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Research Article

The Related Extension and Application of the Ši'lnikov Theorem

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The traditional Ši'lnikov theorems provide analytic criteria for proving the existence of chaos in high-dimensional autonomous systems. We have established one extended version of the Ši'lnikov homoclinic theorem and have given a set of sufficient conditions under which the system generates chaos in the sense of Smale horseshoes. In this paper, the extension questions of the Ši'lnikov homoclinic theorem and its applications are still discussed. We establish another extended version of the Ši'lnikov homoclinic theorem. In addition, we construct a new three-dimensional chaotic system which meets all the conditions in this extended Ši'lnikov homoclinic theorem. Finally, we list all well-known three-dimensional autonomous quadratic chaotic systems and classify them in the light of the Ši'lnikov theorems.

1. Introduction

Over the past decades, chaos has received extensive attention from scientific communities such as physics, biology, chemistry, social sciences, and engineering. Today's science thinks that chaos can be found everywhere; for example, chaos can be obtained in both three-variable model for Ca^{2+} signal and ten-variable model for circadian oscillations in Drosophila [1, 2]. People study chaos from theoretical viewpoints on the one hand, including revealing the essence and basic characteristics of chaos and understanding its dynamical behaviors, and try to control and anticontrol chaos from viewpoints of engineering applications on the other hand. Based on the consideration of two aspects of theory and applications, many new chaotic systems have been constructed, such as the generalized Lorenz system, Rössler system, and Chen system. Note that all these constructed chaotic systems belong to the class of three-dimensional(3D) autonomous quadratic dynamical systems. Since a planar autonomous polynomial system cannot generate chaos, three dimensions are needed for an autonomous system to generate chaos, and moreover, the simplest possible form of chaotic systems is 3D autonomous quadratic dynamical systems. Thus, 3D autonomous polynomial systems, in particular 3D autonomous quadratic dynamical systems, occupy a particular position in the study of chaos.

By far, continuous efforts have been devoted to seeking a unified theory and some kinds of canonical forms for all possible continuous-time 3D autonomous quadratic chaotic systems. In particular, there are many important theoretical results about criteria of chaos; for example, for discrete dynamical systems, there is the famous Marotto theorem. For high-dimensional continuous dynamical systems, there is the famous Ši'lnikov theorem [3–6]. Despite these endeavors, there have not yet been systematic results on how to reasonably classify chaos in autonomous systems, and the related studies are few. Therefore, even for a simple class of 3D autonomous quadratic dynamical systems, finding out all simplest possible forms of chaotic systems is especially important with significant impacts on both basic research and engineering applications.

So far there are only two different algebraic criteria in [7–11] about the classification of 3D autonomous chaotic systems. However, we know that the linear parts of a nonlinear system can only influence the local dynamical behavior and chaos usually is determined by the nonlinear parts. Therefore, although these two algebraic criteria can classify a large set of chaotic systems, they could not reveal the geometric structure and the formation mechanism of chaos well [12].

This paper discusses the extension of Ši'lnikov homoclinic theorem and its applications. We have extended the classical Ši'lnikov homoclinic theorem to one critical case

[13]. Next we will extend it to another critical case in this paper. Moreover, in applications of Ši'lnikov theorems (including the extended versions), the key is to check whether or not a homoclinic orbit exists, but usually, this condition is not easily verified mainly because there are no available efficient methods. Based on the type of eigenvalues of the Jacobian matrix evaluated at the system equilibrium, we have proposed a framework of finding homoclinic orbits for two different cases; that is, first expand homoclinic orbits to be found as a reasonable series form, and then determine all the coefficients in the expansion by using the undetermined coefficient method combined with numerical simulation, thus determining homoclinic orbits [14, 15]. The numerical examples are displayed to examine the effectiveness of this method. Finally, based on the Ši'lnikov criteria, the 3D autonomous quadratic dynamical systems are classified into four types [14]:

(1) chaos of the Ši'lnikov homoclinic orbit type;

(2) chaos of the Ši'lnikov heteroclinic orbit type;

(3) chaos of the hybrid type, that is, those chaotic systems with both Ši'lnikov homoclinic and heteroclinic orbits;

(4) chaos of other types except for the previous three types.

