

Research Article

A Note on Fractional Order Derivatives and Table of Fractional Derivatives of Some Special Functions

Abdon Atangana¹ and Aydin Secer²

¹ *Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa*

² *Yildiz Technical University, Department of Mathematical Engineering, Davutpasa, 34210 İstanbul, Turkey*

Correspondence should be addressed to Aydin Secer; asecer@yildiz.edu.tr

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The purpose of this note is to present the different fractional order derivatives definition that are commonly used in the literature on one hand and to present a table of fractional order derivatives of some functions in Riemann-Liouville sense On other the hand. We present some advantages and disadvantages of these fractional derivatives. And finally we propose alternative fractional derivative definition.

1. Introduction

Fractional calculus has been used to model physical and engineering processes, which are found to be best described by fractional differential equations. It is worth nothing that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases. In the recent years, fractional calculus has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Major topics include anomalous diffusion, vibration and control, continuous time random walk, Levy statistics, fractional Brownian motion, fractional neutron point kinetic model, power law, Riesz potential, fractional derivative and fractals, computational fractional derivative equations, nonlocal phenomena, history-dependent process, porous media, fractional filters, biomedical engineering, fractional phase-locked loops, fractional variational principles, fractional transforms, fractional wavelet, fractional predator-prey system, soft matter mechanics, fractional signal and image processing; singularities analysis and integral representations for fractional differential systems; special functions related to fractional calculus, non-Fourier heat conduction, acoustic dissipation, geophysics, relaxation, creep, viscoelasticity, rheology, fluid dynamics,

chaos and groundwater problems. An excellent literature of this can be found in [1–9]. These entire models are making use of the fractional order derivatives that exist in the literature. However, there are many of these definitions in the literature nowadays, but few of them are commonly used, including Riemann-Liouville [10, 11], Caputo [5, 12], Weyl [10, 11, 13], Jumarie [14, 15], Hadamard [10, 11], Davison and Essex [16], Riesz [10, 11], Erdelyi-Kober [10, 11], and Coimbra [17]. All these fractional derivatives definitions have their advantages and disadvantages. The purpose of this note is to present the result of fractional order derivative for some function and from the results establish the disadvantages and advantages of these fractional order derivative definitions. We shall start with the definitions.

2. Definitions

There exists a vast literature on different definitions of fractional derivatives. The most popular ones are the Riemann-Liouville and the Caputo derivatives. For Caputo we have

$${}^c_0D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt, \quad (1)$$

$n-1 < \alpha \leq n.$

For the case of Riemann-Liouville we have the following definition:

$$D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t) dt. \quad (2)$$

Guy Jumarie proposed a simple alternative definition to the Riemann-Liouville derivative:

$$D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} \{f(t) - f(0)\} dt. \quad (3)$$

For the case of Weyl we have the following definition:

$$D_x^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^\infty (x-t)^{n-\alpha-1} f(t) dt. \quad (4)$$

With the Erdelyi-Kober type we have the following definition:

$$D_{0,\sigma,\eta}^\alpha (f(x)) = x^{-n\sigma} \left(\frac{1}{\sigma x^{\sigma-1}} \frac{d}{dx} \right)^n x^{\sigma(n+\eta)} I_{0,\sigma,\eta+\sigma}^{n-\alpha} (f(x)), \quad \sigma > 0. \quad (5)$$

Here

$$I_{0,\sigma,\eta+\sigma}^\alpha (f(x)) = \frac{\sigma x^{-\sigma(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^x \frac{t^{\sigma\eta+\sigma-1} f(t)}{(t^\sigma - x^\sigma)^{1-\alpha}} dt. \quad (6)$$

With Hadamard type, we have the following definition:

$$D_0^\alpha (f(x)) = \frac{1}{\Gamma(n-\alpha)} \left(x \frac{d}{dx} \right)^n \int_0^x \left(\log \frac{x}{t} \right)^{n-\alpha-1} f(t) \frac{dt}{t}. \quad (7)$$

