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Research Article

Ergodicity of Stochastic Burgers' System with Dissipative Term

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A 2-dimensional stochastic Burgers equation with dissipative term perturbed by Wiener noise is considered. The aim is to prove the well-posedness, existence, and uniqueness of invariant measure as well as strong law of large numbers and convergence to equilibrium.

1. Introduction

The paper is concerned with the 2-dimensional Burgers equation in a bounded domain with Wiener noise as the body forces like this

$$\begin{aligned} du &= (\nu\Delta u + (u\cdot\nabla)u)\,dt + dW, \quad \text{on } [0,T]\times D, \\ u(t,x) &= 0, \quad t\in[0,T],\ x\in\partial D, \\ u(0,x) &= u_0(x), \quad x\in D, \end{aligned} \tag{1}$$

where $u(t,x) = (u^1(t,x), u^2(t,x))$ is the velocity field, $\nu > 0$ is viscid coefficient, Δ denotes the Laplace operator, ∇ represents the gradient operator, W stands for the Q-Wiener process, and D is a regular bounded open domain of $\mathbb{R}^2$. Burgers equation has received an extensive amount of attention since the studies by Burgers in the 1940s (and it has been considered even earlier by Beteman [1] and Forsyth [2]). But it is well known that the Burgers' equation is not a good model for turbulence since it does not perform any chaos. Even if a force is added to equation, all solutions will converge to a unique stationary solution as time goes to infinity. However, if the force is a random one, the result is completely different. So, several authors have indeed suggested to use the stochastic Burgers' equation to model turbulence, see [3–6]. The stochastic equation has also been proposed in [7] to study the dynamics of interfaces.

So far, most of the monographs concerning the equation focus on one-dimensional case, for example, Bertini et al. [8] solved the equation with additive space-time white noise by an adaptation of the Hopf-cole transformation. Da Prato et al. [9] studied the equation via a different approach based on semigroup property for the heat equation on a bounded interval. The more general equation with multiplicative noise was considered by Da Prato and Debussche [10]. With a similar method, Gyöngy and Nualart [11] extended the Burgers equation from bounded interval to real line. A large deviation principle for the solution was obtained by Gourcy [12]. Concerning the ergodicity, an important paper by Weinan et al. [13] proved that there exists a unique stationary distribution for the solutions of the random inviscid Burgers equation, and typical solutions are piecewise smooth with a finite number of jump discontinuities corresponding to shocks. For model with jumps, Dong and Xu [14] proved that the global existence and uniqueness of the strong, weak, and mild solutions for a one-dimensional Burgers equation perturbed by Lévy noise. When the noise is fractal, Wang et al. [15] get the well-posedness.

The main aim in our paper is to study the large time behavior of stochastic system. There are lots of the literature about the topic (see [16–20]).

Burgers system is a well-known model for mechanics problems. But as far as we know, there are no results about the long-term behavior of stochastic Burgers' system. We think that the difficulty lies in the fact that the dissipative term Δu cannot dominate the nonlinear term $(u\cdot\nabla)u$. However, in many practical cases, we cannot ignore the energy dissipation and external forces, especially considering the long-term behavior. Therefore, we introduce dissipative term $f(u)$ and study the ergodicity of the following equation:

$$du = \left[\Delta u + (u \cdot \nabla) u - f(u)\right] dt + dW, \quad \text{on } [0, T] \times D,$$
$$u(t, x) = 0, \quad t \in [0, T], \; x \in \partial D,$$
$$u(0, x) = u_0(x), \quad x \in D, \tag{2}$$

where $f(u) = \vartheta |u(t, x)|^2 u(t, x)$, $\vartheta > 0$, $|\cdot|$ denote the absolute value or norm for the real number or two-dimensional vector, respectively.

We believe that our work is new and is worth researching. The methods and results in this paper can be applied to stochastic reaction diffusion equations and stochastic real valued Ginzburg Landau equation in high dimensions. But we cannot extend our result to dynamical systems with state-delays. Since in order to show the existence of an invariant measure, we should consider the segments of a solution. In contrast to the scalar solution process, the process of segments is a Markov process. We show that the process of segments is also Feller and that there exists a solution of which the segments are tight. Then, we apply the Krylov-Bogoliubov method. Since the segment process has values in the infinite-dimensional space $C([-r, 0], H)$, boundedness in probability does not automatically imply tightness. For solution processes of infinite-dimensional equations, one often uses compactness of the orbits of the underlying deterministic equation to obtain tightness. For an infinite-dimensional formulation of the functional differential equation, however, such a compactness property does not hold. For ergodicity of stochastic delay equations, we can see [21]. We believe that stochastic Burgers' system with state-delays is a very interesting problem.

In order to study ergodicity of problem (2), we use a remarkable dissipativity property of the stochastic dynamic to obtain the existence of the invariant measure. For uniqueness, we try to use the method from [22] to prove that the distributions $P(t, x, \cdot)$ induced by the solution are equivalent. It is well known that the equivalence of the distributions implies uniqueness, a strong law of large numbers, and the convergence to equilibrium.

The remaining of this paper is organized as follows. Some preliminaries are presented in Section 2, the local existence and global existence are presented, respectively, in Sections 3 and 4. In Section 5, we obtain the existence and uniqueness of the invariant measure as well as strong law of large numbers, and convergence to equilibrium. As usual, constants C may change from one line to the next; we denote by C_a a constant which depends on some parameter a.

2. Preliminaries on the Burgers Equation

Let $u(t, x) = (u^1(t, x), u^2(t, x))$ be a row vector valued function on $[0, \infty) \times \mathbb{R}^2$. And it denotes the following:

$$|u|^2 := \sum_{i=1}^{2} \left|u^i\right|^2, \quad \partial_i u^j := \frac{\partial u^j}{\partial x_i}, \quad i, j = 1, 2. \tag{3}$$

Let $[C^\infty(D)]^2$ be infinitely differentiable 2-dimensional vector field on D, and let $[C_0^\infty(D)]^2$ be infinitely differentiable 2-dimensional vector field with compact support strictly contained in D. We denote by H^α the closure of $[C^\infty(D)]^2$ in $[H^\alpha(D)]^2$, whose norms are denoted by $\|\cdot\|_{H^\alpha}$, when $\alpha \neq 0$. Let H_0^1, H be the closure of $[C_0^\infty(D)]^2$ in $[H^1(D)]^2$ and $[L^2(D)]^2$ whose norms are denoted by $\|\cdot\|_{H^1}$ and $\|\cdot\|_H$, respectively. Without confusion, set $\langle \cdot, \cdot \rangle$ as the inner product in H or $L^2(D)$. For $p > 0$, let $\|\cdot\|_{L^p}$ be the norm of vector filed in Lebesgue spaces $[L^p(D)]^2$. $|\cdot|_{H^\alpha}$ represents the norm in the usual sobolev spaces $H^\alpha(D)$ for real valued functions on D and $\alpha \in \mathbb{R}$; $|\cdot|_{L^p}$ stands for the norm in the usual Lebesgue spaces $L^p(D)$ for real valued functions on D. Denote $A := -\Delta$; then $A : D(A) \subset H \to H$ and $D(A) = [H^2(D)]^2 \cap H_0^1$. Since H_0^1 coincides with $D(A^{1/2})$, we can endow H_0^1 with the norm $\|u\|_{H^1} = \|A^{1/2} u\|_H$. The operator A is positive self-adjoint with compact resolvent; we denote by $0 < \alpha_1 \leq \alpha_2 \leq \cdots$ the eigenvalues of A, and by $e_1, e_2, \ldots$ the eigenvectors which is a corresponding complete orthonormal system in H satisfying

