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Research Article

On Kadison-Schwarz Type Quantum Quadratic Operators on $\mathbb{M}_2(\mathbb{C})$

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We study the description of Kadison-Schwarz type quantum quadratic operators (q.q.o.) acting from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$. Note that such kind of operators is a generalization of quantum convolution. By means of such a description we provide an example of q.q.o. which is not a Kadison-Schwartz operator. Moreover, we study dynamics of an associated nonlinear (i.e., quadratic) operators acting on the state space of $\mathbb{M}_2(\mathbb{C})$.

1. Introduction

It is known that one of the main problems of quantum information is the characterization of positive and completely positive maps on C^*-algebras. There are many papers devoted to this problem (see, e.g., [1–4]). In the literature the completely positive maps have proved to be of great importance in the structure theory of C^*-algebras. However, general positive (order-preserving) linear maps are very intractable [2, 5]. It is therefore of interest to study conditions stronger than positivity, but weaker than complete positivity. Such a condition is called *Kadison-Schwarz property*, that is, a map ϕ satisfies the Kadison-Schwarz property if $\phi(a)^*\phi(a) \le \phi(a^*a)$ holds for every a. Note that every unital completely positive map satisfies this inequality, and a famous result of Kadison states that any positive unital map satisfies the inequality for self-adjoint elements a. In [6] relations between n-positivity of a map ϕ and the Kadison-Schwarz property of certain map is established. Certain relations between complete positivity, positive, and the Kadison-Schwarz property have been considered in [7–9]. Some spectral and ergodic properties of Kadison-Schwarz maps were investigated in [10–12].

In [13] we have studied quantum quadratic operators (q.q.o.), that is, maps from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, with the Kadison-Schwarz property. Some necessary conditions for the trace-preserving quadratic operators are found to be the Kadison-Schwarz ones. Since trace-preserving maps arise naturally in quantum information theory (see, e.g., [14]) and other situations in which one wishes to restrict attention to a quantum system that should properly be considered a subsystem of a larger system with which it interacts. Note that in [15, 16] quantum quadratic operators acting on a von Neumann algebra were defined and studied. Certain ergodic properties of such operators were studied in [17, 18] (see for review [19]). In the present paper we continue our investigation; that is, we are going to study further properties of q.q.o. with Kadison-Schwarz property. We will provide an example of q.q.o. which is not a Kadison-Schwarz operator and study its dynamics. We should stress that q.q.o. is a generalization of quantum convolution (see [20]). Some dynamical properties of quantum convolutions were investigated in [21].

Note that a description of bistochastic Kadison-Schwarz mappings from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C})$ has been provided in [22].

2. Preliminaries

In what follows, by $\mathbb{M}_2(\mathbb{C})$ we denote an algebra of 2×2 matrices over complex filed $\mathbb{C}$. By $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ we mean tensor product of $\mathbb{M}_2(\mathbb{C})$ into itself. We note that such a product can be considered as an algebra of 4×4 matrices $\mathbb{M}_4(\mathbb{C})$ over $\mathbb{C}$. In the sequel $\mathbb{1}$ means an identity matrix, that is, $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. By $S(\mathbb{M}_2(\mathbb{C}))$ we denote the set of all states (i.e., linear positive functionals which take value 1 at $\mathbb{1}$) defined on $\mathbb{M}_2(\mathbb{C})$.

Definition 1. A linear operator $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ is said to be

(a) a *quantum quadratic operator (q.q.o.)* if it satisfies the following conditions:

(i) unital, that is, $\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1}$;

(ii) Δ is positive, that is, $\Delta x \geq 0$ whenever $x \geq 0$;

(b) a *Kadison-Schwarz operator (KS)* if it satisfies

$$\Delta\left(x^* x\right) \geq \Delta\left(x^*\right)\Delta\left(x\right), \quad \forall x \in \mathbb{M}_2\left(\mathbb{C}\right). \tag{1}$$

One can see that if Δ is unital and KS operator, then it is a q.q.o. A state $h \in S(\mathbb{M}_2(\mathbb{C}))$ is called *a Haar state* for a q.q.o. Δ if for every $x \in \mathbb{M}_2(\mathbb{C})$ one has

$$(h \otimes \mathrm{id}) \circ \Delta(x) = (\mathrm{id} \otimes h) \circ \Delta(x) = h(x)\mathbb{1}. \tag{2}$$

Remark 2. Note that if a quantum convolution Δ on $\mathbb{M}_2(\mathbb{C})$ becomes a $*$-homomorphic map with a condition

$$\begin{aligned}&\overline{\mathrm{Lin}}\left(\left(\mathbb{1} \otimes \mathbb{M}_2\left(\mathbb{C}\right)\right)\Delta\left(\mathbb{M}_2\left(\mathbb{C}\right)\right)\right) \\ &\quad = \overline{\mathrm{Lin}}\left(\left(\mathbb{M}_2\left(\mathbb{C}\right) \otimes \mathbb{1}\right)\Delta\left(\mathbb{M}_2\left(\mathbb{C}\right)\right)\right) = \mathbb{M}_2\left(\mathbb{C}\right) \otimes \mathbb{M}_2\left(\mathbb{C}\right),\end{aligned} \tag{3}$$

then a pair $(\mathbb{M}_2(\mathbb{C}), \Delta)$ is called a *compact quantum group* [20]. It is known [20] that for any given compact quantum group there exists a unique Haar state w.r.t. Δ.

Remark 3. Let $U : \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a linear operator such that $U(x \otimes y) = y \otimes x$ for all $x, y \in \mathbb{M}_2(\mathbb{C})$. If a q.q.o. Δ satisfies $U\Delta = \Delta$, then Δ is called a *quantum quadratic stochastic operator*. Such a kind of operators was studied and investigated in [17].

Each q.q.o. Δ defines a conjugate operator $\Delta^* : (\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}))^* \to \mathbb{M}_2(\mathbb{C})^*$ by

$$\Delta^*\left(f\right)\left(x\right) = f\left(\Delta x\right), \qquad f \in \left(\mathbb{M}_2\left(\mathbb{C}\right) \otimes \mathbb{M}_2\left(\mathbb{C}\right)\right)^*, \quad x \in \mathbb{M}_2\left(\mathbb{C}\right). \tag{4}$$

One can define an operator V_Δ by

$$V_\Delta\left(\varphi\right) = \Delta^*\left(\varphi \otimes \varphi\right), \quad \varphi \in S\left(\mathbb{M}_2\left(\mathbb{C}\right)\right), \tag{5}$$

which is called a *quadratic operator (q.c.)*. Thanks to conditions (a) (i), (ii) of Definition 1 the operator V_Δ maps $S(\mathbb{M}_2(\mathbb{C}))$ to $S(\mathbb{M}_2(\mathbb{C}))$.

