive integer; then $R[x;\alpha]/(x^n)$ is an α-skew π-McCoy ring, where (x^n) is the ideal generated by x^n.

Proof. (a) Let $f \in R[x;\alpha] \setminus \{0\}$. First, we assume that $f \in I[x;\alpha]$, then for all $r \in R$ we have that $fr \in N(R[x;\alpha])$. Secondly, let $f \notin I[x;\alpha]$, hence for all nonzero $t \in I$, we obtain that $ft \in I[x;\alpha] \subseteq N(R[x;\alpha])$. Thus R is α-skew π-McCoy.

(b) Suppose that $P(R) \neq 0$ for a ring R, and then $0 \neq P(R)[x;\alpha] = P(R[x,\alpha]) \subseteq N(R[x;\alpha])$, hence R is an α-skew π-McCoy ring by (a).

(c) Since $\mathrm{Mat}_n(R)$ is nonsemiprime ring, then it is α-skew π-McCoy by (b).

(d) $U(R)$ and $L(R)$ are nonsemiprime rings, so they are α-skew π-McCoy by (b).

(e) Since $R[x;\alpha]/(x^n)$ is nonsemiprime ring, then $R[x;\alpha]/(x^n)$ is α-skew π-McCoy ring by (b). □

Remark 8. The $n \times n$ matrix ring $\mathrm{Mat}_n(R)$ over the semiprime ring R considered in [4, Example 1.5] is π-McCoy ring. By Lemma 7(c), if R is non-smiprime, then $\mathrm{Mat}_n(R)$ $(n \geq 2)$ is an α-skew π-McCoy ring. This is not necessary mean that "if R is semiprime, then $\mathrm{Mat}_n(R)$ is not α-skew π-McCoy." The matrix ring $\mathrm{Mat}_n(R)$ is also an α-skew π-McCoy ring by Lemma 7(a).

A ring R is called 2-primal by Birkenmeier et al. [9] if $P(R) = N(R)$. Note that a 2-primal ring is reduced and a ring R is 2-primal if and only if $R/P(R)$ is reduced. It is easy to see that every reduced ring is π-McCoy (which is unknown for α-skew π-McCoy ring), for this reason 2-primal rings are π-McCoy, so we have the following for the case of an α-skew π-McCoy ring.

Proposition 9. *Let R be a nonreduced ring. If R is 2-primal, then R is an α-skew π-McCoy ring.*

Proof. Assume that R is 2-primal, and then $P(R) = N(R)$, and since R is nonreduced, then $P(R) = N(R) \neq 0$, hence R is an α-skew π-McCoy ring by Lemma 7(b). □

The converse of Proposition 9 needs not be true because the α-skew π-McCoy ring in Remark 8 is not 2-primal by [10].

Due to Jeon et al. [4], the class of π-McCoy rings contains both McCoy rings and 2-primal rings. However, regular π-McCoy rings are not McCoy or 2-primal [4]. Recall that a ring R is π-regular if there exist a positive integer n and $x \in R$ such that $a^n = a^n x a^n$ for every element $a \in R$. While R is called a right (resp., left) π-regular ring if there exists a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$ (resp., $a^n = xa^{n+1}$) for every element $a \in R$, a ring R is called strongly π-regular if R is both right and left π-regular rings. It is known that every strongly π-regular ring is π-regular and every regular ring is π-regular, but the converse may not be true. Also, note that R is left π-regular if it satisfies the DCC on chains of the form $Ra \supseteq Ra^2 \supseteq Ra^3 \ldots$.

In the following example we show that I_R-skew π-McCoy ring may not be π-regular.

Example 10. Let $W_1[F]$ be the first Weyl algebra over a field F of characteristic zero. Recall that $W_1[F] = F[\mu, \lambda]$, the polynomial ring with indeterminate μ and λ with $\lambda\mu = \mu\lambda+1$. Now, let

$$R = \begin{pmatrix} W_1[F] & W_1[F] \\ 0 & W_1[F] \end{pmatrix}, \tag{12}$$

where R is not π-regular and $P(R) = \left(\begin{smallmatrix} 0 & W_1[F] \\ 0 & 0 \end{smallmatrix}\right) \neq 0$ by [11], so that R is nonsemiprime, and hence R is an I_R-skew π-McCoy ring by Lemma 7(b). Furthermore, we have $P(R) = N(R)$ [11] which implies that R is a 2-primal ring (which is nonreduced), and then R is an I_R-skew π-McCoy ring by Proposition 9.