With Riesz type, we have the following definition:

$$D_x^\alpha (f(x)) = -\frac{1}{2 \cos(\alpha\pi/2)} \times \left\{ \frac{1}{\Gamma(\alpha)} \left(\frac{d}{dx} \right)^m \times \left(\int_{-\infty}^x (x-t)^{m-\alpha-1} f(t) dt + \int_x^\infty (t-x)^{m-\alpha-1} f(t) dt \right) \right\}. \quad (8)$$

We will not mention the Grunward-Letnikov type here because it is in series form. This is not more suitable for analytical purpose. In 1998, Davison and Essex [16] published a paper which provides a variation to the Riemann-Liouville definition suitable for conventional initial value problems

within the realm of fractional calculus. The definition is as follows:

$$D_0^\alpha f(x) = \frac{d^{n+1-k}}{dx^{n+1-k}} \int_0^x \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d^k f(t)}{dt^k} dt. \quad (9)$$

In an article published by Coimbra [17] in 2003, a variable order differential operator is defined as follows:

$$D_0^{\alpha(t)} (f(x)) = \frac{1}{\Gamma(1-\alpha(x))} \int_0^x (x-t)^{-\alpha(t)} \frac{df(t)}{dt} dt + \frac{(f(0^+) - f(0^-)) x^{-\alpha(x)}}{\Gamma(1-\alpha(x))}. \quad (10)$$

3. Table of Fractional Order Derivative for Some Functions

In this section we present the fractional of some special functions. The fractional derivatives in Table 1 are in Riemann-Liouville sense.

In Table 1, HypergeometricPFQ [{}], {} is the generalized hypergeometric function which is defined as follows in the Euler integral representation:

$$F_1(a, b, c, z) = \frac{\Gamma(a)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^-, a, b \in \mathbb{C} (0 < \text{Re}[b] < \text{Re}[c]; |\arg(1-z)| < \pi). \quad (11)$$

The PolyGamma[n, z] and PolyGamma[z] are the logarithmic derivative of gamma function given by

$$\text{PolyGamma}[n, z] = \frac{d^n}{dz^n} \left(\frac{\Gamma'(z)}{\Gamma(z)} \right), \quad (12)$$

$$\text{PolyGamma}[z] = \text{PolyGamma}[0, z].$$

These functions are meromorphic of z with no branch cut discontinuities. $E_\alpha(-t^\alpha)$ is the generalized Mittag-Leffler function and is defined as

$$E_\alpha(-t^\alpha) = \sum_{k=0}^\infty \frac{(-t^\alpha)^k}{\Gamma(k\alpha + 1)}. \quad (13)$$

Γ is denotes the gamma function, which is the Mellin transform of exponential function and is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \text{Re}[z] > 0. \quad (14)$$

$J_n(x)$, $K_n(x)$, and $Y_n(x)$ are Bessel functions first and second kind. Zeta[s] is the zeta function, has no branch cut discontinuities, and is defined as

$$\text{Zeta}[x] = \sum_{n=1}^\infty n^{-x}. \quad (15)$$

TABLE I: Fractional order derivatives for some functions.