3. Quantum Quadratic Operators with Kadison-Schwarz Property on $\mathbb{M}_2(\mathbb{C})$

In this section we are going to describe quantum quadratic operators on $\mathbb{M}_2(\mathbb{C})$ and find necessary conditions for such operators to satisfy the Kadison-Schwarz property.

Recall [23] that the identity and Pauli matrices $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for $\mathbb{M}_2(\mathbb{C})$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

In this basis every matrix $x \in \mathbb{M}_2(\mathbb{C})$ can be written as $x = w_0\mathbb{1} + \mathbf{w}\sigma$ with $w_0 \in \mathbb{C}$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$, here $\mathbf{w}\sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3$.

Lemma 4 (see [3]). *The following assertions hold true:*

(a) *x is self-adjoint if and only if w_0, $\mathbf{w}$ are reals;*

(b) $\mathrm{Tr}(x) = 1$ *if and only if* $w_0 = 0.5$*; here* Tr *is the trace of a matrix x;*

(c) *$x > 0$ if and only if $\|\mathbf{w}\| \leq w_0$, where $\|\mathbf{w}\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2}$.*

Note that any state $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$ can be represented by

$$\varphi\left(w_0\mathbb{1} + \mathbf{w}\sigma\right) = w_0 + \langle \mathbf{w}, \mathbf{f} \rangle, \tag{7}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ with $\| \mathbf{f} \| \leq 1$. Here as before $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{C}^3$. Therefore, in the sequel we will identify a state φ with a vector $\mathbf{f} \in \mathbb{R}^3$.

In what follows by τ we denote a normalized trace, that is, $\tau(x) = (1/2)\ \mathrm{Tr}(x)$, $x \in \mathbb{M}_2(\mathbb{C})$.

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a q.q.o. with a Haar state τ. Then one has

$$\begin{aligned}\tau \otimes \tau\left(\Delta x\right) &= \tau\left(\tau \otimes id\right)\left(\Delta\left(x\right)\right) \\ &= \tau\left(x\right)\tau\left(\mathbb{1}\right) = \tau\left(x\right), \quad x \in \mathbb{M}_2\left(\mathbb{C}\right),\end{aligned} \tag{8}$$

which means that τ is an invariant state for Δ.

Let us write the operator Δ in terms of a basis in $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ formed by the Pauli matrices, namely,

$$\begin{aligned}&\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1}, \\ &\Delta\left(\sigma_i\right) = b_i\left(\mathbb{1} \otimes \mathbb{1}\right) + \sum_{j=1}^{3} b_{ji}^{(1)}\left(\mathbb{1} \otimes \sigma_j\right) \\ &\qquad + \sum_{j=1}^{3} b_{ji}^{(2)}\left(\sigma_j \otimes \mathbb{1}\right) + \sum_{m,l=1}^{3} b_{ml,i}\left(\sigma_m \otimes \sigma_l\right), \quad i = 1, 2, 3,\end{aligned} \tag{9}$$

where $b_i, b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ijk} \in \mathbb{C}$ $(i, j, k \in \{1, 2, 3\})$.

One can prove the following.

Theorem 5 (see [13, Proposition 3.2]). *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a q.q.o. with a Haar state τ, then it has the following form:*

$$\Delta\left(x\right) = w_0\mathbb{1} \otimes \mathbb{1} + \sum_{m,l=1}^{3} \langle \mathbf{b}_{ml}, \overline{\mathbf{w}} \rangle \sigma_m \otimes \sigma_l, \tag{10}$$

where $x = w_0 + \mathbf{w}\sigma$, $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3}) \in \mathbb{R}^3$, $m, n, k \in \{1, 2, 3\}$.

Let us turn to the positivity of Δ. Given vector $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ put

$$\beta(\mathbf{f})_{ij} = \sum_{k=1}^{3} b_{ki,j} f_k. \tag{11}$$

Define a matrix $\mathbb{B}(\mathbf{f}) = (\beta(\mathbf{f})_{ij})_{ij=1}^{3}$.

By $\|\mathbb{B}(\mathbf{f})\|$ we denote a norm of the matrix $\mathbb{B}(\mathbf{f})$ associated with Euclidean norm in $\mathbb{C}^3$. Put

$$S = \{\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3 : p_1^2 + p_2^2 + p_3^2 \le 1\} \tag{12}$$

and denote

$$|\|\mathbb{B}\|| = \sup_{\mathbf{f}\in S} \|\mathbb{B}(\mathbf{f})\|. \tag{13}$$

Proposition 6 (see [13, Proposition 3.3]). *Let Δ be a q.q.o. with a Haar state τ, then $|\|\mathbb{B}\|| \le 1$.*

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a liner operator with a Haar state τ. Then due to Theorem 5 Δ has the form (10). Take arbitrary states $\varphi, \psi \in S(\mathbb{M}_2(\mathbb{C}))$ and let $\mathbf{f}, \mathbf{p} \in S$ be the corresponding vectors (see (7)). Then one finds that

$$\Delta^* (\varphi \otimes \psi)(\sigma_k) = \sum_{i,j=1}^{3} b_{ij,k} f_i p_j, \quad k = 1, 2, 3. \tag{14}$$

Thanks to Lemma 4 the functional $\Delta^*(\varphi \otimes \psi)$ is a state if and only if the vector

$$\mathbf{f}_{\Delta^*(\varphi,\psi)} = \left(\sum_{i,j=1}^{3} b_{ij,1} f_i p_j, \sum_{i,j=1}^{3} b_{ij,2} f_i p_j, \sum_{i,j=1}^{3} b_{ij,3} f_i p_j \right) \tag{15}$$

satisfies $\| \mathbf{f}_{\Delta^*(\varphi,\psi)} \| \le 1$.

So, we have the following.

Proposition 7 (see [13, Proposition 4.1]). *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a liner operator with a Haar state τ. Then $\Delta^*(\varphi \otimes \psi) \in S(\mathbb{M}_2(\mathbb{C}))$ for any $\varphi, \psi \in S(\mathbb{M}_2(\mathbb{C}))$ if and only if the following holds:*

$$\sum_{k=1}^{3} \left| \sum_{i,j=1}^{3} b_{ij,k} f_i p_j \right|^2 \le 1, \quad \forall \mathbf{f}, \mathbf{p} \in S. \tag{16}$$

From the proof of Proposition 6 and the last proposition we conclude that $|\|\mathbb{B}\|| \le 1$ holds if and only if (16) is satisfied.

Remark 8. Note that characterizations of positive maps defined on $\mathbb{M}_2(\mathbb{C})$ were considered in [24] (see also [25]). Characterization of completely positive mappings from $\mathbb{M}_2(\mathbb{C})$ into itself with invariant state τ was established in [3] (see also [26]).