$$\text{(i)} \; e_i \in \left[C_0^\infty(D)\right]^2,$$
$$\text{(ii)} \; \left|e_i(x)\right| \leq C, \qquad \left|\nabla e_i(x)\right| \leq C\sqrt{\alpha_i}, \tag{4}$$
$$x \in D, \; i = 1, 2, \ldots,$$

for some positive constant C. We remark that $\|u\|_{H^1}^2 \geq \alpha_1 \|u\|_H^2$. We define the bilinear operator $B(u, v) : H^1 \times H^1 \to H^{-1}$ as

$$\langle B(u, v), z \rangle = \int_D z(x) \cdot (u(x) \cdot \nabla) v(x) \, dx, \tag{5}$$

for all $z \in H^1$. Then, (2) is equivalent to the following abstract equation:

$$du(t) + \left[Au(t) + B(u(t), u(t)) + f(u(t))\right] dt = dW(t). \tag{6}$$

W is the Q Wiener process having the following representative:

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n(t), \quad t \in [0, T], \tag{7}$$

in which $\sum_{n=1}^{\infty} \lambda_n < \infty$ and β_k are a sequence of mutually independent 1-dimensional Brownian motions in a fixed probability space $(\Omega, \mathcal{F}, P)$ adapted to a filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

It can be derived from [23] that the solution to the linear problem corresponding to (2) with the following initial condition:

$$du = \Delta u dt + dW,$$
$$u(t, x) = 0, \quad t \in [0, T], \; x \in \partial D, \tag{8}$$
$$u(0, x) = u_0(x), \quad x \in D,$$

is unique, and when $u_0 = 0$, it has the form of

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s). \tag{9}$$

Let

$$v(t) = u(t) - W_A(t), \quad t \geq 0, \tag{10}$$

then u is a solution to (2) if and only if it solves the following evolution equation:

$$\begin{gathered} \frac{\partial v}{dt} + Av + B(v + W_A, v + W_A) + f(v + W_A) = 0, \\ v(t,x) = 0, \quad t \in [0,T], \ x \in \partial D, \\ v(0) = u_0. \end{gathered} \tag{11}$$

So, we see that when $w \in \Omega$ is fixed, this equation is in fact a deterministic equation. From now on, we will study the equation of the form (11) to get the existence and uniqueness of the solution a.s. $w \in \Omega$.

3. Local Existence in Time

Definition 1 (see Definition 5.1.1 in [24]). We say a $(\mathcal{F}(t))_{t\geq 0}$ adapted process $v(t)$ is a mild solution to (11), if $v(t) \in C([0,T]; H_0^1)$ and it satisfies

$$\begin{aligned} v(t) = e^{tA}v_0 &+ \int_0^t e^{(t-s)A} B(v + W_A, v + W_A)\, ds \\ &- \int_0^t e^{(t-s)A} f(v + W_A)\, ds, \quad t \in [0,T]. \end{aligned} \tag{12}$$

Lemma 2. *For any $\theta \in (0,1)$, if $\sum_{i=1}^{\infty} \lambda_i (\alpha_i)^\theta < \infty$, then $A^{1/2}W_A$ has a version which is α-Hölder continuous with respect to $t \in [0,T]$, $x \in D$ with any $\alpha \in]0, \theta/2[$.*

Proof. Let $T > 0$ and $s, t \in [0,T]$; then

$$\begin{aligned} &E\left|A^{1/2}W_A(t,x) - A^{1/2}W_A(s,x)\right|^2 \\ &\quad = \sum_{i=1}^{\infty} \lambda_i \int_s^t \left|A^{1/2}S(t-\tau)e_i(x)\right|^2 ds \\ &\quad\quad + \sum_{i=1}^{\infty} \lambda_i \int_0^s \left|A^{1/2}[S(t-\tau) - S(s-\tau)]e_i(x)\right|^2 d\tau \\ &\quad =: I_1(t,s,x) + I_2(t,s,x). \end{aligned} \tag{13}$$

Then, we have

$$\begin{aligned} &I_1(t,s,x) \\ &\quad \leq C\sum_{i=1}^{\infty} \lambda_i \alpha_i \int_s^t e^{-2(t-\tau)\alpha_i} d\tau \\ &\quad = C\sum_{i=1}^{\infty} \lambda_i \alpha_i \left(\frac{1 - e^{-2(t-s)\alpha_i}}{2\alpha_i}\right) \\ &\quad \leq C\sum_{i=1}^{\infty} \lambda_i (\alpha_i)^\theta |t-s|^\theta, \\ &I_2(t,s,x) \\ &\quad \leq \frac{1}{2} C\sum_{i=1}^{\infty} \lambda_i \alpha_i \int_0^s \left|\left[e^{-(t-\tau)\alpha_i} - e^{-(s-\tau)\alpha_i}\right]\right|^2 d\tau \\ &\quad = C\sum_{i=1}^{\infty} \lambda_i \alpha_i \frac{1}{2\alpha_i} \left[\left(e^{-(t-s)\alpha_i} - 1\right)^2 - \left(e^{-t\alpha_i} - e^{-s\alpha_i}\right)^2\right] \\ &\quad \leq C\sum_{i=1}^{\infty} \lambda_i (\alpha_i)^\theta |t-s|^\theta. \end{aligned} \tag{14}$$

So, by the estimate of I_1 and I_2, we arrive at

$$E\left|A^{1/2}W_A(t,x) - A^{1/2}W_A(s,x)\right|^2 \leq C\sum_{i=1}^{\infty} \lambda_i (\alpha_i)^\theta |t-s|^\theta. \tag{15}$$

For $t \in [0,T]$, $x, y \in D$, we get

$$\begin{aligned} &E\left|A^{1/2}W_A(t,x) - A^{1/2}W_A(t,y)\right|^2 \\ &\quad = \sum_{i=1}^{\infty} \lambda_i \alpha_i \int_0^t e^{-2\alpha_i(t-s)} \left|e_i(x) - e_i(y)\right|^2 ds \\ &\quad \leq \sum_{i=1}^{\infty} \lambda_i \left|e_i(x) - e_i(y)\right|^2 \\ &\quad \leq \sum_{i=1}^{\infty} \lambda_i (\alpha_i)^\theta |x-y|^\theta. \end{aligned} \tag{16}$$