Functions	L-fractional derivatives
$x^\beta, \beta > -1$	$\frac{x^{-\alpha+\beta}\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)}$
$\text{Cos}(ax), a \in \mathbb{R}$	$\frac{x^{-\alpha} \text{HypergeometricPFQ}[\{1\}, \{1-\alpha/2, 3/2-\alpha/2\} - (1/4)a^2x^2] - (1/4)a^2x^2 \text{HypergeometricPFQ}[\{2\}, \{2-\alpha/2, 5/2-\alpha/2\} - (1/4)a^2x^2]}{\Gamma(1-\alpha)}$
$\text{Sin}(ax), a \in \mathbb{R}$	$\frac{ax^{1-\alpha} (2-\alpha) \text{HypergeometricPFQ}[\{1\}, \{2-\alpha/2, 3/2-\alpha/2\} - (1/4)a^2x^2] - 2a^3x^{3-\alpha} \text{HypergeometricPFQ}[\{2\}, \{5/2-\alpha/2, 3-\alpha/2\} - (1/4)a^2x^2]}{\Gamma(2-\alpha)}$
$\ln(x)$	$-\frac{x^{-\alpha} (\text{EulerGamma} + \pi \text{Cot}[\pi\alpha] - \ln(x) + \text{PolyGamma}[0, \alpha])}{\Gamma(1-\alpha)}$
$e^{ax}, a \in \mathbb{R}$	$\left(\frac{a^\alpha ((ax)^{-\alpha} + e^{ax} (\Gamma(1-\alpha) - \Gamma(1-\alpha, ax)))}{\Gamma(1-\alpha)} \right)$
$\text{Sinh}(ax)$	$\left(\frac{e^{-ax} (-a^2x^2)^{-\alpha}}{2\Gamma(1-\alpha)} - (-a^2x^2)^{-\alpha} ((-a)^\alpha (\alpha\Gamma(-\alpha) + \Gamma(1-\alpha, -ax)) + a^\alpha e^{2ax} (\Gamma(1-\alpha) - \Gamma(1-\alpha, ax))) \right)$
$\text{Cosh}(ax)$	$\left(\frac{1}{2\Gamma(1-\alpha)} (e^{-ax} (-a^2x^2)^{-\alpha} ((-a)^\alpha (ax)^\alpha (e^{ax} + ax \text{ExpIntegralE}[\alpha, -ax]) + a^\alpha e^{ax} (-ax)^\alpha (1 - ae^{ax} xax \text{ExpIntegralE}[\alpha, ax]) + ((-a)^\alpha + a^\alpha e^{2ax})^\alpha \Gamma(1-\alpha))) \right)$
$\text{Arcsin}(x), 0 < x < 1$	$\left(\frac{x^{1-\alpha} \text{HypergeometricPFQ}[\{1/2, 1/2, 1\}, \{3/2-\alpha/2, 2-\alpha/2\}, x^2]}{\Gamma(2-\alpha)} + \frac{2x^{3-\alpha} \text{HypergeometricPFQ}[\{3/2, 3/2, 2\}, \{5/2-\alpha/2, 3-\alpha/2\}, x^2]}{\Gamma(5-\alpha)} \right)$
$\text{Arccos}(x), 0 < x < 1$	$\left(-\frac{x^{-\alpha} (\pi(-2+\alpha) + 2x \text{HypergeometricPFQ}[\{1/2, 1/2, 1\}, \{3/2-\alpha/2, 2-\alpha/2\}, x^2])}{2(2-\alpha)\Gamma(1-\alpha)} - \frac{x^{3-\alpha} \text{HypergeometricPFQ}[\{3/2, 3/2, 2\}, \{5/2-\alpha/2, 3-\alpha/2\}, x^2]}{2(2-3\alpha+\alpha^2)\Gamma(1-\alpha)} \right)$
$\text{Arctan}(x)$	$-\frac{x^{1-\alpha}}{\Gamma(5-\alpha)} \left((-4+\alpha)(-3+\alpha)(-2+\alpha) \text{HypergeometricPFQ}[\{1/2, 1, 1\}, \{3-\alpha/2, 2-\alpha/2\}, x^2] + 4x^2 \text{HypergeometricPFQ}[\{3/2, 2, 2\}, \{5-\alpha/2, 3-\alpha/2\}, x^2] \right)$
$\int_x^\infty \frac{e^{-y}}{y} dy$	$\frac{1}{\Gamma(3-\alpha)} (e^{-x} x^{-1-\alpha} ((-x)^\alpha (-2+\alpha)(\Gamma(2-\alpha) - \Gamma(2-\alpha, -x)) + e^x x(-1+\alpha)(x \text{HypergeometricPFQ}[\{1, 1\}, \{2, 3-\alpha\}, -x](-2+\alpha)(\pi \text{Cot}(\pi\alpha) - \ln(x) + \text{PolyGamma}(0, \alpha)))))$
$E_\alpha(-t^\alpha)$	$\frac{x^{-\alpha}}{\Gamma(1-\alpha)} - E_\alpha(-t^\alpha)$
$J_n(x), \text{Re}[n] > -1$	$\frac{2^{-n} x^{n-\alpha} (1+n-\alpha)}{\Gamma(2+n-\alpha)} \text{HypergeometricPFQ}[\{1+\frac{n}{2}, 1+\frac{n}{2}\}, \{1+n, 1+\frac{n}{2}-\frac{\alpha}{2}, 2-\frac{\alpha}{2}-\frac{\alpha}{2}\}, \frac{-x^2}{4}] + \frac{2^{-1-n} (1/2+n/2)(1+n/2)x^{2+n-\alpha}}{2^{1-n} (1+n/2-\alpha/2)(3/2+n/2-\alpha/2)\Gamma(2+n-\alpha)} \text{HypergeometricPFQ}[\{3+\frac{n}{2}, 2+\frac{n}{2}\}, \{2+n, 2+\frac{n}{2}-\frac{\alpha}{2}, 2+\frac{n}{2}-\frac{\alpha}{2}\}, \frac{-x^2}{4}]$
$K_n(x), 1 > \text{Re}[n] > -1$	$2^{-1-n} \pi x^{1-n-\alpha} \text{Csc}(\pi n) \left(\frac{4^n}{\Gamma(2+n-\alpha)} \text{HypergeometricPFQ}[\{1-\frac{n}{2}, 1-\frac{n}{2}\}, \{1-n, 1-\frac{n}{2}-\frac{\alpha}{2}, 2-\frac{\alpha}{2}-\frac{\alpha}{2}\}, \frac{x^2}{4}] + \frac{x^{2n}}{\Gamma(2+n-\alpha)} \text{HypergeometricPFQ}[\{1+\frac{n}{2}, 1+\frac{n}{2}\}, \{1+n, 1+\frac{n}{2}-\frac{\alpha}{2}, 2-\frac{\alpha}{2}-\frac{\alpha}{2}\}, \frac{x^2}{4}] \right)$