2. Revisiting Ši'lnikov Homoclinic Theorems

One of the rare theories for proving the existence of chaos in the high-dimensional continuous dynamical systems is Ši'lnikov homoclinic (heteroclinic) theorem. Its core idea is that supposing there exists a homoclinic orbit based at the equilibrium point, then constructing a Poincaré map, defined in the neighborhood of the homoclinic orbit, and then proving that the map has Smale horseshoes. One version of the classical Ši'lnikov homoclinic theorem can refer to [6].

Theoretical value notwithstanding, many concrete chaotic systems do not meet some conditions stated in the classical Ši'lnikov theorem, such as the Sprott(c) system [16]. In fact, one can intentionally construct this kind of chaotic systems that do not meet the required conditions. Thus, further relaxing the restrictive conditions of the classical Ši'lnikov theorems is in order. We have extended the classical Ši'lnikov homoclinic theorem to one critical case where three eigenvalues of the corresponding Jacobian evaluated at the equilibrium point are zero and $\rho \pm i\omega$. This extended version of Ši'lnikov homoclinic theorem is given in [13]. The proof of the extended theorem is basically similar to that of the original Ši'lnikov homoclinic theorem. The detailed proof and the numerical example can refer to [13].

Here, we extend the classical Ši'lnikov homoclinic theorem to another critical case where a 3D autonomous polynomial system with one equilibrium at which the eigenvalues of the Jacobian are given by λ and $\pm i\omega$. Consider the following 3D autonomous system:

$$\begin{aligned}\frac{dx}{dt} &= -\omega y + P(x,y,z),\\ \frac{dy}{dt} &= \omega x + Q(x,y,z),\\ \frac{dz}{dt} &= \lambda z + R(x,y,z),\end{aligned} \tag{1}$$

where P, Q, and R are C^2 vector fields, vanishing at the origin $(x,y,z) = (0,0,0)$, and are nonlinear in x, y, and z. According to the normal form theory [17], the normal form of system (1) is

$$\begin{aligned}\frac{dx}{dt} &= -\omega y + (ax - by)(x^2+y^2) + o(4),\\ \frac{dy}{dt} &= \omega x + (ay + bx)(x^2+y^2) + o(4),\\ \frac{dz}{dt} &= \lambda z + cz(x^2+y^2) + o(4),\end{aligned} \tag{2}$$

where a, b, and c are constants. In this case, the conclusion of the extended version of the Ši'lnikov homoclinic theorem is stated as follows.

Theorem 1. *Suppose that*

(1) $a < 0$, $\lambda > 0$;

(2) *there exists a homoclinic orbit connected at* $(0,0,0)$.

Then

(1) *the Ši'lnikov map, defined in a neighborhood of the homoclinic orbit of the system, possesses a countable number of Smale horseshoes in its corresponding discrete system;*

(2) *system* (2) *possesses horseshoe chaos.*

The proof of the theorem is similar to that of other two Ši'lnikov homoclinic theorems. The detailed proof which is omitted here can refer to [13]. The according numerical example is

$$\begin{aligned}\frac{dx}{dt} &= -y + z^2 + ax(x^2+y^2),\\ \frac{dy}{dt} &= x + ay(x^2+y^2),\\ \frac{dz}{dt} &= 4z - z(x^2+y^2),\end{aligned} \tag{3}$$

where a is a constant. The system (3) is chaotic when the parameter $a = -0.02$. Similarly, we can show that system (3) has Ši'lnikov homoclinic chaos.

3. A Framework of Finding Homoclinic Orbits

We know that the difficult part in applying the Ši'lnikov theorems (including the extended versions) is how to determine

Table 1: The systems with one equilibrium point.

System	Typical parameter values	The equivalent point	The eigenvalues
Sprott(d)	$a = 4$	$(1/4, 1/16, 0)$	$-1.937, 0.4685 \pm 1.3585i$
Sprott(e)	$a = 0.5$	$(-2, -4, 4)$	$-1.872, 0.686 \pm 0.7732i$
Sprott(f)	$a = 0.4$	$(5/2, 5/2, -1)$	$-2.1347, 0.3674 \pm 0.5775i$
Sprott(i)	$a = 2$	$(0, 0, 0)$	$-1.3532, 0.1766 \pm 1.2028i$
Sprott(k)	$a = 3.9,\ b = 0.9$	$(1, 0.9, -39/9)$	$-1.4333, 0.2166 \pm 1.6353i$
Sprott(m)	$a = 2,\ b = 2$	$(-1/4, 0, 1/2)$	$-2.3146, 0.1573 \pm 1.3052i$
Sprott(q)	$a = 0.9,\ b = 0.4$	$(4/9, 0.9, -0.4)$	$-1.6183, 0.3091 \pm 0.6787i$
Sprott(s)	$a = 2.017$	$(0, 0, 0)$	$-2.2199, 0.1015 \pm 0.6635i$
Zhou(a)	$a_1 = -1.221,\ a_2 = 1.5,\ a_3 = 2$ $b = -1.3,\ c_1 = -1.5,\ c_2 = -1$	$(-0.281, -1.0201, 1.3521)$	$-1.8716, 0.8354 \pm 1.3325i$
Hc	$a = 0.25,\ b = 4,\ F = 8,\ G = 1$	$(7.9679, -0.0065, 0.0297)$	$-0.2498, 6.9678 \pm 31.8716i$