Therefore,

$$\begin{aligned} &E\left|A^{1/2}W_A(t,x) - A^{1/2}W_A(s,y)\right|^2 \\ &\quad \leq C\left(|t-s|^\theta + |x-y|^\theta\right). \end{aligned} \tag{17}$$

As $A^{1/2}W_A(t,x) - A^{1/2}W_A(s,y)$ is a Gaussian random variable, we obtain

$$\begin{aligned} &E\left|A^{1/2}W_A(t,x) - A^{1/2}W_A(s,y)\right|^{2m} \\ &\quad \leq C\left(|t-s|^{m\theta} + |x-y|^{m\theta}\right), \end{aligned} \tag{18}$$

for $m = 1, 2, \ldots$ By Kolmogorov' test theorem, we get the conclusion. □

Remark 3. An example of the noise satisfying condition of Lemma 2 is

$$dW(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n d\beta_n(t), \tag{19}$$

where $\{\beta_n\}$ is a sequence of independent 1-dimensional Brownian motion, and $\{\lambda_n\}$ satisfies

$$\lambda_n = n^{-(1+2\theta)}, \quad \alpha_n = n \quad \forall n \in \mathbb{N}. \tag{20}$$

It is so because the eigenvalues α_n of the operator A, in 2-dimensional space, behave like n.

Remark 4. Another example of stochastic noise satisfying Lemma 2 is

$$A^{-\gamma} L dW(t), \tag{21}$$

where $W(t) = \sum_{n=1}^{\infty} e_n \beta_n(t)$, L is an isomorphism in H, and

$$\gamma \geq \frac{1}{2} + \theta. \tag{22}$$

To prove the local existence of the solution of (1) in sense of Definition 1, we introduce the space $\mathscr{B}_m$ defined by

$$\mathscr{B}_m = \left\{ v : v \in C\left(\left[0, T^*\right]; H_0^1\right), \|v\|_{H^1} \leq m, \forall t \in \left[0, T^*\right] \right\}, \tag{23}$$

where $T^* \geq 0$ which in fact is a stopping time and $m > 0$, $p > 0$.

Lemma 5. *For $u_0 = (u^1(0), u^2(0))$, $\|u_0\|_{H^1} < m$, and $u^i(0)$ is adapted to $\mathscr{F}_0$, $i = 1, 2$; then there exists a unique mild solution v in sense of Definition 1 to* (11) *in $\mathscr{B}_m$.*

Proof. Choose a v in $\mathscr{B}_m$, and set

$$\begin{aligned} \mathscr{L}(v) &:= e^{-tA} u_0 \\ &+ \int_0^t e^{-(t-s)A} \left[(v + W_A) \cdot \nabla \right] (v + W_A) \, ds \\ &- \int_0^t e^{-(t-s)A} f(v + W_A) \, ds. \end{aligned} \tag{24}$$

Then,

$$\begin{aligned} \|\mathscr{L}(v)\|_{H^1} \leq & \left\| e^{-tA} u_0 \right\|_{H^1} \\ &+ \left\| \int_0^t e^{-(t-s)A} \left[(v + W_A) \cdot \nabla \right] (v + W_A) \, ds \right\|_{H^1} \\ &+ \left\| \int_0^t e^{-(t-s)A} f(v + W_A) \, ds \right\|_{H^1}. \end{aligned} \tag{25}$$

For the second term on the right hand side of (25),

$$\begin{aligned} &\left\| e^{-(t-s)A} \left[(v + W_A) \cdot \nabla \right] (v + W_A) \right\|_{H^1} \\ &= \left\| e^{-(t-s)A} \left[u \cdot \nabla \right] u \right\|_{H^1} \\ &\leq \frac{1}{2} \left| e^{-(t-s)A} \partial_1 \left(u^1\right)^2 \right|_{H^1} \\ &\quad + \frac{1}{2} \left| e^{-(t-s)A} \partial_2 \left(u^2\right)^2 \right|_{H^1} \\ &\quad + \left| e^{-(t-s)A} u^2 \partial_2 u^1 \right|_{H^1} \\ &\quad + \left| e^{-(t-s)A} u^1 \partial_1 u^2 \right|_{H^1} \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{26}$$

In the following, we will estimate I_i, respectively, $i = 1, 2, 3, 4$. Since $\{e^{-tA}\}_{t \geq 0}$ is contraction on $L^p(D)$, $p \geq 1$, it is known that

$$\left| e^{-tA} z \right|_{W^{s_2, r}} \leq C_1 t^{(s_1 - s_2)/2} |z|_{W^{s_1, r}}, \tag{27}$$

for all $z \in W^{s_1, r}(D)$, $s_1, s_2 \in \mathbb{R}$, $s_1 \leq s_2$, $r \geq 1$, and C_1 only depends on s_1, s_2, and r. Before calculating each I_i, we outline the Sobolev embedding principle in fractional Sobolev spaces as follows:

$$W^{\eta_1, p_1}(D) \subset W^{\eta_2, q_1}(D), \tag{28}$$

when

$$\frac{1}{p_1} - \frac{1}{n}(\eta_1 - \eta_2) \leq \frac{1}{q_1} \leq \frac{1}{p_1}, \tag{29}$$

where n is the dimension of the spatial. Let $\eta_1 = 3/4$, $p_1 = 2$, $\eta_2 = 1/4$, $q_1 = 4$ satisfying (29) such that

$$W^{3/4, 2}(D) \subset W^{1/4, 4}(D). \tag{30}$$

For I_1, by (27) and Theorem A.8 in [25], we get

$$\begin{aligned} I_1 &\leq C_1 |t - s|^{-7/8} \left| \partial_1 \left(u^1\right)^2 \right|_{H^{-3/4}} \\ &= C_1 |t - s|^{-7/8} \left| A^{1/8} \left(u^1\right)^2 \right|_H \\ &= C_1 |t - s|^{-7/8} \left| 2u^1 A^{1/8} u^1 + R \right|_H, \end{aligned} \tag{31}$$

where

$$R = A^{1/8} \left(u^1\right)^2 - 2A^{1/8} u^1, \tag{32}$$

satisfying

$$|R|_H \leq \left| A^{1/16} u^1 \right|_{L^4}^2 \leq \left| u^1 \right|_{H^1}^2. \tag{33}$$

The last inequality follows by (30). For the other term added to R, we have

$$\left| 2u^1 A^{1/8} u^1 \right|_H \leq |u_1|_{L^4}^2 + \left| A^{1/8} u^1 \right|_{L^4}^2 \leq 2 \left| u^1 \right|_{H^1}^2. \tag{34}$$