TABLE 1: Continued.

Functions	L-fractional derivatives
$Y_n(x),$ $-1 < \text{Re}[n] < 1$	$2^{-1-n} x^{1-n-\alpha} \left(-\frac{\text{Csc}(n\pi) 4^n}{\Gamma(2+n-\alpha)} \text{HypergeometricPFQ} \left[\left\{ \frac{1}{2}, \frac{n}{2}, 1 - \frac{n}{2} \right\}, \left\{ 1-n, 1 - \frac{n}{2}, \frac{\alpha}{2}, \frac{n}{2}, \frac{n}{2}, \frac{\alpha}{2} \right\}, -\frac{x^2}{4} \right] \right.$ $\left. + \frac{x^{2n}}{\Gamma(2+n-\alpha)} \text{HypergeometricPFQ} \left[\left\{ \frac{1}{2} + \frac{n}{2}, 1 + \frac{n}{2} \right\}, \left\{ 1+n, 1 + \frac{n}{2}, \frac{\alpha}{2}, \frac{n}{2}, \frac{n}{2}, \frac{\alpha}{2} \right\}, -\frac{x^2}{4} \right] \right)$
Zeta[x]	$\sum_{n=1}^{\infty} \left[\frac{(-1)^n (-x)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{n^{-x} \alpha \Gamma(-\alpha) (-\ln(n))^\alpha}{\Gamma(1-\alpha)} - \frac{n^{-x} \alpha \Gamma(1-\alpha, -x \ln(n)) (-\ln(n))^\alpha}{\Gamma(1-\alpha)} \right]$
Erfi(t)	$2^{-1+x} x^{1-\alpha} (2-\alpha) \text{HypergeometricPFQRegularized} \left[\left\{ \frac{1}{2}, 1 \right\}, \left\{ \frac{3}{2}, \frac{\alpha}{2}, 2 - \frac{\alpha}{2} \right\}, -x^2 \right] + 2^{-1+x} x^{3-\alpha} \text{HypergeometricPFQRegularized} \left[\left\{ \frac{3}{2}, 2 \right\}, \left\{ \frac{5}{2}, \frac{\alpha}{2}, 3 - \frac{\alpha}{2} \right\}, -x^2 \right]$

The above obtained special functions as derivation of Riemann-Liouville fractional derivative are solution of some fractional ordinary differential equation, for instance, Cauchy type.