Table 2: The systems with two equilibrium points.

System	Typical parameter values	The equivalent point	The eigenvalues
Sprott(r)	$a = 4$	$(-1, 1/4, \pm 1)$	$-1.6075, 0.3038 \pm 2.2101i$ $1.203, -1.1015 \pm 2.3317i$
Sprott(g)	$a = 0.5$	$(0, 0, 0)$ $(-2, 4, -2)$	$-1, 0.25 \pm 0.9682i$ $0.2149, -0.3574 \pm 2.1274i$
Sprott(j)	$a = 0.3$	$(0, 0, 0)$ $(-1/0.3, -1/0.3, 1/0.09)$	$-1, 0.15 \pm 0.9887i$ $-0.7868, -0.1066 \pm 1.0153i$
Sprott(l)	$a = 1.7,\ b = 1.7$	$(2.4064, 5.7908, 0)$ $(-0.7064, 0.499, 0)$	$0.4087, -0.7543 \pm 2.6547i$ $-0.4693, 0.3179 \pm 2.5698i$
Sprott(n)	$a = 2.7$	$(0, 0, 0)$ $(-1, 0, -1)$	$-0.5101, 0.2551 \pm 1.3767i$ $0.4315, -0.7157 \pm 1.3436i$
Sprott(o)	$a = 2.7$	$(0, 0, 0)$ $(1, -1, 2.7)$	$-0.5101, 0.2551 \pm 1.3763i$ $0.3828, -1.1914 \pm 1.0121i$
Sprott(p)	$a = 3.1,\ b = 0.5$	$(0, 0, 0)$ $(-3.1, -3.1, 0)$	$-1, 0.25 \pm 1.7428i$ $0.8347, -0.6673 \pm 1.8079i$
Rössler	$a = b = 0.2,\ c = 5.7$	$(0.007, -0.0351, 0.0351)$ $(5.693, -28.4649, 28.4649)$	$-5.687, 0.097 \pm 0.9952i$ $0.1935, 0.068 \pm 5.428i$
ζ^3	$a = 3.5,\ b = 2$	$(0, 0, 0)$ $(3.5, 0, 0)$	$0.9255, -0.9627 \pm 1.6897i$ $-1.3833, 0.1916 \pm 1.5791i$
Rössler Toroidal	$a = 0.386,\ b = 0.2$	$(0, 0, 0)$ $(0, 1.5181, -1.5181)$	$-0.5071, 0.1535 \pm 1.064i$ $0.4524, -0.3262 \pm 1.0903i$
BS	$a = 10,\ b = 13$	$(\pm\sqrt{1.3}, \mp\sqrt{1.3}, 0.1)$	$-14.4526, 1.7263 \pm 13.3011i$

the existence of one homoclinic orbit. We have presented a framework of finding the possible homoclinic orbits in two classes of 3D autonomous systems (saddle-focus type or one-dimensional degenerate type) [15]. First, if the eigenvalues of the Jacobian $A = Df$, evaluated at equilibrium point, are $\gamma > 0$, $\rho < 0$. According to the form of the flow linearized at the neighborhood of the origin, we specifically suppose that, for $t < 0$, the homoclinic orbit of the system has the following form:

$$x(t) = \sum_{k=1}^{+\infty} a_k e^{k\gamma t}, \qquad y(t) = \sum_{k=1}^{+\infty} b_k e^{k\gamma t},$$
$$z(t) = \sum_{k=1}^{+\infty} c_k e^{k\gamma t}, \tag{4}$$

where a_k, b_k, and c_k are constants to be determined, whereas for $t > 0$, it has the form

$$x(t) = \sum_{k=1}^{+\infty} a'_{k^{e^{k(\rho+\omega i)t}}}, \qquad y(t) = \sum_{k=1}^{+\infty} b'_{k^{e^{k(\rho+\omega i)t}}},$$
$$z(t) = \sum_{k=1}^{+\infty} c'_{k^{e^{k(\rho+\omega i)t}}}, \tag{5}$$

where a'_k, b'_k, and c'_k are also constants to be determined. In addition, note that a'_k, b'_k, and c'_k are complex values. Denote the real and imaginary parts of a'_k, b'_k, and c'_k by a'_{k1}, b'_{k1}, and c'_{k1} and a'_{k2}, b'_{k2}, and c'_{k2}, respectively.