Next we would like to recall (see [13]) some conditions for q.q.o. to be the Kadison-Schwarz ones.

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a linear operator with a Haar state τ; then it has the form (10). Now we are going to find some conditions to the coefficients $\{b_{ml,k}\}$ when Δ is a Kadison-Schwarz operator. Given $x = w_0 + \mathbf{w}\sigma$ and state $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$, let us denote

$$\mathbf{x}_m = (\langle \mathbf{b}_{m1}, \mathbf{w} \rangle, \langle \mathbf{b}_{m2}, \mathbf{w} \rangle, \langle \mathbf{b}_{m3}, \mathbf{w} \rangle), \qquad f_m = \varphi(\sigma_m), \tag{17}$$

$$\alpha_{ml} = \langle \mathbf{x}_m, \mathbf{x}_l \rangle - \langle \mathbf{x}_l, \mathbf{x}_m \rangle, \qquad \gamma_{ml} = [\mathbf{x}_m, \overline{\mathbf{x}}_l] + [\overline{\mathbf{x}}_m, \mathbf{x}_l], \tag{18}$$

where $m, l = 1, 2, 3$. Here and in what follows $[\cdot, \cdot]$ stands for the usual cross-product in $\mathbb{C}^3$. Note that here the numbers α_{ml} are skew symmetric, that is, $\overline{\alpha_{ml}} = -\alpha_{ml}$. By π we will denote mapping $\{1, 2, 3, 4\}$ to $\{1, 2, 3\}$ defined by $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 1$, $\pi(4) = \pi(1)$.

Denote

$$\mathbf{q}(\mathbf{f}, \mathbf{w}) = (\langle \beta(\mathbf{f})_1, [\mathbf{w}, \overline{\mathbf{w}}] \rangle, \langle \beta(\mathbf{f})_2, [\mathbf{w}, \overline{\mathbf{w}}] \rangle, \langle \beta(\mathbf{f})_3, [\mathbf{w}, \overline{\mathbf{w}}] \rangle), \tag{19}$$

where $\beta(\mathbf{f})_m = (\beta(\mathbf{f})_{m1}, \beta(\mathbf{f})_{m2}, \beta(\mathbf{f})_{m3})$ (see (11)).

Theorem 9 (see [13, Theorem 3.6]). *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a Kadison-Schwarz operator with a Haar state τ; then it has the form* (10) *and the coefficients $\{b_{ml,k}\}$ satisfy the following conditions:*

$$\|\mathbf{w}\|^2 \ge i \sum_{m=1}^{3} f_m \alpha_{\pi(m),\pi(m+1)} + \sum_{m=1}^{3} \|\mathbf{x}_m\|^2, \tag{20}$$

$$\left\| \mathbf{q}(\mathbf{f}, \mathbf{w}) - i \sum_{m=1}^{3} f_m \gamma_{\pi(m),\pi(m+1)} - [\mathbf{x}_m, \overline{\mathbf{x}}_m] \right\| \le \|\mathbf{w}\|^2 - i \sum_{k=1}^{3} f_k \alpha_{\pi(k),\pi(k+1)} - \sum_{m=1}^{3} \|\mathbf{x}_m\|^2 \tag{21}$$

for all $\mathbf{f} \in S$, $\mathbf{w} \in \mathbb{C}^3$. Here as before $\mathbf{x}_m = (\langle \mathbf{b}_{m1}, \mathbf{w} \rangle, \langle \mathbf{b}_{m2}, \mathbf{w} \rangle, \langle \mathbf{b}_{m3}, \mathbf{w} \rangle)$; $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$, and $\mathbf{q}(\mathbf{f}, \mathbf{w})$, α_{ml}, and γ_{ml} are defined in (19), (17), *and* (18), *respectively.*

Remark 10. The provided characterization with [2, 3] allows us to construct examples of positive or Kadison-Schwarz operators which are not completely positive (see next section).

Now we are going to give a general characterization of KS operators. Let us first give some notations. For a given mapping $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, by $\Delta(\sigma)$ we denote the vector $(\Delta(\sigma_1), \Delta(\sigma_2), \Delta(\sigma_3))$, and by $\mathbf{w}\Delta(\sigma)$ we mean the following:

$$\mathbf{w}\Delta(\sigma) = w_1 \Delta(\sigma_1) + w_2 \Delta(\sigma_2) + w_3 \Delta(\sigma_3), \tag{22}$$

where $\mathbf{w} \in \mathbb{C}^3$. Note that the last equality (22), due to the linearity of Δ, can also be written as $\mathbf{w}\Delta(\sigma) = \Delta(\mathbf{w}\sigma)$.

Theorem 11. *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a unital $*$-preserving linear mapping. Then Δ is a KS operator if and only if one has*

$$i [\mathbf{w}, \overline{\mathbf{w}}] \Delta(\sigma) + (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)) \le \mathbb{1} \otimes \mathbb{1}, \tag{23}$$

for all $\mathbf{w} \in \mathbb{C}^3$ with $\|\mathbf{w}\| = 1$.

Proof. Let $x \in M_2(\mathbb{C})$ be an arbitrary element, that is, $x = w_0\mathbb{1} + \mathbf{w}\sigma$. Then $x^* = \overline{w_0}\mathbb{1} + \overline{\mathbf{w}}\sigma$. Therefore

$$x^*x = \left(|w_0|^2 + \|\mathbf{w}\|^2\right)\mathbb{1} + \left(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w} - i[\mathbf{w}, \overline{\mathbf{w}}]\right)\sigma. \quad (24)$$

Consequently, we have

$$\begin{aligned}\Delta(x) &= w_0\mathbb{1}\otimes\mathbb{1} + \mathbf{w}\Delta(\sigma),\\ \Delta(x^*) &= \overline{w_0}\mathbb{1}\otimes\mathbb{1} + \overline{\mathbf{w}}\Delta(\sigma),\end{aligned} \quad (25)$$

$$\begin{aligned}\Delta(x^*x) = &\left(|w_0|^2 + \|\mathbf{w}\|^2\right)\mathbb{1}\otimes\mathbb{1}\\ &+ \left(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w} - i[\mathbf{w}, \overline{\mathbf{w}}]\right)\Delta(\sigma),\end{aligned} \quad (26)$$

$$\begin{aligned}\Delta(x)^*\Delta(x) = &|w_0|^2\mathbb{1}\otimes\mathbb{1} + \left(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w}\right)\Delta(\sigma)\\ &+ (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)).\end{aligned} \quad (27)$$