4. Advantages and Disadvantages

4.1. Advantages. It is very important to point out that all these fractional derivative order definitions have their advantages and disadvantages; here we will include Caputo, variational order, Riemann-Liouville Jumarie and Weyl. We will examine first the Variational order differential operator. Anomalous diffusion phenomena are extensively observed in physics, chemistry, and biology fields [18–21]. To characterize anomalous diffusion phenomena, constant-order fractional diffusion equations are introduced and have received tremendous success. However, it has been found that the constant order fractional diffusion equations are not capable of characterizing some complex diffusion processes, for instance, diffusion process in inhomogeneous or heterogeneous medium [22]. In addition, when we consider diffusion process in porous medium, if the medium structure or external field changes with time, in this situation, the constant-order fractional diffusion equation model cannot be used to well characterize such phenomenon [23, 24]. Still in some biology diffusion processes, the concentration of particles will determine the diffusion pattern [25, 26]. To solve the above problems, the variable-order (VO) fractional diffusion equation models have been suggested for use [27]. The ground-breaking work of VO operator can be traced to Samko et al. by introducing the variable order integration and Riemann-Liouville derivative in [27]. It has been recognized as a powerful modelling approach in the fields of viscoelasticity [17–32] viscoelastic deformation [28], viscous fluid [29] and anomalous diffusion [30]. With the Jumarie definition which is actually the modified Riemann-Liouville fractional derivative, an arbitrary continuous function needs not to be differentiable; the fractional derivative of a constant is equal to zero and more importantly it removes singularity at the origin for all functions for which $f(0) = \text{constant}$ for instance, the exponentials functions and Mittag-Leffler functions. With the Riemann-Liouville fractional derivative, an arbitrary function needs not to be continuous at the origin and it needs not to be differentiable. One of the great advantages of the Caputo fractional derivative is that it allows traditional initial and boundary conditions to be included in the formulation of the problem [5, 12]. In addition its derivative for a constant is zero. It is customary in groundwater investigations to choose a point on the centerline of the pumped borehole as a reference for the observations and therefore neither the drawdown nor its derivatives will vanish at the origin, as required [33]. In such situations where the distribution of the piezometric head in the aquifer is a decreasing function of the distance from the borehole, the problem may be circumvented by rather using the complementary, or Weyl, fractional order derivative [33].

4.2. Disadvantages. Although these fractional derivative display great advantages, they are not applicable in all the

situations. We shall begin with the Liouville-Riemann type. The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. The Riemann-Liouville derivative of a constant is not zero. In addition, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin for instant exponential and Mittag-Leffler functions. Theses disadvantages reduce the field of application of the Riemann-Liouville fractional derivative. Caputo’s derivative demands higher conditions of regularity for differentiability: to compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative. Caputo derivatives are defined only for differentiable functions while functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann-Liouville sense. With the Jumarie fractional derivative, if the function is not continuous at the origin, the fractional derivative will not exist, for instance what will be the fractional derivative of $\ln(x)$ and many other ones. Variational order differential operator cannot easily be handled analytically. Numerical approach is sometimes needs to deal with the problem under investigation. Although Weyl fractional derivative found its place in groundwater investigation, it still displays a significant disadvantage; because the integral defining these Weyl derivatives is improper, greater restrictions must be placed on a function. For instance, the Weyl derivative of a constant is not defined. On the other hand, general theorem about Weyl derivatives are often more difficult to formulate and be proved than are corresponding theorems for Riemann-Liouville derivatives.

5. Derivatives Revisited

5.1. Variational Order Differential Operator Revisited. Let $f : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow f(x)$ denotes a continuous but necessary differentiable, let $\alpha(x)$ be a continuous function in $(0, 1]$. Then its variational order differential in $[a, \infty)$ is defined as

$$\begin{aligned}
 D_a^{\alpha(t)}(f(x)) &= \text{FP} \left(\frac{1}{\Gamma(1-\alpha(x))} \frac{d}{dx} \right. \\
 &\quad \left. \times \int_a^x (x-t)^{-\alpha(t)} (f(t) - f(a)) dt \right), \tag{16}
 \end{aligned}$$

where FP means finite part of the variational order operator. Notice that the above derivative meets all the requirements of the variational order differential operator; in additional, the derivative of the constant is zero, which was not possible with the standard version.