Table 3: The systems with three and more equilibrium points.

System	Typical parameter values	The equivalent points	The eigenvalues
Lorenz	$a = 10,\ b = 8/3,\ c = 28$	$(0, 0, 0)$ $(\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$	$-22.8277, -2.6667, 11.8277$ $-13.8546, 0.094 \pm 1.3585i$
Rucklidge	$a = -2,\ b = -6.7$	$(0, 0, 0)$ $(0, \pm 2.5884, 6.7)$	$-3.7749, -1, 1.7749$ $-3.5154, 0.2577 \pm 1.9353i$
Chen	$a = 0.3$	$(0, 0, 0)$ $(\pm 3\sqrt{7}, \pm 3\sqrt{7}, 21)$	$-30.8359, -1, 1.7749$ $-18.4288, 4.214 \pm 14.8846i$
SM	$a = 0.85,\ b = 0.5$	$(0, 0, 0)$ $(\pm\sqrt{2}/2, 0, 1)$	$-1.5116, -0.5, 0.6616$ $1.5079, 0.079 \pm 0.8105i$
Lü	$a = 36,\ b = 3,\ c = 20$	$(0, 0, 0)$ $(\pm 2\sqrt{15}, \pm 2\sqrt{15}, 20)$	$-36, 20, -3$ $-22.6516, 1.8258 \pm 13.6867i$
Modified Lorenz	$a = 10,\ b = 8/3,\ c = 28$	$(0, 0, 0)$ $(\pm 3\sqrt{2}/5, \pm 3\sqrt{2}/5, 1.35)$	$-22.8277, 11.8277, -2.6667$ $-13.8546, 0.094 \pm 10.1943i$
The united system	$\alpha = 0.5$	$(0, 0, 0)$ $(\pm 2\sqrt{17}, \pm 2\sqrt{17}, 24)$	$-28.1696, 19.1696, -2.8333$ $-16.9593, 2.563 \pm 13.1857i$
Zhou(b)	$a_1 = -4.1,\ a_2 = 1.2,\ a_3 = 13.45$ $c_1 = 2.76,\ c_2 = 0.6,\ c_3 = 13.13$ $d = 1.6$	$(0.9211, -1.9226, 0.1092)$ $(-0.7106, 0.8407, -0.1416)$ $(-0.112, -9.6873, -0.8984)$ $(-0.0985, -11.1136, -1.0216)$	$-0.9776, 4.427 \pm 3.8675i$ $15.0739, -5.7162, 0.1768$ $0.0041, 1.6068 \pm 18.1354i$ $-0.004, 1.1829 \pm 19.6666i$
Zhou(d)	$a_1 = 0.5,\ a_2 = 0.1,$ $b_1 = -12,\ b_2 = -0.062,$ $b_3 = -6.79$	$(0, 0, 0)$ $(\pm 8.9676, \mp 1.8808, -2.484)$ $(\mp 9.0944, \pm 1.8056, 3.327)$	$-12, -6.79, 0.5$ $-18.6583, 0.2611 \pm 2.9646i$ $-18.8001, 0.1519 \pm 3.4702i$
liu	$a = 5,\ b = -10$ $c = -3.4,\ d_1 = -1$ $d_2 = d_3 = 1$	$(0, 0, 0)$ $(\sqrt{34}, \pm\sqrt{17}, \pm 5\sqrt{2})$ $(-\sqrt{34}, \pm\sqrt{17}, \mp 5\sqrt{2})$	$-10, -3.4, 5$ $-12.6496, 2.1248 \pm 7.0172i$
CL	$a = 5,\ b = 10,\ c = 3.8$	$(0, 0, 0)$ $(\pm\sqrt{114}, \pm\sqrt{57}, \sqrt{50})$ $(\pm\sqrt{114}, \mp\sqrt{57}, -\sqrt{50})$	$-10, -3.8, 5$ $-13.177, 2.1885 \pm 7.2723i$
NL	$a = 0.4,\ b = 0.175$	$(0, 0, 0)$ $(\mp 0.0705, \pm 0.2737, -0.1103)$ $(\pm 0.5352, \pm 0.0689, 0.2107)$	$0.175, -0.4 \pm i$ $-0.7997, 0.0874 \pm 0.8754i$ $-0.803, 0.089 \pm 1.2118i$