From (26) and (27) one gets

$$\begin{aligned}&\Delta(x^*x) - \Delta(x)^*\Delta(x)\\ &\quad = \|\mathbf{w}\|^2\mathbb{1}\otimes\mathbb{1} - i[\mathbf{w}, \overline{\mathbf{w}}]\Delta(\sigma) - (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)).\end{aligned} \quad (28)$$

So, the positivity of the last equality implies that

$$\|\mathbf{w}\|^2\mathbb{1}\otimes\mathbb{1} - i[\mathbf{w}, \overline{\mathbf{w}}]\Delta(\sigma) - (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)) \geq 0. \quad (29)$$

Now dividing both sides by $\|\mathbf{w}\|^2$ we get the required inequality. Hence, this completes the proof. □

4. An Example of Q.Q.O. Which Is Not Kadison-Schwarz One

In this section we are going to study dynamics of (57) for a special class of quadratic operators. Such class operators are associated with the following matrix $\{b_{ij,k}\}$ given by

$$\begin{aligned}&b_{11,1} = \varepsilon, && b_{11,2} = 0, && b_{11,3} = 0,\\ &b_{12,1} = 0, && b_{12,2} = 0, && b_{12,3} = \varepsilon,\\ &b_{13,1} = 0, && b_{13,2} = \varepsilon, && b_{13,3} = 0,\\ &b_{22,1} = 0, && b_{22,2} = \varepsilon, && b_{22,3} = 0,\\ &b_{23,1} = \varepsilon, && b_{23,2} = 0, && b_{23,3} = 0,\\ &b_{33,1} = 0, && b_{33,2} = 0, && b_{33,3} = \varepsilon,\end{aligned} \quad (30)$$

and $b_{ij,k} = b_{ji,k}$.

Via (10) we define a liner operator Δ_ε, for which τ is a Haar state. In the sequel we would like to find some conditions to ε which ensures positivity of Δ_ε.

It is easy that for given $\{b_{ijk}\}$ one can find a form of Δ_ε as follows.

$$\begin{aligned}\Delta_\varepsilon(x) = &w_0\mathbb{1}\otimes\mathbb{1} + \varepsilon\omega_1\sigma_1\otimes\sigma_1 + \varepsilon\omega_3\sigma_1\otimes\sigma_2\\ &+ \varepsilon\omega_2\sigma_1\otimes\sigma_3 + \varepsilon\omega_3\sigma_2\otimes\sigma_1 + \varepsilon\omega_2\sigma_2\otimes\sigma_2\\ &+ \varepsilon\omega_1\sigma_2\otimes\sigma_3 + \varepsilon\omega_2\sigma_3\otimes\sigma_1 + \varepsilon\omega_1\sigma_3\otimes\sigma_2\\ &+ \varepsilon\omega_3\sigma_3\otimes\sigma_3,\end{aligned} \quad (31)$$

where, as before, $x = w_0\mathbb{1} + \mathbf{w}\sigma$.

Theorem 12. *A linear operator Δ_ε given by (31) is a q.q.o. if and only if $|\varepsilon| \leq 1/3$.*

Proof. Let $x = w_0\mathbb{1} + \mathbf{w}\sigma$ be a positive element from $\mathbb{M}_2(\mathbb{C})$. Let us show positivity of the matrix $\Delta_\varepsilon(x)$. To do it, we rewrite (31) as follows: $\Delta_\varepsilon(x) = w_0\mathbb{1} + \varepsilon\mathbf{B}$; here

$$\mathbf{B} = \begin{pmatrix}\omega_3 & \omega_2 - i\omega_1 & \omega_2 - i\omega_1 & \omega_1 - 2i\omega_3 - \omega_2\\ \omega_2 + i\omega_1 & -\omega_3 & \omega_1 + \omega_2 & -\omega_2 + i\omega_1\\ \omega_2 + i\omega_1 & \omega_1 + \omega_2 & -\omega_3 & -\omega_2 + i\omega_1\\ \omega_1 + 2i\omega_3 - \omega_2 & -\omega_2 - i\omega_1 & -\omega_2 - i\omega_1 & \omega_3\end{pmatrix}, \quad (32)$$

where positivity of x yields that $w_0, \omega_1, \omega_2, \omega_3$ are real numbers. In what follows, without loss of generality, we may assume that $w_0 = 1$, and therefore $\|\mathbf{w}\| \leq 1$. It is known that positivity of $\Delta_\varepsilon(x)$ is equivalent to positivity of the eigenvalues of $\Delta_\varepsilon(x)$.

Let us first examine eigenvalues of $\mathbf{B}$. Simple algebra shows us that all eigenvalues of $\mathbf{B}$ can be written as follows:

$$\begin{aligned}\lambda_1(\mathbf{w}) = &\omega_1 + \omega_2 + \omega_3\\ &+ 2\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3},\\ \lambda_2(\mathbf{w}) = &\omega_1 + \omega_2 + \omega_3\\ &- 2\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3},\\ \lambda_3(\mathbf{w}) = &\lambda_4(\mathbf{w}) = -\omega_1 - \omega_2 - \omega_3.\end{aligned} \quad (33)$$

Now examine maximum and minimum values of the functions $\lambda_1(\mathbf{w}), \lambda_2(\mathbf{w}), \lambda_3(\mathbf{w}), \lambda_4(\mathbf{w})$ on the ball $\|\mathbf{w}\| \leq 1$.

One can see that

$$\begin{aligned}|\lambda_3(\mathbf{w})| = |\lambda_4(\mathbf{w})| \leq \sum_{k=1}^{3}|\omega_k| \leq \sqrt{3}\sum_{k=1}^{3}|\omega_k|^2\\ \leq \sqrt{3}.\end{aligned} \quad (34)$$

Note that the functions λ_3, λ_4 can reach values $\pm\sqrt{3}$ at $\pm(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Now let us rewrite $\lambda_1(\mathbf{w})$ and $\lambda_2(\mathbf{w})$ as follows:

$$\begin{aligned}\lambda_1(\mathbf{w}) = &\omega_1 + \omega_2 + \omega_3\\ &+ \frac{2}{\sqrt{2}}\sqrt{3(\omega_1^2 + \omega_2^2 + \omega_3^2) - (\omega_1 + \omega_2 + \omega_3)^2},\end{aligned} \quad (35)$$

$$\begin{aligned}\lambda_2(\mathbf{w}) = &\omega_1 + \omega_2 + \omega_3\\ &- \frac{2}{\sqrt{2}}\sqrt{3(\omega_1^2 + \omega_2^2 + \omega_3^2) - (\omega_1 + \omega_2 + \omega_3)^2}.\end{aligned} \quad (36)$$

One can see that

$$\lambda_k(h\omega_1, h\omega_2, h\omega_3) = h\lambda_k(\omega_1, \omega_2, w_3), \quad \text{if } h \geq 0, \quad (37)$$

$$\lambda_1(h\omega_1, h\omega_2, h\omega_3) = h\lambda_2(\omega_1, \omega_2, w_3), \quad \text{if } h \leq 0. \quad (38)$$

where $k = 1, 2$. Therefore, the functions $\lambda_k(\mathbf{w})$, $k = 1, 2$ reach their maximum and minimum on the sphere $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$

(i.e., $\|\mathbf{w}\| = 1$). Hence, denoting $t = \omega_1 + \omega_2 + \omega_3$ from (37) and (36) we introduce the following functions:

$$g_1(t) = t + \frac{2}{\sqrt{2}}\sqrt{3 - t^2}, \qquad g_2(t) = t - \frac{2}{\sqrt{2}}\sqrt{3 - t^2}, \quad (39)$$

where $|t| \le \sqrt{3}$.