So, by (31)–(34), we have

$$I_1 \leq 3C_1 |t - s|^{-7/8} \left| u^1 \right|_{H^1}^2. \tag{35}$$

Similarly, we get for I_2 that

$$I_2 \leq 3C_1 |t - s|^{-7/8} \left| u^2 \right|_{H^1}^2. \tag{36}$$

For I_3, by Theorem A.8 in [25], we get

$$\begin{aligned} I_3 &\leq \left| e^{-(t-s)A} u^2 A^{1/2} u^1 \right|_{H^1} \\ &= \left| e^{-(t-s)A} \left[A^{1/4} \left(u^2 A^{1/4} u^1 \right) - \left(A^{1/4} u^1 \right) \left(A^{1/4} u^2 \right) - R_1 \right] \right|_{H^1}, \end{aligned} \tag{37}$$

where

$$R_1 = A^{1/4}\left(u^2 A^{1/4} u^1\right) - \left[A^{1/4} u^2\right]\left[A^{1/4} u^1\right] - u^2 A^{1/2} u^1. \tag{38}$$

For R_1, we have

$$\begin{aligned}\left|e^{-(t-s)A} R_1\right|_{H^1} &\le C_1 |t-s|^{-1/2} \left|R_1\right|_H \\ &\le C_1 |t-s|^{-1/2} \left|A^{1/4} u^1\right|_{L^4} \cdot \left|A^{1/4} u^2\right|_{L^4} \\ &\le C_1 |t-s|^{-1/2} \left(\left|u^1\right|_{H^1}^2 + \left|u^2\right|_{H^1}^2\right).\end{aligned} \tag{39}$$

For the first term on the right hand side of (37), by (27), we have

$$\begin{aligned}&\left|e^{-(t-s)A} A^{1/4}\left(u^2 A^{1/4} u^1\right)\right|_{H^1} \\ &\quad= \left|e^{-(t-s)A} A^{3/4}(u^2 A^{1/4} u^1)\right|_H \\ &\quad\le C_1 |t-s|^{-3/4} \left|u^2 A^{1/4} u^1\right|_H \\ &\quad\le C_1 |t-s|^{-3/4} \left(\left|u^2\right|_{L^4}^2 + \left|A^{1/4} u^1\right|_{L^4}^2\right) \\ &\quad\le C_1 |t-s|^{-3/4} \left(\left|u^2\right|_{H^1}^2 + \left|u^1\right|_{H^1}^2\right).\end{aligned} \tag{40}$$

For the second term on the right hand side of (37), by (27), we obtain

$$\begin{aligned}&\left|e^{-(t-s)A} [A^{1/4} u^2 \cdot A^{1/4} u^1]\right|_{H^1} \\ &\quad\le C_1 |t-s|^{-1/2} \left|A^{1/4} u^2 \cdot A^{1/4} u^1\right|_H \\ &\quad\le C_1 |t-s|^{-1/2} \left(\left|u^1\right|_{H^1}^2 + \left|u^2\right|_{H^1}^2\right).\end{aligned} \tag{41}$$

From (37) to (41), we get for I_3 that

$$I_3 \le C\left(|t-s|^{-1/2} + |t-s|^{-3/4}\right)\left(\left|u^1\right|_{H^1}^2 + \left|u^2\right|_{H^1}^2\right). \tag{42}$$

Analogously, for I_4, we get

$$I_4 \le C\left(|t-s|^{-1/2} + |t-s|^{-3/4}\right)\left(\left|u^1\right|_{H^1}^2 + \left|u^2\right|_{H^1}^2\right). \tag{43}$$

By (26), (35), (36), (42), and (43), we have

$$\begin{aligned}&\left\|e^{-(t-s)A}\left[(v + W_A) \cdot \nabla\right](v + W_A)\right\|_{H^1} \\ &\quad\le C\left(|t-s|^{-1/2} + |t-s|^{-3/4} + |t-s|^{-7/8}\right) \\ &\qquad\times \left(\left|u^1\right|_{H^1}^2 + \left|u^2\right|_{H^1}^2\right).\end{aligned} \tag{44}$$

As $u = v + W_A$, by (44), for $t \le T^*$, we have

$$\begin{aligned}&\int_0^t ds \left\|e^{-(t-s)A}\left[(v + W_A) \cdot \nabla\right](v + W_A)\right\|_{H^1} \\ &\quad\le C\left(t^{1/8} + t^{1/4} + t^{1/2}\right)\left(\sup_{t\in[0,T^*]} \|v\|_{H^1}^2 + \sup_{t\in[0,T]} \left\|W_A\right\|_{H^1}^2\right).\end{aligned} \tag{45}$$

Since by Lemma 2,

$$\sup_{t\in[0,T]} \left\|W_A\right\|_{H^1}^2 < \infty. \tag{46}$$

For the last term on the right hand side of (25), we have

$$\begin{aligned}&\left\|e^{-(t-s)A} f\left(v + W_A\right)\right\|_{H^1} \\ &\quad\le C|t-s|^{-1/2}\left(\left\|v + W_A\right\|_{L^6}^3\right) \\ &\quad\le C|t-s|^{-1/2}\left(\left\|W_A\right\|_{H^1}^3 + \|v\|_{H^1}^3\right).\end{aligned} \tag{47}$$

Therefore,

$$\begin{aligned}&\left\|\int_0^t e^{-(t-s)A} f\left(v + W_A\right) ds\right\|_{H^1} \\ &\quad\le C\left(1 + m^3\right)\int_0^t |t-s|^{-1/2} ds \\ &\quad\le C\left(1 + m^3\right) T^{*\,1/2}.\end{aligned} \tag{48}$$

So by (25), (45), and (48), when T^* is small enough,

$$\|\mathscr{L}(v)\|_{H^1} \le m. \tag{49}$$

For each $v_1, v_2 \in \mathscr{B}_m$, set $u_i = v_i + W_A$, $i = 1, 2$. To simplify the notation in the following calculation, we denote $u_i = (u_i^1, u_i^2)$, $i = 1, 2$. Then,

$$\begin{aligned}&\mathscr{L}\left(v_1\right) - \mathscr{L}\left(v_2\right) \\ &\quad= \int_0^t e^{-(t-s)A}\left[\left(u_1 \cdot \nabla\right) u_1 - \left(u_2 \cdot \nabla\right) u_2\right] ds \\ &\qquad+ \int_0^t e^{-(t-s)A}\left[f\left(u_1\right) - f\left(u_2\right)\right] ds.\end{aligned} \tag{50}$$

So,

$$\begin{aligned}&\left\|\mathscr{L}\left(v_1\right) - \mathscr{L}\left(v_2\right)\right\|_{H^1} \\ &\quad\le \int_0^t \left\|e^{-(t-s)A}\left[\left(u_1 \cdot \nabla\right) u_1 - \left(u_2 \cdot \nabla\right) u_2\right]\right\|_{H^1} ds \\ &\qquad+ \int_0^t \left\|e^{-(t-s)A}\left[f\left(u_1\right) - f\left(u_2\right)\right]\right\|_{H^1} ds.\end{aligned} \tag{51}$$