Then, through a rigorously mathematical calculation, a set of equations with respect to the free parameters is

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a'_{k1} = x(0),$$
$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} b'_{k1} = y(0), \tag{6}$$
$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} c'_{k1} = z(0),$$

where $x(0)$, $y(0)$, and $z(0)$ are initial values for the homoclinic orbit. In general, it is difficult to find an analytic solution of the infinite series form (6). Here, we first truncate infinite terms of (6). Then, we numerically solve algebraic equations of the truncated form by applying a numerical method, for example, Newton's method. Once these estimates of the unknown parameters are obtained, the approximate expression of the homoclinic orbit with the form of (4) and (5) is found. Moreover, the more the truncated terms are, the better the approximation is. On the other hand, we can show the uniform convergence of the series expansions (4) and (5) of the homoclinic orbit. The detailed discussions refer to [15]. Next we consider system where the eigenvalues of the Jacobian $A = Df$, evaluated at equilibrium point, satisfy $c > 0$, $\rho < 0$. For $t > 0$, the homoclinic orbit still has the expansion form (4), whereas for $t < 0$, it has the following form:

$$x(t) = \sum_{k=1}^{+\infty} a_k (t+1)^{-k}, \qquad y(t) = \sum_{k=1}^{+\infty} b_k (t+1)^{-k},$$
$$z(t) = \sum_{k=1}^{+\infty} c_k (t+1)^{-k}. \tag{7}$$

The treatment of the left questions for this case is completely similar to that for the first kind of system and thus is omitted here. For the saddle-focus form, we consider the Sprott(h) system. The one-dimensional degenerate example is the system constructed by us in [13]. The corresponding numerical results refer to [15].

4. Listing All Found Chaotic Systems of Ši'lnikov Type

In this section, we list all well-known 3D autonomous quadratic chaotic systems and classify them according to the Ši'lnikov theorems (including the extended versions). First, since the Nose-Hoover system has no equilibrium point, it belongs to the fourth type. Secondly, the systems with one equilibrium point refer to Table 1.

The equilibrium point of all the systems in Table 1 belongs to saddle-focus form. The eigenvalues of the Jacobian evaluated at the equilibrium point of these systems all satisfy the algebraic condition, and their chaotic attractors are one-scroll except the Hadley circulation (Hc) system. So they belong to the first type. The Hc system belongs to the fourth type. Thirdly, for the systems with two equilibrium points, see Table 2.

In Table 2, the Sprott(r) system, the Sprott(p) system, and the Rössler Toroidal who have one-scroll attractors belong to the first type. The Burke-Shaw system has a two-scroll attractor, so it belongs to the second type. The left systems who do not satisfy the algebraic condition belong to the fourth type. Finally, for the systems with three and more equilibrium points, see Table 3.

In Table 3, the systems with three equilibrium points who have two-scroll attractors all belong to the second type. The Zhou (b) system who do not satisfy the algebraic condition belongs to the fourth type. The Zhou (d) system, the Chen-Lee system, the Newton-Leipnik system, and the Liu system all belong to the second type.

5. Conclusion and Comments

We have established two extended versions of the Ši'lnikov homoclinic theorem. According to the Ši'lnikov criteria, all well-known 3D autonomous quadratic chaotic systems have been classified. Here, we should point out that there exist 3D autonomous nonlinear systems with eigenvalues of the corresponding Jacobian matrices calculated at the equilibria of interest not belonging to two types considered in this paper, for example, for those whose Jacobians have one zero eigenvalue and two conjugated purely-imaginary roots (three-dimensional degenerate case). How to expand the Ši'lnikov homoclinic theorem to this degenerate case will be our next tasks. Moreover, how to determine the homoclinic orbits for the two-or three-dimensional degenerate cases will also be further investigated in the future. Finally, it should be pointed out that the essential significance of the extension of the Ši'lnikov theorems is in helping to classify chaos in 3D autonomous polynomial systems, an interesting and yet challenging and long-standing problem in the mathematical chaos theory.

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