One can find that the critical values of g_1 are $t = \pm 1$, and the critical value of g_2 is $t = -1$. Consequently, extremal values of g_1 and g_2 on $|t| \le \sqrt{3}$ are the following:

$$\begin{aligned} &\min_{|t|\le\sqrt{3}} g_1(t) = -\sqrt{3}, \qquad \max_{|t|\le\sqrt{3}} g_1(t) = 3, \\ &\min_{|t|\le\sqrt{3}} g_2(t) = -3, \qquad \max_{|t|\le\sqrt{3}} g_2(t) = \sqrt{3}. \end{aligned} \quad (40)$$

Therefore, from (37) and (38) we conclude that

$$-3 \le \lambda_k(\mathbf{w}) \le 3, \quad \text{for any } \|\mathbf{w}\| \le 1,\ k = 1, 2. \quad (41)$$

It is known that for the spectrum of $\mathbb{1} + \varepsilon\mathbf{B}$ one has

$$Sp(\mathbb{1} + \varepsilon\mathbf{B}) = 1 + \varepsilon Sp(\mathbf{B}). \quad (42)$$

Therefore,

$$Sp(\mathbb{1} + \varepsilon\mathbf{B}) = \left\{1 + \varepsilon\lambda_k(\mathbf{w}) : k = \overline{1,4}\right\}. \quad (43)$$

So, if

$$|\varepsilon| \le \frac{1}{\max_{\|\mathbf{w}\|\le 1}|\lambda_k(\mathbf{w})|}, \quad k = \overline{1,4}, \quad (44)$$

then one can see $1 + \varepsilon\lambda_k(\mathbf{w}) \ge 0$ for all $\|\mathbf{w}\| \le 1$, $k = \overline{1,4}$. This implies that the matrix $\mathbb{1} + \varepsilon\mathbf{B}$ is positive for all $\mathbf{w}$ with $\|\mathbf{w}\| \le 1$.

Now assume that Δ_ε is positive. Then $\Delta_\varepsilon(x)$ is positive whenever x is positive. This means that $1 + \varepsilon\lambda_k(\mathbf{w}) \ge 0$ for all $\|\mathbf{w}\| \le 1 (k = \overline{1,4})$. From (34) and (41) we conclude that $|\varepsilon| \le 1/3$. This completes the proof. □

Theorem 13. *Let $\varepsilon = 1/3$ then the corresponding q.q.o. Δ_ε is not KS operator.*

Proof. It is enough to show the dissatisfaction of (21) at some values of $\mathbf{w}$ ($\|\mathbf{w}\| \le 1$) and $\mathbf{f} = (f_1, f_1, f_2)$.

Assume that $\mathbf{f} = (1, 0, 0)$; then a little algebra shows that (21) reduces to the following one:

$$\sqrt{A + B + C} \le D, \quad (45)$$

where

$$\begin{aligned} A &= \left|\varepsilon\left(\overline{\omega}_2\omega_3 - \overline{\omega}_3\omega_2\right) - i\varepsilon^2\left(2\overline{\omega}_2\omega_3 - 2|\omega_1|^2 - \overline{\omega}_2\omega_1 \right.\right. \\ &\qquad \left.\left. + \overline{\omega}_1\omega_2 - \overline{\omega}_1\omega_3 + \overline{\omega}_3\omega_1\right)\right|^2, \\ B &= \left|\varepsilon\left(\overline{\omega}_1\omega_2 - \overline{\omega}_2\omega_1\right) - i\varepsilon^2\left(2\overline{\omega}_1\omega_2 - 2|\omega_3|^2 - \overline{\omega}_1\omega_3 \right.\right. \\ &\qquad \left.\left. + \overline{\omega}_3\omega_1 - \overline{\omega}_3\omega_2 + \overline{\omega}_2\omega_3\right)\right|^2, \\ C &= \left|\varepsilon\left(\overline{\omega}_3\omega_1 - \overline{\omega}_1\omega_3\right) - i\varepsilon^2\left(2\overline{\omega}_3\omega_1 - 2|\omega_2|^2 - \overline{\omega}_3\omega_2 \right.\right. \\ &\qquad \left.\left. + \overline{\omega}_2\omega_3 - \overline{\omega}_2\omega_1 + \overline{\omega}_1\omega_2\right)\right|^2, \\ D &= \left(1 - 3|\varepsilon|^2\right)\left(|\omega_1|^2 + |\omega_2|^2 + |\omega_3|^2\right) \\ &\quad - i\varepsilon^2\left(\overline{\omega}_3 w_2 - \overline{\omega}_2\omega_3 + \overline{\omega}_2\omega_1 - \overline{\omega}_1\omega_2 + \overline{\omega}_1\omega_3 - \overline{\omega}_3\omega_1\right). \end{aligned} \quad (46)$$

Now choose $\mathbf{w}$ as follows:

$$\omega_1 = -\frac{1}{9}, \qquad \omega_2 = \frac{5}{36}, \qquad \omega_3 = \frac{5i}{27}. \quad (47)$$

Then calculations show that

$$\begin{aligned} A &= \frac{9594}{19131876}, \qquad B = \frac{19625}{86093442}, \\ C &= \frac{1625}{3779136}, \qquad D = \frac{589}{17496}. \end{aligned} \quad (48)$$

Hence, we find

$$\sqrt{\frac{9594}{19131876} + \frac{19625}{86093442} + \frac{1625}{3779136}} > \frac{589}{17496}, \quad (49)$$

which means that (45) is not satisfied. Hence, Δ_ε is not a KS operator at $\varepsilon = 1/3$. □

Recall that a linear operator $T : \mathbb{M}_k(\mathbb{C}) \to \mathbb{M}_m(\mathbb{C})$ is *completely positive* if for any positive matrix $(a_{ij})_{i,j=1}^n \in \mathbb{M}_k(\mathbb{M}_n(\mathbb{C}))$ the matrix $(T(a_{ij}))_{i,j=1}^n$ is positive for all $n \in \mathbb{N}$. Now we are interested when the operator Δ_ε is completely positive. It is known [1] that the complete positivity of Δ_ε is equivalent to the positivity of the following matrix:

$$\widehat{\Delta}_\varepsilon = \begin{pmatrix} \Delta_\varepsilon(e_{11}) & \Delta_\varepsilon(e_{12}) \\ \Delta_\varepsilon(e_{21}) & \Delta_\varepsilon(e_{22}) \end{pmatrix}, \quad (50)$$

here e_{ij} $(i, j = 1, 2)$ are the standard matrix units in $\mathbb{M}_2(\mathbb{C})$.