5.2. Variational Order Fractional Derivatives via Fractional Difference. Let $f : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow f(x)$ denotes a continuous but necessary differentiable, let $\alpha(x)$ be a continuous function

in $(0, 1]$, and $h > 0$ denote a constant discretization span. Define the forward operator $\text{FW}h$ by the expression

$$\text{FW}(h)(f(x)) := f(x+h) \quad (17)$$

Note that, the symbol means that the left side is defined by the right side. Then the variational order fractional difference of order $\alpha(x)$ of $f(x)$ is defined by the expression

$$\begin{aligned} \Delta^{\alpha(x)} f(x) &= (\text{FW} - 1)^{\alpha(x)} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(k - \alpha(x))} f(x - (\alpha(x) - k)h). \end{aligned} \quad (18)$$

And its variational order fractional derivative of order $\alpha(x)$ is defined by the limit

$$f^{\alpha(x)}(x) = \lim_{h \rightarrow 0} \frac{\Delta^{\alpha(x)}(f(x) - f(0))}{h^{\alpha(x)}}. \quad (19)$$

5.3. Jumarie Fractional Derivative Revisited. Recently, Guy Jumarie proposed a simple alternative definition to the Riemann-Liouville derivative. His modified Riemann-Liouville derivative has the advantage of both standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (nondifferentiable) functions and the fractional derivative of a constant is equal to zero. However if the function is not defined at the origin, the fractional derivative will not exist, therefore in order to circumvent this defeat we propose the following definition. Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $x \rightarrow f(x)$ denotes a continuous but necessary at the origin and not necessary differentiable, then its fractional derivative is defined as:

$$\begin{aligned} D_0^\alpha(f(x)) &= \text{FP} \left(\frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} (f(t) - f(0)) dt \right), \end{aligned} \quad (20)$$

where FP means finite part of the fractional derivative order operator. Notice that, the above derivative meets all the requirement of the fractional derivative operator; the derivative of the constant is zero, in addition the function needs not to be continuous at the origin. With this definition, the fractional derivative of $\ln(x)$ is given as

$$\begin{aligned} D_0^\alpha(\ln(x)) &= -\frac{x^{-\alpha} (\text{EulerGamma} + \pi \text{Cot}[\pi \alpha] - \ln(x) + \text{PolyGamma}[0, \alpha])}{\Gamma(1-\alpha)}. \end{aligned} \quad (21)$$

The above fractional order derivative definition can be used in many field for instance in the field of groundwater. Because this definition does not produce a fractional derivative with any kind of singularity as in the case of Jumarie and the traditional Riemann-Liouville fractional order derivative. This concept was introduced by Hadamard [34–36].

The Hadamard regularization [34–36], based on the concept of finite part (“partie finie”) of a singular function or a divergent integral, plays an important role in several branches of Mathematical Physics see [29–37]. Typically one deals with functions admitting some non-integrable singularities on a discrete set of isolated points located at finite distances from the origin. The regularization consists of assigning by definition a value for the function at the location of one of the singular points, and for the generally divergent integral of that function. The definition may not be fully deterministic, as the Hadamard “partie finie” depends in general on some arbitrary constants [38].