Example 11. If R denotes the 2×2 upper triangle matrix ring over a field, then R is a π-regular ring [12] and R is an α-skew π-McCoy ring by Lemma 7(d).

In case that R is a commutative ring, the concept of π-regular rings coincides with the concept of strongly π-regular rings. Also, every nonreduced ring is an α-skew π-McCoy ring. It is well known that if R is commutative ring, then R is regular if and only if R is π-regular and $P(R) = 0$. In addition, every Artinian ring is π-regular [13], so we have the following.

Proposition 12. *Let R be a commutative ring. If R is a π-regular ring but not regular, then R is α-skew π-McCoy.*

Proof. Since R is a π-regular ring but not regular, then $P(R) \neq 0$, hence R is α-skew π-McCoy by Lemma 7(b). □

Corollary 13. *Let R be a commutative not regular ring. If R is Artinian, then R is an α-skew π-McCoy ring.*

Corollary 14. *Let R be a commutative ring. If R is π-regular, then R is a π-McCoy ring.*

Corollary 15. *If R is commutative Artinian ring, then R is π-McCoy.*

Let R be a ring, and Let α be an endomorphism of R; Kwak in [14] defines an $\alpha(*)$-ring to be a ring in which $a\alpha(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$. Also he called an ideal P of a ring R by completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$.

Proposition 16. *Let R be a nonreduced ring, and Let α be an automorphism of R. If R is an $\alpha(*)$-ring, then R is α-skew π-McCoy.*

Proof. Since R is an $\alpha(*)$-ring, then by [15] R is a 2-primal ring, therefore R is α-skew π-McCoy by Proposition 9. □

Corollary 17. *Let R be a Noetherian nonreduced ring, and Let α be an automorphism of R. If for each minimal prime ideal P of R, $\alpha(P) = P$ and P is completely prime ideal of R, then R is an α-skew π-McCoy ring.*

Proof. By [15] and Proposition 16. □

Proposition 18. *If R is a nonreduced, 2-primal ring with a nilpotent prime ideal, then $R[x;\alpha]$ is an α-skew π-McCoy ring.*

Proof. By [16] R is a 2-primal ring, hence by Proposition 9 R is an α-skew π-McCoy ring. □

Chen [17] introduced the notion of semiabelian rings. A ring R is semiabelian if $\mathrm{Id}(R) = S_r(R) \cup S_l(R)$ where (i) $\mathrm{Id}(R)$ is the set of idempotents in R, (ii) $S_r(R)$ (resp., $S_l(R)$) is the set of right (resp., left) semicentral idempotents of R, (iii) an idempotent e in a ring R is right (resp., left) semicentral if for every $x \in R$, $ex = exe$ (resp., $xe = exe$). Recall that a ring R is Abelian if every idempotent element of R is central and that a ring R is right (resp., left) quasiduo if every maximal right (resp., left) ideal is an ideal, and a ring R is quasiduo if it is right and left quasiduos.

Theorem 19. *Let R be a right Noetherian ring. If R is an Abelian π-regular ring, then R is 2-primal.*

Proof. Since R is a right Noetherian ring, then every nil right or left ideal of R is nilpotent [18], therefore $P(R)$ contains all nil right or left ideals of R, but $N(R)$ is two sided [19] because R is an Abelian π-regular ring, hence $N(R) \subseteq P(R)$ which implies that R is a 2-primal ring. □

Badawi [19] and Chen [17] proved that if R satisfies any one of the following: (a) an Abelian π-regular ring; (b) a right (resp., left) quasiduo π-regular ring; (c) a semiabelian π-regular ring, then $N(R)$ is an ideal of R, so we have the following.