From (31) one can calculate that

$$\begin{aligned} \Delta_\varepsilon(e_{11}) &= \frac{1}{2}\mathbb{1} \otimes \mathbb{1} + \varepsilon B_{11}, \qquad \Delta_\varepsilon(e_{22}) = \frac{1}{2}\mathbb{1} \otimes \mathbb{1} - \varepsilon B_{11}, \\ \Delta_\varepsilon(e_{12}) &= \varepsilon B_{12}, \qquad \Delta_\varepsilon(e_{21}) = \varepsilon B_{12}^*, \end{aligned} \quad (51)$$

where

$$B_{11} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -i \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ i & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$B_{12} = \begin{pmatrix} 0 & 0 & 0 & \frac{1-i}{2} \\ i & 0 & \frac{1+i}{2} & 0 \\ i & \frac{1+i}{2} & 0 & 0 \\ \frac{1-i}{2} & -i & -i & 0 \end{pmatrix}. \tag{52}$$

Hence, we find that

$$2\widehat{\Delta}_{\varepsilon} = \mathbb{1}_8 + \varepsilon\mathbb{B}, \tag{53}$$

where $\mathbb{1}_8$ is the unit matrix in $\mathbb{M}_8(\mathbb{C})$ and

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 & -2i & 0 & 0 & 0 & 1-i \\ 0 & -1 & 0 & 0 & 2i & 0 & 1+i & 0 \\ 0 & 0 & -1 & 0 & 2i & 1+i & 0 & 0 \\ 2i & 0 & 0 & 1 & 1-i & -2i & -2i & 0 \\ 0 & -2i & -2i & 1+i & -1 & 0 & 0 & 2i \\ 0 & 0 & 1-i & 2i & 0 & 1 & 0 & 0 \\ 0 & 1-i & 0 & 2i & 0 & 0 & 1 & 0 \\ 1+i & 0 & 0 & 0 & -2i & 0 & 0 & -1 \end{pmatrix}. \tag{54}$$

So, the matrix $\widehat{\Delta}_{\varepsilon}$ is positive if and only if

$$|\varepsilon| \le \frac{1}{\lambda_{\max}(\mathbb{B})}, \tag{55}$$

where $\lambda_{\max}(\mathbb{B}) = \max_{\lambda \in Sp(\mathbb{B})} |\lambda|$.

One can easily calculate that $\lambda_{\max}(\mathbb{B}) = 3\sqrt{3}$. Therefore, we have the following.

Theorem 14. *Let $\Delta_{\varepsilon} : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be given by* (31). *Then Δ_{ε} is completely positive if and only if $|\varepsilon| \le 1/3\sqrt{3}$.*

5. Dynamics of Δ_{ε}

Let Δ be a q.q.o. on $\mathbb{M}_2(\mathbb{C})$. Let us consider the corresponding quadratic operator defined by $V_{\Delta}(\varphi) = \Delta^*(\varphi \otimes \varphi)$, $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$. From Theorem 5 one can see that the defined operator V_{Δ} maps $S(\mathbb{M}_2(\mathbb{C}))$ into itself if and only if $|||\mathbb{B}||| \le 1$ or equivalently (16) holds. From (14) we find that

$$V_{\Delta}(\varphi)(\sigma_k) = \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad \mathbf{f} \in S. \tag{56}$$

Here, as before, $S = \{\mathbf{f} = (f_1, f_2 f p_3) \in \mathbb{R}^3 : f_1^2 + f_2^2 + f_3^2 \le 1\}$.

So, (56) suggests that we consider the following nonlinear operator $V : S \to S$ defined by

$$V(\mathbf{f})_k = \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad k = 1, 2, 3, \tag{57}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in S$.

It is worth to mention that uniqueness of the fixed point (i.e., $(0,0,0)$) of the operator given by (57) was investigated in [13, Theorem 4.4].

In this section, we are going to study dynamics of the quadratic operator V_{ε} corresponding to Δ_{ε} (see (31)), which has the following form

$$\begin{aligned} V_{\varepsilon}(f)_1 &= \varepsilon\left(f_1^2 + 2f_2 f_3\right), \\ V_{\varepsilon}(f)_2 &= \varepsilon\left(f_2^2 + 2f_1 f_3\right), \\ V_{\varepsilon}(f)_3 &= \varepsilon\left(f_3^2 + 2f_1 f_2\right). \end{aligned} \tag{58}$$

Let us first find some condition on ε which ensures (16).

Lemma 15. *Let V_{ε} be given by* (58). *Then V_{ε} maps S into itself if and only if $|\varepsilon| \le 1/\sqrt{3}$ is satisfied.*

Proof. "If" Part. Assume that V_{ε} maps S into itself. Then (16) is satisfied. Take $\mathbf{f} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $\mathbf{p} = \mathbf{f}$. Then from (16) one finds that

$$\sum_{k=1}^{3} \left| \sum_{i,j=1}^{3} b_{ij,k} f_i p_j \right|^2 = 3\varepsilon^2 \le 1 \tag{59}$$

which yields $|\varepsilon| \le 1/\sqrt{3}$.

"Only If" Part. Assume that $|\varepsilon| \le 1/\sqrt{3}$. Take any $\mathbf{f} = (f_1, f_2, f_3)$, $\mathbf{p} = (p_1, p_2, p_3) \in S$. Then one finds that

$$\begin{aligned} &\sum_{k=1}^{3} \left| \sum_{i,j=1}^{3} b_{ij,k} f_i p_j \right|^2 \\ &\quad = \varepsilon^2 \left(\left| f_1 p_1 + f_3 p_2 + f_2 p_3 \right|^2 \right. \\ &\qquad \left. + \left| f_3 p_1 + f_2 p_2 + f_1 p_3 \right|^2 + \left| f_2 p_1 + f_1 p_2 + f_3 p_3 \right|^2 \right) \\ &\quad \le \varepsilon^2 \left(\left(f_1^2 + f_2^2 + f_3^2\right)\left(p_1^2 + p_2^2 + p_3^2\right) \right. \\ &\qquad + \left(f_3^2 + f_2^2 + f_1^2\right)\left(p_1^2 + p_2^2 + p_3^2\right) \\ &\qquad \left. + \left(p_1^2 + p_2^2 + p_3^2\right)\left(f_2^2 + f_1^2 + f_3^2\right) \right) \\ &\quad \le \varepsilon^2 (1 + 1 + 1) = 3\varepsilon^2 \le 1. \end{aligned} \tag{60}$$

This completes the proof. □

Remark 16. We stress that condition (16) is necessary for Δ to be a positive operator. Namely, from Theorem 12 and Lemma 15 we conclude that if $\varepsilon \in (1/3, 1/\sqrt{3}]$ then the operator Δ_{ε} is not positive, while (16) is satisfied.