In order to simplify the notation, we set

$$\left(u_1 \cdot \nabla\right) u_1 - \left(u_2 \cdot \nabla\right) u_2 = \left(f_1 + f_2, f_3 + f_4\right), \tag{52}$$

where

$$\begin{aligned}f_1 &= \frac{1}{2}\partial_1\left[\left(u_1^1\right)^2 - \left(u_2^1\right)^2\right], \\ f_2 &= u_1^2 \partial_2 u_1^1 - u_2^2 \partial_2 u_2^1, \\ f_3 &= \frac{1}{2}\partial_2\left[\left(u_1^2\right)^2 - \left(u_2^2\right)^2\right], \\ f_4 &= u_1^1 \partial_1 u_1^2 - u_2^1 \partial_1 u_2^2.\end{aligned} \tag{53}$$

Then, we estimate f_i, $i = 1,2,3,4$, respectively. For f_1, we have

$$\begin{aligned}&\left|e^{-(t-s)A}f_1\right|_{H^1}\\&\quad=\left|e^{-(t-s)A}A^{1/2}\left[\left(u_1^1-u_2^1\right)\left(u_1^1+u_2^1\right)\right]\right|_{H^1}\\&\quad\le C|t-s|^{-7/8}\left|A^{1/8}\left[\left(u_1^1\right)^2-\left(u_2^1\right)^2\right]\right|_{H}\\&\quad=C|t-s|^{-7/8}\left|A^{1/8}\left[\left(u_1^1-u_1^2\right)\left(u_1^1+u_1^2\right)\right]\right|_{H}\\&\quad=C|t-s|^{-7/8}\left|\left[A^{1/8}\left(u_1^1-u_1^2\right)\right]\left(u_1^1+u_1^2\right)\right|_{H}\\&\qquad+C|t-s|^{-7/8}\left|\left[A^{1/8}\left(u_1^1+u_1^2\right)\right]\left(u_1^1-u_1^2\right)+R_2\right|_{H}.\end{aligned}\tag{54}$$

We first consider

$$\begin{aligned}|R_2|_H&\le C\left|A^{1/16}\left(u_1^1-u_2^1\right)\right|_{L^4}\cdot\left|A^{1/16}\left(u_1^1+u_2^1\right)\right|_{L^4}\\&\le C\left|u_1^1-u_2^1\right|_{H^1}\cdot\left|u_1^1+u_2^1\right|_{H^1}.\end{aligned}\tag{55}$$

For the other term added to R_2,

$$\begin{aligned}&\left|\left[A^{1/8}\left(u_1^1+u_2^1\right)\right]\left(u_1^1-u_2^1\right)\right|_{H}\\&\qquad\le\left|u_1^1+u_2^1\right|_{H^1}\cdot\left|u_1^1-u_2^1\right|_{H^1}.\end{aligned}\tag{56}$$

By (54)–(56),

$$\left|e^{-(t-s)A}f_1\right|_{H^1}\le C|t-s|^{-7/8}\left|u_1^1-u_2^1\right|_{H^1}\cdot\left|u_1^1+u_2^1\right|_{H^1}.\tag{57}$$

Analogously, for f_3,

$$\left|e^{-(t-s)A}f_3\right|_{H^1}\le C|t-s|^{-7/8}\left|u_1^1-u_2^1\right|_{H^1}\cdot\left|u_1^1+u_2^1\right|_{H^1}.\tag{58}$$

For f_2, by (53), we have

$$\begin{aligned}\left|e^{-(t-s)A}f_2\right|_{H^1}&=\left|e^{-(t-s)A}\left(u_1^2\partial_2u_1^1-u_2^2\partial_2u_2^1\right)\right|_{H^1}\\&\le\left|e^{-(t-s)A}\left(u_1^2\left(\partial_2u_1^1-\partial_2u_2^1\right)\right)\right|_{H^1}\\&\quad+\left|e^{-(t-s)A}\left(\left(u_1^2-u_2^2\right)\partial_2u_2^1\right)\right|_{H^1}.\end{aligned}\tag{59}$$

For the first term on the right hand side of (59), we have

$$\begin{aligned}&\left|e^{-(t-s)A}\left(u_1^2\left(\partial_2u_1^1-\partial_2u_2^1\right)\right)\right|_{H^1}\\&\quad\le\left|e^{-(t-s)A}\left(u_1^2A^{1/2}\left(u_1^1-u_2^1\right)\right)\right|_{H^1}\\&\quad=\left|e^{-(t-s)A}\left\{A^{1/4}\left[u_1^2A^{1/4}\left(u_1^1-u_2^1\right)\right]\right.\right.\\&\qquad\left.\left.-\left[A^{1/4}u_1^2,A^{1/4}\left(u_1^1-u_2^1\right)\right]-R_3\right\}\right|_{H^1}.\end{aligned}\tag{60}$$

For R_3,

$$\begin{aligned}&\left|e^{-(t-s)A}R_3\right|_{H^1}\\&\quad\le|t-s|^{-1/2}|R_3|_H\\&\quad\le|t-s|^{-1/2}\left|A^{1/4}u_1^2\right|_{L^4}\cdot\left|A^{1/4}\left(u_1^1-u_2^1\right)\right|_{L^4}\\&\quad=|t-s|^{-1/2}\left|u_1^2\right|_{H^1}\cdot\left|u_1^1-u_2^1\right|_{H^1}.\end{aligned}\tag{61}$$

For the first term on the right hand side of (60), we arrive at

$$\begin{aligned}&\left|e^{-(t-s)A}A^{1/4}\left[u_1^2A^{1/4}\left(u_1^1-u_2^1\right)\right]\right|_{H^1}\\&\quad=\left|e^{-(t-s)A}A^{3/4}\left[u_1^2A^{1/4}\left(u_1^1-u_2^1\right)\right]\right|_{H}\\&\quad\le|t-s|^{-3/4}\left|u_1^2A^{1/4}\left(u_1^1-u_2^1\right)\right|_{H}\\&\quad\le|t-s|^{-3/4}\left|u_1^2\right|_{L^4}\cdot\left|A^{1/4}\left(u_1^1-u_2^1\right)\right|_{L^4}\\&\quad\le|t-s|^{-3/4}\left|u_1^2\right|_{H^1}\cdot\left|u_1^1-u_2^1\right|_{H^1}.\end{aligned}\tag{62}$$

For the second term on the right hand side of (60), we obtain

$$\begin{aligned}&\left|e^{-(t-s)A}\left[\left(A^{1/4}u_1^2\right)\left(A^{1/4}\left(u_1^1-u_2^1\right)\right)\right]\right|_{H^1}\\&\quad\le|t-s|^{-1/2}\left|\left[A^{1/4}u_1^2\right]\cdot\left[A^{1/4}\left(u_1^1-u_2^1\right)\right]\right|_{H}\\&\quad\le|t-s|^{-1/2}\left|A^{1/4}u_1^2\right|_{L^4}\cdot\left|A^{1/4}\left(u_1^1-u_2^1\right)\right|_{L^4}\\&\quad\le|t-s|^{-1/2}\left|u_1^2\right|_{H^1}\cdot\left|u_1^1-u_2^1\right|_{H^1}.\end{aligned}\tag{63}$$