Corollary 20. *Let R be a nonreduced right Noetherian ring. If R is an Abelian π-regular ring, then R is α-skew π-McCoy.*

Corollary 21. *Let R be a nonreduced, right Noetherian ring. If R is a right (resp., left) quasiduo π-regular ring, then R is α-skew π-McCoy.*

Corollary 22. *Let R be a nonreduced right Noetherian ring. If R is a semiabelian π-regular ring, then R is α-skew π-McCoy.*

3. Two Generalizations of α-Skew McCoy Rings

As mentioned before that a ring R with an endomorphism α is called α-skew McCoy ring; if two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ such that $f(x)g(x) = 0$, then $f(x)c = 0$ for some $c \in R \setminus \{0\}$ [3]. In fact Song et al. in [20] introduced a concept of α-skew McCoy rings in another way as a generalization of McCoy rings and α-rigid rings (a ring with an endomorphism α such that $a\alpha(a) = 0$ implies $a = 0$ for a in the ring). Let α be an endomorphism of a ring R, and let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ with $f(x)g(x) = 0$, R is called a left α-skew McCoy ring if there exists $r \in R \setminus \{0\}$ such that $rb_j = 0$ for all $0 \leq j \leq m$, and R is called a right α-skew

McCoy ring if there exists $s \in R \setminus \{0\}$ such that $a_i\alpha^i(s) = 0$ for all $0 \leq i \leq n$. If a ring R is both left α-skew McCoy and right α-skew McCoy, then R is called an α-skew McCoy ring. Every McCoy ring R is an I_R-skew McCoy ring, where I_R is the identity endomorphism of R. Here an α-skew Armendariz ring may not be α-skew McCoy in general [20], but if R is an α-skew Armendariz ring, then R is right α-skew McCoy [20].

As a generalization of the concept of α-skew McCoy rings in the sense of Song et al. [20], we motivated by the previous to introduce the concepts of α-skew π_1-McCoy rings and α-skew π_2-McCoy rings taking into consideration the set of nilpotent elements of $R[x;\alpha]$, $N(R[x;\alpha])$. We gave examples to show that these two new concepts are not equivalent to each others and not equivalent to the concept of α-skew π-McCoy rings. Furthermore, we studied the relationship between each others as well as between them and α-skew π-McCoy rings on the other hand. We showed that if a certain property satisfies for α-skew π-McCoy rings may be this is not true for α-skew $\pi_1(\pi_2)$-McCoy rings and vise versa. Also, we investigate some of their properties and characterizations.

Definition 23. Let R be a ring, and Let α be an endomorphism of R. We say that R is right α-skew π_1-McCoy; if two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ with $f(x)g(x) \in N(R[x,\alpha])$, then for any $i \in \{0, 1, 2, \ldots, n\}$ there exists $c = c(a_i) \in R \setminus \{0\}$ (i.e., c depending on a_i) such that $a_i c \in N(R)$. A left α-skew π_1-McCoy ring is defined similarly. If R is both left and right α-skew π_1-McCoy, then R is called an α-skew π_1-McCoy ring.

Every left α-skew McCoy ring (in the sense of [20]) is left α-skew π_1-McCoy, but the converse may not be true in general. Also, if R is a skew π-Armendariz ring, then it is α-skew π_1-McCoy, so every α-skew Armendariz ring is α-skew π_1-McCoy, again the converse needs not be true as in the following.

Example 24. Let $R = Z_2 \oplus Z_2$ and $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. R is not left α-skew McCoy [20] and R is not α-skew Armendariz [6]. However, R is an α-skew π_1-McCoy ring if $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) \in R[x;\alpha] \setminus \{0\}$ with $f(x)g(x) \in N(R[x,\alpha])$, then it is clear that there exists $(1, 1) \in R$ such that $a_i(1, 1) \in N(R)$ for each $0 \leq i \leq n$.

We mention that there is no example of a ring which is not α-skew π_1-McCoy so far. However, it is convenient to show that the concept of α-skew π-McCoy rings and the concept of α-skew π_1-McCoy rings are not equivalent. In fact an α-skew π_1-McCoy ring may not be α-skew π-McCoy as we see in the following.

Example 25. Let R be the subring as in Example 4 which is not α-skew π-McCoy. However, always we can find $c = c(a_i) \in R$ such that $a_i c \in N(R)$. So R is an α-skew π_1-McCoy ring.