In what follows, to study dynamics of V_ε we assume $|\varepsilon| \le 1/\sqrt{3}$. Recall that a vector $\mathbf{f} \in S$ is a fixed point of V_ε if $V_\varepsilon(\mathbf{f}) = \mathbf{f}$. Clearly $(0,0,0)$ is a fixed point of V_ε. Let us find others. To do it, we need to solve the following equation:

$$\begin{aligned} \varepsilon\left(f_1^2 + 2f_2f_3\right) &= f_1, \\ \varepsilon\left(f_2^2 + 2f_1f_3\right) &= f_2, \\ \varepsilon\left(f_3^2 + 2f_1f_2\right) &= f_3. \end{aligned} \tag{61}$$

We have the following.

Proposition 17. *If $|\varepsilon| < 1/\sqrt{3}$ then V_ε has a unique fixed point $(0,0,0)$ in S. If $|\varepsilon| = 1/\sqrt{3}$ then V_ε has the following fixed points: $(0,0,0)$ and $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ in S.*

Proof. It is clear that $(0,0,0)$ is a fixed point of V_ε. If $f_k = 0$, for some $k \in \{1,2,3\}$ then due to $|\varepsilon| \le 1/\sqrt{3}$, one can see that the only solution of (61) belonging to S is $f_1 = f_2 = f_3 = 0$. Therefore, we assume that $f_k \ne 0$ $(k = 1,2,3)$. So, from (61) one finds

$$\begin{aligned} \frac{f_1^2 + 2f_2f_3}{f_2^2 + 2f_1f_3} &= \frac{f_1}{f_2}, \\ \frac{f_1^2 + 2f_2f_3}{f_3^2 + 2f_1f_2} &= \frac{f_1}{f_3}, \\ \frac{f_2^2 + 2f_1f_3}{f_3^2 + 2f_1f_2} &= \frac{f_2}{f_3}. \end{aligned} \tag{62}$$

Denoting

$$x = \frac{f_1}{f_2}, \qquad y = \frac{f_1}{f_3}, \qquad z = \frac{f_2}{f_3}. \tag{63}$$

From (62) it follows that

$$\begin{aligned} x\left(\frac{x(1+2/xy)}{1+2x/z} - 1\right) &= 0, \\ y\left(\frac{y(1+2/xy)}{1+2yz} - 1\right) &= 0, \\ z\left(\frac{z(1+2x/z)}{1+2yz} - 1\right) &= 0. \end{aligned} \tag{64}$$

According to our assumption x, y, z are nonzero, so from (64) one gets

$$\begin{aligned} \frac{x(1+2/xy)}{1+2x/z} &= 1, \\ \frac{y(1+2/xy)}{1+2yz} &= 1, \\ \frac{z(1+2x/z)}{1+2yz} &= 1, \end{aligned} \tag{65}$$

where $2x \ne -z$ and $2yz \ne -1$.

Dividing the second equality of (65) to the first one of (65) we find that

$$\frac{y(1+2x/z)}{x(1+2yz)} = 1, \tag{66}$$

which with $xz = y$ yields

$$y + 2x^2 = x + 2y^2. \tag{67}$$

Simplifying the last equality one gets

$$(y - x)(1 - 2(y + x)) = 0. \tag{68}$$

This means that either $y = x$ or $x + y = 1/2$.

Assume that $x = y$. Then from $xz = y$, one finds $z = 1$. Moreover, from the second equality of (65) we have $y + 2/y = 1 + 2y$. So, $y^2 + y - 2 = 0$; therefore, the solutions of the last one are $y_1 = 1$, $y_2 = -2$. Hence, $x_1 = 1$, $x_2 = -2$.

Now suppose that $x + y = 1/2$; then $x = 1/2 - y$. We note that $y \ne 1/2$, since $x \ne 0$. So, from the second equality of (65) we find

$$y + \frac{4}{1-2y} = 1 + \frac{4y^2}{1-2y}. \tag{69}$$

So, $2y^2 - y - 1 = 0$ which yields the solutions $y_3 = -1/2$, $y_4 = 1$. Therefore, we obtain $x_3 = 1$, $z_3 = -1/2$ and $x_4 = -1/2$, $z_4 = -2$.

Consequently, solutions of (65) are the following ones:

$$(1,1,1), \qquad \left(1,-\frac{1}{2},-\frac{1}{2}\right), \qquad \left(-\frac{1}{2},1,-2\right), \qquad (-2,-2,1). \tag{70}$$

Now owing to (63) we need to solve the following equations:

$$\begin{aligned} \frac{f_1}{f_2} &= x_k, \\ \frac{f_2}{f_3} &= z_k. \end{aligned} \qquad k = \overline{1,4}, \tag{71}$$

According to our assumption $f_k \ne 0$, we consider cases when $x_k z_k \ne 0$.

Now let us start to consider several cases.

Case 1 . Let $x_2 = 1$, $z_2 = 1$. Then from (71) one gets $f_1 = f_2 = f_3$. So, from (61) we find $3\varepsilon f_1^2 = f_1$, that is, $f_1 = 1/3\varepsilon$. Now taking into account $f_1^2 + f_2^2 + f_3^2 \le 1$ one gets $1/3\varepsilon^2 \le 1$. From the last inequality we have $|\varepsilon| \ge 1/\sqrt{3}$. Due to Lemma 15 the operator V_ε is well defined if and only if $|\varepsilon| \le 1/\sqrt{3}$; therefore, one gets $|\varepsilon| = 1/\sqrt{3}$. Hence, in this case a solution is $(\pm 1/\sqrt{3}; \pm 1/\sqrt{3}; \pm 1/\sqrt{3})$.