6. Discussions and Conclusions

We presented the definitions of the commonly used fractional derivatives operators which are ranging from Riemann-Liouville to Guy Jumarie. We presented the disadvantages and advantages of each definition. No definition has fulfilled the entire requirement needed; for example, the Jumarie definition fulfills some interesting requirements including the derivative of a constant is zero, and a nondifferentiable function may have a fractional derivative. However, if the function is not defined at the origin, it may not have a fractional derivative in Jumarie sense. With the Riemann-Liouville fractional derivative, the function needs not to be continuous at the origin and needs not to be differentiable; however, the derivative of a constant is not zero; in addition, his has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Also if an arbitrary function is a zero at the origin, its fractional derivation has a singularity at the origin, for instance exponential and Mittag-Leffler functions. Theses disadvantages reduce the field of application of the Riemann-Liouville fractional derivative. The Caputo derivative is very useful when dealing with real-world problem because, it allows traditional initial and boundary conditions to be included in the formulation of the problem and in addition the derivative of a constant is zero; however, functions that are not differentiable do not have fractional derivative, which reduces the field of application of Caputo derivative. It is in addition important to notice that, to characterize anomalous diffusion phenomena, constant-order fractional diffusion equations have been introduced and have received tremendous success. However, it has been found that the constant order fractional diffusion equations are not capable of characterizing some complex diffusion processes. To solve the above problems, the variable-order (VO) fractional diffusion equation models have been suggested for use; however, the calculations involved in these definitions are very difficult to handle analytically; therefore, numerical attentions are needed for these cases. To solve the problem found in Jumarie definition, we proposed an alternative fractional derivative and we extended the definition to the case of variational differential operator. We provided a table of Liouville fractional derivative of some special functions. Now we can conclude here by observing, that all fractional derivatives examined here are all useful, and they have to be used according to the support of the function.

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

Abdon Atangana wrote the first draft and Aydin Secer corrected final version. All authors read and approved the final draft