Proposition 26. *Let S be any ring, and $R = T_n(S)$ is an α-skew π_1-McCoy ring, for all $n \geq 2$.*

Proof. For any

$$X = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \in R, \qquad (13)$$

take

$$Y = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \in R, \qquad (14)$$

then we have $XY \in N(R)$ and $YX \in N(R)$, hence R is an α-skew π_1-McCoy ring. □

Recall that a ring R is semicommutative if $ab = 0$ implies $aRb = 0$ for $a, b \in R$, and that a ring R is said to satisfy the α-condition for an endomorphism α of R in case $ab = 0$ if and only if $a\alpha(b) = 0$ where $a, b \in R$ [21]. In the following we show how may an α-skew π_1-McCoy ring imply α-skew π-McCoy.

Theorem 27. *Let α be an endomorphism of a ring R and R be a semicommutative ring satisfies the α-condition. If R is an α-skew π_1-McCoy ring, then R is α-skew π-McCoy.*

Proof. Let R be an α-skew π_1-McCoy ring, and let $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ such that $f(x)g(x) \in N(R[x;\alpha])$, and then there exists $c \in R \setminus \{0\}$ such that $a_i c \in N(R)$ for each $0 \leq i \leq n$, hence $f(x)c = a_0 c + a_1 x c + a_2 x^2 c + \cdots + a_n x^n c = a_0 c + a_1 \alpha(c) x + a_2 \alpha^2(c) x^2 + \cdots + a_n \alpha^n(c) x^n$. Since $a_i c \in N(R)$, then by [22], we have that $a_i^n(c) \in N(R)$ for any positive integer n, hence $a_i \alpha^i(c) \in N(R)$ for each $0 \leq i \leq n$. Again by [22], we have $N(R)[x;\alpha] = N(R[x;\alpha])$, hence $f(x)c \in N(R)[x;\alpha] = N(R[x;\alpha])$, so that R is an α-skew π-McCoy ring. □

Theorem 28. *Let R be a Noetherian ring, and Let α be an automorphism of R which satisfies the α-condition. If R is an α-skew π_1-McCoy ring, then R is α-skew π-McCoy.*

Proof. The proof is in the same steps of the proof of theorem 27 by using [22, Corollary 3.2] and [23, Proposition 2]. □

Definition 29. Let α be an endomorphism of a ring R. We say that R is right α-skew π_2-McCoy; if two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ with $f(x)g(x) \in N(R[x;\alpha])$, then for any $i \in \{0, 1, 2, \ldots, n\}$, there exists $c = c(a_i) \in R \setminus \{0\}$ (i.e., c depending on a_i) such that $a_i \alpha^i(c) \in N(R)$. A left α-skew π_2-McCoy ring is defined similarly. A ring R is called α-skew π_2-McCoy if it is both left and right α-skew π_2-McCoy rings.

Every right α-skew McCoy ring (in the sense of [20]) is right α-skew π_2-McCoy, but the converse may not be true in general, and likewise, every α-skew Armendariz ring is α-skew π_2-McCoy, but the converse needs not be true as in the following.

Example 30. Let $R = Z_2 \oplus Z_2$, and let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a,b)) = (b,a)$. The ring R is not α-skew McCoy [20], and R is not α-skew Armendariz [6]. However R is an α-skew π_2-McCoy ring; if $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=m}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ such that $f(x)g(x) \in N(R[x,\alpha])$, then there exists $(1,1) \in R$ such that $a_i\alpha^i((1,1)) \in N(R)$ for each $0 \le i \le n$.

Also here we mention that there is no example of a ring which is not α-skew π_2-McCoy so far.

Remark 31. As in the case of an α-skew π_1-McCoy ring, the concept of an α-skew π-McCoy ring is not equivalent to the concept of an α-skew π_2-McCoy ring, since the subring R referred to in Example 25 is an α-skew π_2-McCoy ring because always we can find $c = c(a_i) \in R$ such that $a_i\alpha^i(c) \in N(R)$.