Case 2 . Let $x_2 = 1$, $z_2 = -1/2$. Then from (71) one finds $f_1 = f_2$, $2f_2 = -f_3$. Substituting the last ones to (61) we get $f_1 + 3f_1^2\varepsilon = 0$. Then, we have $f_1 = -1/3\varepsilon$, $f_2 = -1/3\varepsilon$, $f_3 = 2/3\varepsilon$. Taking into account $f_1^2 + f_2^2 + f_3^2 \le 1$ we find $1/9\varepsilon^2 + 4/9\varepsilon^2 + 1/9\varepsilon^2 \le 1$. This means $|\varepsilon| \ge \sqrt{2/3}$; due to Lemma 15 in this case the operator V_ε is not well defined; therefore, we conclude that there is no fixed point of V_ε belonging to S.

Using the same argument for the rest of the cases we conclude the absence of solutions. This shows that if $|\varepsilon| < 1/\sqrt{3}$ the operator V_ε has unique fixed point in S. If $|\varepsilon| = 1/\sqrt{3}$, then V_ε has three fixed points belonging to S. This completes the proof. □

Now we are going to study dynamics of operator V_ε.

Theorem 18. *Let V_ε be given by* (58). *Then the following assertions hold true:*

(i) *if $|\varepsilon| < 1/\sqrt{3}$, then for any $\mathbf{f} \in S$ one has $V_\varepsilon^n(\mathbf{f}) \to (0,0,0)$ as $n \to \infty$.*

(ii) *if $|\varepsilon| = 1/\sqrt{3}$, then for any $\mathbf{f} \in S$ with $\mathbf{f} \notin \{(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})\}$ one has $V_\varepsilon^n(\mathbf{f}) \to (0,0,0)$ as $n \to \infty$.*

Proof. Let us consider the following function $\rho(\mathbf{f}) = f_1^2 + f_2^2 + f_3^2$. Then we have

$$\begin{aligned}\rho(V_\varepsilon(\mathbf{f})) &= \varepsilon^2\left(\left(f_1^2 + 2f_2f_3\right)^2 + \left(f_2^2 + 2f_1f_3\right)^2 + \left(f_3^2 + 2f_1f_2\right)^2\right)\\ &\le \varepsilon^2\left(f_1^2 + 2|f_2||f_3| + f_2^2 + 2|f_1||f_3| + f_3^2 + 2|f_1||f_2|\right)\\ &\le \varepsilon^2\left(f_1^2 + f_2^2 + f_3^2 + f_2^2 + f_1^2 + f_3^2 + f_3^2 + f_1^2 + f_2^2\right)\\ &= 3\varepsilon^2\left(f_1^2 + f_2^2 + f_3^2\right) = 3\varepsilon^2\rho(\mathbf{f}).\end{aligned} \tag{72}$$

This means

$$\rho(V_\varepsilon(\mathbf{f})) \le 3\varepsilon^2\rho(\mathbf{f}). \tag{73}$$

Due to $\varepsilon^2 \le 1/3$ from (73) one finds that

$$\rho\left(V_\varepsilon^{n+1}(\mathbf{f})\right) \le \rho\left(V_\varepsilon^{n}(\mathbf{f})\right), \tag{74}$$

which yields that the sequence $\{\rho(V_\varepsilon^n(\mathbf{f}))\}$ is convergent. Next we would like to find the limit of $\{\rho(V_\varepsilon^n(\mathbf{f}))\}$.

(i) First we assume that $|\varepsilon| < 1/\sqrt{3}$; then from (73) we obtain

$$\rho\left(V_\varepsilon^{n}(\mathbf{f})\right) \le 3\varepsilon^2\rho\left(V_\varepsilon^{n-1}(\mathbf{f})\right) \le \cdots \le \left(3\varepsilon^2\right)^n\rho(\mathbf{f}). \tag{75}$$

This yields that $\rho(V_\varepsilon^n(\mathbf{f})) \to 0$ as $n \to \infty$, for all $\mathbf{f} \in S$.

(ii) Now let $|\varepsilon| = 1/\sqrt{3}$. Then consider two distinct subcases.

Case A. Let $f_1^2 + f_2^2 + f_3^2 < 1$ and denote $d = f_1^2 + f_2^2 + f_3^2$. Then one gets

$$\begin{aligned}\rho(V_\varepsilon(\mathbf{f})) &\le \varepsilon^2\left(\left(f_1^2 + 2|f_2||f_3|\right)^2 + \left(f_2^2 + 2|f_1||f_3|\right)^2 + \left(f_3^2 + 2|f_1||f_2|\right)^2\right)\\ &\le \varepsilon^2\left(\left(f_1^2 + f_2^2 + f_3^2\right)^2 + \left(f_2^2 + f_1^2 + f_3^2\right)^2 + \left(f_3^2 + f_1^2 + f_2^2\right)^2\right)\\ &= 3\varepsilon^2 d^2 = dd = d\rho(\mathbf{f}).\end{aligned} \tag{76}$$

Hence, we have $\rho(V_\varepsilon(\mathbf{f})) \le d\rho(\mathbf{f})$. This means $\rho(V_\varepsilon^n(\mathbf{f})) \le d^n\rho(\mathbf{f}) \to 0$. Hence, $V_\varepsilon^n(\mathbf{f}) \to 0$ as $n \to \infty$.

Case B. Now take $f_1^2 + f_2^2 + f_3^2 = 1$ and assume that $\mathbf{f}$ is not a fixed point. Therefore, we may assume that $f_i \ne f_j$ for some $i \ne j$, otherwise from Proposition 17 one concludes that $\mathbf{f}$ is a fixed point. Hence, from (58) one finds

$$\begin{aligned}V_\varepsilon(\mathbf{f})_1 &= \varepsilon\left(f_1^2 + 2f_2f_3\right) = \varepsilon\left(1 - f_2^2 - f_3^2 + 2f_2f_3\right)\\ &= \varepsilon\left(1 - \left(f_2 - f_3\right)^2\right).\end{aligned} \tag{77}$$

Similarly, one gets

$$\begin{aligned}V_\varepsilon(\mathbf{f})_2 &= \varepsilon\left(1 - \left(f_1 - f_3\right)^2\right),\\ V_\varepsilon(\mathbf{f})_3 &= \varepsilon\left(1 - \left(f_1 - f_2\right)^2\right).\end{aligned} \tag{78}$$

It is clear that $|V_\varepsilon(\mathbf{f})_k| \le |\varepsilon|$ $(k = 1,2,3)$. According to our assumption $f_i \ne f_j$ $(i \ne j)$ we conclude that one of $|V_\varepsilon(\mathbf{f})_k|$ is strictly less than $1/\sqrt{3}$; this means $V_\varepsilon(\mathbf{f})_1^2 + V_\varepsilon(\mathbf{f})_2^2 + V_\varepsilon(\mathbf{f})_3^2 < 1$. Therefore, from Case A, one gets that $V_\varepsilon^n(\mathbf{f}) \to 0$ as $n \to \infty$. □

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