By (59)–(63), we get for f_2 that

$$\begin{aligned}&\left|e^{(t-s)A}f_2\right|_{H^1}\\&\quad\le C\left(|t-s|^{-1/2}+|t-s|^{-3/4}\right)\\&\qquad\times\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)\|u_1-u_2\|_{H^1}.\end{aligned}\tag{64}$$

Similarly, we get for f_4 that

$$\begin{aligned}&\left|e^{(t-s)A}f_4\right|_{H^1}\\&\quad\le C\left(|t-s|^{-1/2}+|t-s|^{-3/4}\right)\\&\qquad\times\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)\|u_1-u_2\|_{H^1}.\end{aligned}\tag{65}$$

By (52), (53), (57), (58), (64), and (65), we have

$$\begin{aligned}&\left\|e^{-(t-s)A}\left[\left(u_1\cdot\nabla\right)u_1-\left(u_2\cdot\nabla\right)u_2\right]\right\|_{H^1}\\&\quad\le\sum_{i=1}^{4}\left|e^{-(t-s)A}f_i\right|_{H^1}\\&\quad\le C\left(|t-s|^{-1/2}+|t-s|^{-3/4}+|t-s|^{-7/8}\right)\\&\qquad\times\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)\|u_1-u_2\|_{H^1}\\&\quad\le C\left(2m+1\right)\left(|t-s|^{-1/2}+|t-s|^{-3/4}+|t-s|^{-7/8}\right)\\&\qquad\times\|v_1-v_2\|_{H^1}.\end{aligned}\tag{66}$$

For the second term on the right hand side of (51), we have

$$f\left(u_1\right)-f\left(u_2\right)=\left(h_1,h_2\right),\tag{67}$$

where

$$\begin{aligned}h_1&=\left|u_1\right|^2u_1^1-\left|u_2\right|^2u_2^1,\\h_2&=\left|u_1\right|^2u_1^2-\left|u_2\right|^2u_2^2.\end{aligned}\tag{68}$$

Then,

$$\begin{aligned}
&\left|e^{(t-s)A}h_1\right|_{H^1}\\
&\quad\le C|t-s|^{-1/2}\left||u_1|^2u_1^1-|u_2|^2u_2^1\right|_H\\
&\quad\le C|t-s|^{-1/2}\left||u_1|^2u_1^1-|u_2|^2u_1^1\right|_H\\
&\qquad+C|t-s|^{-1/2}\left||u_2|^2u_1^1-|u_2|^2u_2^1\right|_H\\
&\quad\le C|t-s|^{-1/2}\left||v_1-v_2|\cdot(|v_1|+|v_2|+2|W_A|)\cdot\left|u_1^1\right|\right|_H\\
&\qquad+C|t-s|^{-1/2}\left||u_2|^2\cdot\left|v_1^1-v_2^1\right|\right|_H\\
&\quad\le C|t-s|^{-1/2}\|v_1-v_2\|_{L^4}\cdot\left|(|v_1|+|v_2|+2|W_A|)\cdot\left|u_1^1\right|\right|_{L^4}\\
&\qquad+C|t-s|^{-1/2}\|v_1-v_2\|_{L^4}\cdot\|u_2\|_{L^4}^2\\
&\quad\le C|t-s|^{-1/2}\|v_1-v_2\|_{L^4}\left(\|v_1\|_{L^8}^2+\|v_2\|_{L^8}^2+1\right)\\
&\quad\le C|t-s|^{-1/2}\|v_1-v_2\|_{H^1}\left(\|v_1\|_{H^1}^2+\|v_2\|_{H^1}^2+1\right).
\end{aligned}\tag{69}$$

Similarly, we can get the same estimate for h_2. So, we have

$$\begin{aligned}
&\int_0^t\left\|e^{(t-s)A}\left[f(u_1)-f(u_2)\right]\right\|_{H^1}ds\\
&\quad\le C\left(1+m^2\right)T^{*1/2}\sup_{t\in[0,T^*]}\|v_1(t)-v_2(t)\|_{H^1}.
\end{aligned}\tag{70}$$

By (51), (66), and (70), we have

$$\begin{aligned}
&\|\mathscr{L}(v_1)-\mathscr{L}(v_2)\|_{H^1}\\
&\quad\le C\left[T^{*1/2}+T^{*1/4}+T^{*1/8}\right]\cdot\left(\sup_{t\in[0,T^*]}\|v_1-v_2\|_{H^1}\right).
\end{aligned}\tag{71}$$

By (49), (71), and fixed point principle, we get the conclusion. □

Remark 6. By making some minor modifications in the proof of Lemma 5, we can see that the conclusion in Lemma 5 is also true for (1). Our original aim is to get the global well-posedness of (1), but we find that the dissipative term Δu cannot dominate the nonlinear term $(u\cdot\nabla)u$. So, we introduce the dissipative term $|u|^2u$ which will also play an important role in obtaining the ergodicity.

4. Global Existence

Theorem 7. *With conditions in Lemma 2, for $v\in C([0,T];H_0^1)$ satisfying* (12)*, when $\vartheta>1/16$, one has*

$$\|v\|_{H^1}\le\left(C_T+\|v_0\|_{H^1}^2\right)e^{C_T}.\tag{72}$$

Subsequently, one gets the existence of the global solution belonging to $C([0,T];H_0^1)$.

Proof. Let $\{u_n^0\}_{n\ge1}$ be a sequence of vectors which satisfies $u_n^0=(u_n^{0,1},u_n^{0,2})$ and $u_n^{0,i}\in C_0^\infty(D)$, $i=1,2$, $n\ge1$, such that

$$u_n^0\longrightarrow u_0,\quad\text{as } n\longrightarrow\infty,\tag{73}$$

in sense of $\|\cdot\|_{H^1}$. Let $\{W_n\}_{n\ge1}$ be a sequence of regular process, such that

$$\begin{aligned}
A^{a/2}W_A^n:=A^{a/2}\int_0^te^{(t-s)A}dW_n(s)\longrightarrow A^{a/2}W_A(t),&\\
\text{as } n\to\infty,&
\end{aligned}\tag{74}$$

in $C(T\times D)$ when $a=0$ or $a=1$. For $h=(h_1,h_2)$, $h_i\in C([0,T]\times D;\mathbb{R})$, $\|h\|_{C(T\times D)}:=\sum_{i=1}^2|h_i|_{C(T\times D)}$, where $|h_i|_{C(T\times D)}=\sup_{(t,x)\in[0,T]\times D}|h_i|$. Then, by (74), we have

$$\sup_{\{n\ge1\}}\|W_A^n\|_{C(T\times D)}<\infty,\tag{75}$$

$$\sup_{\{n\ge1\}}\sup_{t\in[0,T]}\left|A^{1/2}W_A^n\right|<\infty.\tag{76}$$

If v_n satisfies

$$\begin{aligned}
v_n=e^{tA}u_n^0&+\int_0^te^{(t-s)A}\left[(v_n+W_A)\cdot\nabla\right](v_n+W_A)\,ds\\
&-\int_0^te^{(t-s)A}f(v_n+W_A),
\end{aligned}\tag{77}$$

then, v_n is regular, such that

$$\frac{\partial v_n}{\partial t}+Av_n+B\left(v_n+W_A^n,v_n+W_A^n\right)+f\left(v_n+W_A^n\right)=0.\tag{78}$$