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References

- [1] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.
- [2] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.
- [3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.
- [4] Abdon Atangana and J. F. Botha, "Generalized groundwater flow equation using the concept of variable order derivative," *Boundary Value Problems*, vol. 2013, 53 pages, 2013.
- [5] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent, part II," *Geophysical Journal International*, vol. 13, no. 5, pp. 529–539, 1967.
- [6] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, 1993.
- [7] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.
- [8] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, 2005.
- [9] A. Atangana and A. Secer, "Time-fractional coupled-the Korteweg-de Vries equations," *Abstract Applied Analysis*, vol. 2013, Article ID 947986, 2013.
- [10] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Integrals and Derivatives of the Fractional Order and Some of Their Applications*, in Russian, Nauka i Tekhnika, Minsk, Belarus, 1987.
- [11] I. Podlubny, "Geometric and physical interpretation of fractional integration and fractional differentiation," *Fractional Calculus and Applied Analysis*, vol. 5, no. 4, pp. 367–386, 2002.
- [12] A. Atangana and A. Kilicman, "Analytical solutions the Space-time-Fractional Derivative of advection dispersion equation," *Mathematical Problem in Engineering*. In press.
- [13] A. Atangana, "Numerical solution of space-time fractional derivative of groundwater flow equation," in *Proceedings of the International Conference of Algebra and Applied Analysis*, p. 20, Istanbul, Turkey, June 2012.
- [14] G. Jumarie, "On the solution of the stochastic differential equation of exponential growth driven by fractional Brownian motion," *Applied Mathematics Letters*, vol. 18, no. 7, pp. 817–826, 2005.
- [15] G. Jumarie, "Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results," *Computers & Mathematics with Applications*, vol. 51, no. 9-10, pp. 1367–1376, 2006.
- [16] M. Davison and C. Essex, "Fractional differential equations and initial value problems," *The Mathematical Scientist*, vol. 23, no. 2, pp. 108–116, 1998.
- [17] C. F. M. Coimbra, "Mechanics with variable-order differential operators," *Annalen der Physik*, vol. 12, no. 11-12, pp. 692–703, 2003.
- [18] T. H. Solomon, E. R. Weeks, and H. L. Swinney, "Observation of anomalous diffusion and Lévy flights in a two-dimensional rotating flow," *Physical Review Letters*, vol. 71, no. 24, pp. 3975–3978, 1993.
- [19] S. Bhalekar, V. Daftardar-Gejji, D. Baleanu, and R. Magin, "Fractional Bloch equation with delay," *Computers & Mathematics with Applications*, vol. 61, no. 5, pp. 1355–1365, 2011.
- [20] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House, Connecticut, UK, 2006.
- [21] R. L. Magin, O. Abdullah, D. Baleanu, and X. J. Zhou, "Anomalous diffusion expressed through fractional order differential operators in the Bloch-Torrey equation," *Journal of Magnetic Resonance*, vol. 190, no. 2, pp. 255–270, 2008.
- [22] A. V. Chechkin, R. Gorenflo, and I. M. Sokolov, "Fractional diffusion in inhomogeneous media," *Journal of Physics A*, vol. 38, no. 42, pp. L679–L684, 2005.
- [23] F. Santamaria, S. Wils, E. de Schutter, and G. J. Augustine, "Anomalous diffusion in Purkinje cell dendrites caused by spines," *Neuron*, vol. 52, no. 4, pp. 635–648, 2006.
- [24] H. G. Sun, W. Chen, and Y. Q. Chen, "Variable order fractional differential operators in anomalous diffusion modeling," *Physica A*, vol. 388, no. 21, pp. 4586–4592, 2009.
- [25] H. G. Sun, Y. Q. Chen, and W. Chen, "Random order fractional differential equation models," *Signal Processing*, vol. 91, no. 3, pp. 525–530, 2011.
- [26] Y. Q. Chen and K. L. Moore, "Discretization schemes for fractional-order differentiators and integrators," *IEEE Transactions on Circuits and Systems I*, vol. 49, no. 3, pp. 363–367, 2002.
- [27] E. N. Azevedo, P. L. de Sousa, R. E. de Souza et al., "Concentration-dependent diffusivity and anomalous diffusion: a magnetic resonance imaging study of water ingress in porous zeolite," *Physical Review E*, vol. 73, no. 1, part 1, Article ID 011204, 2006.
- [28] S. Umarov and S. Steinberg, "Variable order differential equations with piecewise constant order-function and diffusion with changing modes," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 28, no. 4, pp. 431–450, 2009.
- [29] B. Ross and S. Samko, "Fractional integration operator of variable order in the holder spaces $H\lambda(x)$," *International Journal of Mathematics and Mathematical Sciences*, vol. 18, no. 4, pp. 777–788, 1995.
- [30] H. T. C. Pedro, M. H. Kobayashi, J. M. C. Pereira, and C. F. M. Coimbra, "Variable order modeling of diffusive-convective effects on the oscillatory flow past a sphere," *Journal of Vibration and Control*, vol. 14, no. 9-10, pp. 1659–1672, 2008.
- [31] D. Ingman and J. Suzdalnitsky, "Application of differential operator with servo-order function in model of viscoelastic deformation process," *Journal of Engineering Mechanics*, vol. 131, no. 7, pp. 763–767, 2005.
- [32] Y. L. Kobelev, L. Y. Kobelev, and Y. L. Klimontovich, "Statistical physics of dynamic systems with variable memory," *Doklady Physics*, vol. 48, no. 6, pp. 285–289, 2003.

- [33] A. H. Clout and J. P. Botha, "A generalized groundwater flow equation using the concept of non-integer order," *Water SA*, vol. 32, no. 1, pp. 1–7, 2006.
- [34] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, France, 1978.
- [35] R. Estrada and R. P. Kanwal, "Regularization and distributional derivatives of $(x_1^2 + x_2^2 + \dots + x_p^2)^{-(1/2)n}$ in \mathbb{R}^p ," *Proceedings of the Royal Society A*, vol. 401, no. 1821, pp. 281–297, 1985.
- [36] R. Estrada and R. P. Kanwal, "Regularization, pseudofunction, and Hadamard finite part," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 1, pp. 195–207, 1989.
- [37] A. Sellier, "Hadamard's finite part concept in dimension $n \geq 2$, distributional definition, regularization forms and distributional derivatives," *Proceedings of the Royal Society A*, vol. 445, no. 1923, pp. 69–98, 1994.
- [38] L. Bel, T. Damour, N. Deruelle, J. Ibañez, and J. Martin, "Poincaré-invariant gravitational field and equations of motion of two pointlike objects: the postlinear approximation of general relativity," *General Relativity and Gravitation*, vol. 13, no. 10, pp. 963–1004, 1981.