Ouyang [24] introduced the concept of weak (α,δ)-compatible rings. For an endomorphism α and α-derivation δ, we say that R is weak α-compatible; if each $a, b \in R$, then $ab \in N(R)$ if and only if $a\alpha(b) \in N(R)$. Moreover, R is said to be weak δ-compatible; if each $a, b \in R$, $ab \in N(R)$, then $a\delta(b) \in N(R)$. If R is both weak α-compatible and weak δ-compatible, then R is said to be weak (α,δ)-compatible. Now, it is clear that every α-skew π_2-McCoy ring is α-skew π_1-McCoy. In the following we show how we can make the converse true.

Proposition 32. *Let R be a weak (α,δ)-compatible ring. If R is an α-skew π_1-McCoy ring, then R is α-skew π_2-McCoy.*

Proof. Since R is an α-skew π_1-McCoy ring, then for two polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) \in R[x;\alpha]$, $f(x)g(x) \in N(R[x;\alpha])$, there exists $c \in R$ such that $a_i c \in N(R)$, but R is weak (α,δ)-compatible ring, thus $a_i\alpha^m(c) \in N(R)$ for every positive integer m [24], hence $a_i\alpha^i(c) \in N(R)$ for each $0 \le i \le n$, therefore R is an α-skew π_2-McCoy ring. □

Theorem 33. *Let R be a weak (α,δ)-compatible NI ring. If R satisfies any one of the following:*

(a) *R is an α-skew π-McCoy ring;*

(b) *R is an α-skew π_1-McCoy ring;*

(c) *R is an α-skew π_2-McCoy ring;*

then R is a skew π-Armendariz ring.

Proof. Let R be any ring satisfies any one of (a), (b), and (c), hence for $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ with $f(x)g(x) \in N(R[x;\alpha])$ and by [24], we have $a_i b_j \in N(R)$ for each i, j. Therefore R is a skew π-Armendariz ring. □

Corollary 34. *Let R be a weak (α,δ)-compatible, Abelian π-regular ring. If R satisfies any one of the following:*

(a) *R is an α-skew π-McCoy ring;*

(b) *R is an α-skew π_1-McCoy ring;*

(c) *R is an α-skew π_2-McCoy ring;*

then R is a skew π-Armendariz ring.

Corollary 35. *Let R be a weak (α,δ)-compatible, right (resp., left) quasiduo π-regular ring. If R satisfies any one of the following:*

(a) *R is an α-skew π-McCoy ring;*

(b) *R is an α-skew π_1-McCoy ring;*

(c) *R is an α-skew π_2-McCoy ring;*

then R is a skew π-Armendariz ring.

Corollary 36. *Let R be a weak (α,δ)-compatible, semiabelian π-regular ring. If R satisfies any one of the following:*

(a) *R is an α-skew π-McCoy ring;*

(b) *R is an α-skew π_1-McCoy ring;*

(c) *R is an α-skew π_2-McCoy ring;*

then R is a skew π-Armendariz ring.

Theorem 37. *Let R be a weak (α,δ)-compatible NI ring. If R satisfies any one of the following:*

(a) *R is an α-skew π-McCoy ring;*

(b) *R is a skew π-Armendariz ring;*

then R is an α-skew π_1-McCoy ring.

Proof. Let R be any ring satisfies any one of (a), (b), and (c), hence for $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\alpha] \setminus \{0\}$ with $f(x)g(x) \in N(R[x;\alpha])$ and by [24], there exists $r \in R \setminus \{0\}$ such that $a_i r \in N(R)$ for all $0 \le i \le n$, therefore R is an α-skew π_1-McCoy ring. □

Corollary 38. *Let R be a weak (α,δ)-compatible, Abelian π-regular ring. If R satisfies any one of the following:*

(a) *R is an α-skew π-McCoy ring;*

(b) *R is a skew π-Armendariz ring;*

then R is an α-skew π_1-McCoy ring.

Corollary 39. *Let R be a weak (α,δ)-compatible, right (resp., left) quasiduo π-regular ring. If R satisfies any one of the following:*

(a) *R is an α-skew π-McCoy ring;*

(b) *R is a skew π-Armendariz ring;*

then R is an α-skew π_1-McCoy ring.

Corollary 40. *Let R be a weak (α, δ)-compatible, semiabelian π-regular ring. If R satisfies any one of the following:*

(a) *R is an α-skew π-McCoy ring;*

(b) *R is a skew π-Armendariz ring;*

then R is an α-skew π_1-McCoy ring.

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