Taking inner product with respect to v_n in (78), we have

$$\begin{aligned}
&\left\langle\frac{\partial v_n}{\partial t},v_n\right\rangle+\langle Av_n,v_n\rangle\\
&\quad+\left\langle B\left(v_n+W_A^n,v_n+W_A^n\right),v_n\right\rangle\\
&\quad+\left\langle f\left(v_n+W_A^n\right),v_n\right\rangle=0.
\end{aligned}\tag{79}$$

For simplicity, we calculate the third term on the left hand side of (79) first as follows:

$$\begin{aligned}
&\left\langle B\left(v_n+W_A^n,v_n+W_A^n\right),v_n\right\rangle\\
&\quad=\left\langle\left(v_n^1+W_{A,1}^n\right)\partial_1\left(v_n^1+W_{A,1}^n\right),v_n^1\right\rangle\\
&\qquad+\left\langle\left(v_n^2+W_{A,2}^n\right)\partial_2\left(v_n^1+W_{A,1}^n\right),v_n^1\right\rangle\\
&\qquad+\left\langle\left(v_n^1+W_{A,1}^n\right)\partial_1\left(v_n^2+W_{A,2}^n\right),v_2^n\right\rangle\\
&\qquad+\left\langle\left(v_n^2+W_{A,2}^n\right)\partial_2\left(v_n^2+W_{A,2}^n\right),v_n^2\right\rangle\\
&\quad=I_1+I_2+I_3+I_4,
\end{aligned}\tag{80}$$

where $W_A^n=(W_{A,1}^n,W_{A,2}^n)$. For I_1, we have

$$\begin{aligned} I_1 &= \left\langle \left(v_n^1+W_{A,1}^n\right)\partial_1\left(v_n^1+W_{A,1}^n\right), v_n^1\right\rangle \\ &= \left\langle v_n^1\partial_1 v_n^1, v_n^1\right\rangle + \left\langle W_{A,1}^n\partial_1 v_n^1, v_n^1\right\rangle \\ &\quad + \left\langle v_n^1\partial_1 W_{A,1}^n, v_n^1\right\rangle + \left\langle W_{A,1}^n\partial_1 W_{A,1}^n, v_n^1\right\rangle . \end{aligned} \tag{81}$$

In the following, we estimate the four terms for I_1, respectively. For the first term,

$$\begin{aligned} \left\langle v_n^1\partial_1 v_n^1, v_n^1\right\rangle &= \int_D \left(v_n^1\right)^2\partial_1 v_n^1 dx \\ &= \int_D \partial_1\left[\frac{\left(v_n^1\right)^3}{3}\right]dx = 0. \end{aligned} \tag{82}$$

For the second term, by (75), we have

$$\begin{aligned} &\langle W_{A,1}^n\partial_1 v_n^1, v_n^1\rangle \\ &\quad \le C\left|v_n^1\right|_H^2 + \varepsilon\int_D\left(\partial_1 v_n^1\right)^2 dx \\ &\quad \le C\left|v_n^1\right|_H^2 + \varepsilon\left|v_n^1\right|_{H^1}^2. \end{aligned} \tag{83}$$

similarly, for the third term,

$$\begin{aligned} &\left|\left\langle v_n^1\partial_1 W_{A,1}^n, v_n^1\right\rangle\right| \\ &\quad = \left|\int_D \left(v_n^1\right)^2\partial_1 W_{A,1}^n dx\right| \\ &\quad = \left|\int_D W_{A,1}^n\partial_1\left(v_n^1\right)^2 dx\right| \\ &\quad \le C\left|\int_D v_n^1\partial_1 v_n^1 dx\right| \\ &\quad \le C\left|v_n^1\right|_H^2 + \varepsilon\left|v_n^1\right|_{H^1}^2. \end{aligned} \tag{84}$$

For the last term, by (75) and (76),

$$\begin{aligned} &\left|\left\langle W_{A,1}^n\partial_1 W_{A,1}^n, v_n^1\right\rangle\right| \\ &\quad \le C\left|\int_D \partial_1 v_n^1 dx\right| \\ &\quad \le C + \varepsilon\left|v_n^1\right|_{H^1}^2. \end{aligned} \tag{85}$$

By (81)–(85), it follows that

$$I_1 \le C\left(1+\|v_n\|_H^2\right) + 4\varepsilon\|v_n\|_{H^1}^2. \tag{86}$$

Similarly,

$$I_4 \le C\left(1+\|v_n\|_H^2\right) + 4\varepsilon\|v_n\|_{H^1}^2. \tag{87}$$

For I_3,

$$\begin{aligned} I_3 &= \langle v_n^1\partial_1 v_n^2, v_n^2\rangle + \langle v_n^1\partial_1 W_{A,2}^n, v_n^2\rangle \\ &\quad + \langle W_{A,1}^n\partial_1 v_n^2, v_n^2\rangle + \langle W_{A,1}^n\partial_1 W_{A,2}^n, v_n^2\rangle. \end{aligned} \tag{88}$$

For the first term on the right hand side of (88), we deduce that

$$\begin{aligned} \left|\left\langle v_n^1\partial_1 v_n^2, v_n^2\right\rangle\right| &= \frac{1}{2}\left|\int_D v_n^1\partial_1\left(v_n^2\right)^2 dx\right| \\ &= \frac{1}{2}\left|\int_D \partial_1 v_n^1\cdot\left(v_n^2\right)^2 dx\right| \\ &\le \frac{1}{2}\left|v_n^2\right|_{L^4}^2\cdot\left|v_n^1\right|_{H^1} \\ &\le \frac{1}{4}\epsilon\left|v_n^2\right|_{L^4}^4 + \frac{1}{4\epsilon}\left|v_n^1\right|_{H^1}^2, \end{aligned} \tag{89}$$

where $\epsilon>0$. For the second term on the right hand side of (88), we have

$$\left|\left\langle v_n^1\partial_1 W_{A,2}^n, v_n^2\right\rangle\right| \le \varepsilon\|v^n\|_{H^1}^2 + C\|v^n\|_H^2. \tag{90}$$

Analogously, for the third term on the right hand side of (88), we see that

$$\left|\left\langle W_{A,1}^n\partial_1 v_n^2, v_n^2\right\rangle\right| \le C\|v^n\|_H^2 + \varepsilon\|v^n\|_{H^1}^2. \tag{91}$$

For the last term, by (75) and (76), we have

$$\left|\left\langle W_{A,1}^n\partial_x W_{A,2}^n, v_n^2\right\rangle\right| \le C + \varepsilon\|v^n\|_{H^1}^2. \tag{92}$$

By (88)–(92), we get

$$\begin{aligned} I_3 &\le \frac{1}{4\epsilon}\left\|v_n^2\right\|_{L^4}^4 + \frac{\epsilon}{4}\left\|v_n^1\right\|_{H^1}^2 \\ &\quad + 3\varepsilon\|v_n\|_{H^1}^2 + C\|v_n\|_H^2 + C. \end{aligned} \tag{93}$$

Analogously, for I_2, it follows that

$$\begin{aligned} I_2 &\le \frac{1}{4\epsilon}\left\|v_n^1\right\|_{L^4}^4 + \frac{\epsilon}{4}\left\|v_n^2\right\|_{H^1}^2 \\ &\quad + 3\varepsilon\|v_n\|_{H^1}^2 + C\|v_n\|_H^2 + C. \end{aligned} \tag{94}$$

By (80) and the estimates of I_1, I_2, I_3, and I_4, see (86), (87), (93), and (94), we have

$$\begin{aligned} &\langle B\left(v_n+W_A^n, v_n+W_A^n\right), v_n\rangle \\ &\quad \le C\left(1+\|v_n\|_H^2\right) + \left(\frac{\epsilon}{4}+14\varepsilon\right)\|v_n\|_{H^1}^2 \\ &\qquad + \frac{1}{4\epsilon}\|v_n\|_{L^4}^4. \end{aligned} \tag{95}$$

For the last term on the left hand side of (79), we have

$$\begin{aligned}
&\langle f(v_v + W_A^n), v_n\rangle \\
&\quad = \vartheta\|v_n\|_{L^4}^4 + 3\vartheta\int_D |v_n|^2 \left(v_n^1 W_{A,1}^n + v_n^2 W_{A,2}^n\right) dx \\
&\qquad + \vartheta\int_D |v_n|^2 |W_A^n|^2 dx \\
&\qquad + \vartheta\int_D \left(v_n^1 W_{A,1}^n + v_n^2 W_{A,2}^n\right) |W_A^n|^2 dx \\
&\qquad + 2\vartheta\int_D \left(\left|W_{A,1}^n\right|^2 \left|v_n^1\right|^2 + \left|W_{A,2}^n\right|^2 \left|v_n^2\right|^2\right) dx \\
&\qquad + 4\vartheta\int_D W_{A,1}^n W_{A,2}^n v_n^1 v_n^2 dx \\
&\quad \le (\vartheta + \varepsilon)\|v_n\|_{L^4}^4 + C\left(1 + \|v_n\|_H^2\right).
\end{aligned} \tag{96}$$

By (79), (95), and (96), we get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|v_n\|_H^2 + \|v_n\|_{H^1}^2 + \vartheta\|v_n\|_{L^4}^4 \\
&\qquad \le C\left(1 + \|v_n\|_H^2\right) + \left(\frac{\epsilon}{4} + 14\varepsilon\right)\|v_n\|_{H^1}^2 \\
&\qquad\quad + \left(\frac{1}{4\epsilon} + \varepsilon\right)\|v_n\|_{L^4}^4.
\end{aligned} \tag{97}$$

Rearranging the above inequality, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|v_n\|_H^2 + \left(1 - \frac{\epsilon}{4} - 14\varepsilon\right)\|v_n\|_{H^1}^2 \\
&\quad + \left(\vartheta - \frac{1}{4\epsilon} - \varepsilon\right)\|v_n\|_{L^4}^4 \le C\left(1 + \|v_n\|_H^2\right).
\end{aligned} \tag{98}$$

Let $\epsilon \in (1/4\vartheta, 4)$, and ε be small enough, such that

$$1 - \frac{\epsilon}{4} - 14\varepsilon > 0, \qquad \vartheta - \frac{1}{4\epsilon} - \varepsilon > 0. \tag{99}$$

So, we integrate with respect to t on both sides of (98) to obtain

$$\begin{aligned}
&\|v_n(t)\|_H^2 + C_\epsilon\int_0^t \|v_n(s)\|_{H^1}^2 ds \\
&\qquad \le \|v_n(0)\|_H^2 + Ct + C\int_0^t \|v_n(s)\|_H^2 ds,
\end{aligned} \tag{100}$$

where $C_\epsilon = 2(1 - \epsilon/4 - 14\varepsilon)$, by Gronwall's inequality, we arrive at

$$\|v_n(t)\|_H^2 \le \left(\|v_n(0)\|_H^2 + Ct\right)e^{Ct} \le C_T. \tag{101}$$

By (100) and (101), we have

$$\int_0^t \|v_n(s)\|_{H^1}^2 ds \le C_T. \tag{102}$$

Multiplying Av_n on both sides of (78), and integrating with respect to $x \in D$, we have

$$\begin{aligned}
&\left\langle \frac{\partial v_n}{\partial t}, Av_n\right\rangle + \langle Av_n, Av_n\rangle + \langle f(v_n + W_A^n), Av_n\rangle \\
&\qquad = \langle B(v_n + W_A^n, v_n + W_A^n), Av_n\rangle,
\end{aligned} \tag{103}$$

which is equivalent to

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|v_n\|_{H^1}^2 + \|v_n\|_{H^2}^2 \\
&\quad = -\langle f(v_n + W_A^n), Av_n\rangle \\
&\qquad + \langle B(v_n + W_A^n, v_n + W_A^n), Av_n\rangle.
\end{aligned} \tag{104}$$

We first estimate the second term on the right hand side of (104) as follows:

$$\begin{aligned}
&\langle B(v_n + W_A^n, v_n + W_A^n), Av_n\rangle \\
&\quad = \left\langle v_n^1 + W_{A,1}^n\partial_1\left(v_n^1 + W_{A,1}^n\right), Av_n^1\right\rangle \\
&\qquad + \left\langle v_n^2 + W_{A,2}^n\partial_2\left(v_n^1 + W_{A,1}^n\right), Av_n^1\right\rangle \\
&\qquad + \left\langle v_n^1 + W_{A,1}^n\partial_1\left(v_n^2 + W_{A,2}^n\right), Av_n^2\right\rangle \\
&\qquad + \left\langle v_n^2 + W_{A,2}^n\partial_2\left(v_n^2 + W_{A,2}^n\right), Av_n^2\right\rangle \\
&\quad = J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{105}$$

For J_1, we have

$$\begin{aligned}
J_1 &= \left\langle v_n^1\partial_1 v_n^1, Av_n^1\right\rangle + \left\langle v_n^1\partial_1 W_{A,1}^n, Av_n^1\right\rangle \\
&\quad + \left\langle W_{A,1}^n\partial_1 v_n^1, Av_n^1\right\rangle + \left\langle W_{A,1}^n\partial_1 W_{A,1}^n, Av_n^1\right\rangle \\
&= k_1 + k_2 + k_3 + k_4.
\end{aligned} \tag{106}$$

For k_1, we have

$$k_1 \le \varepsilon\left|v_n^1\right|_{H^2}^2 + C\left|v_n^1\right|_{L^4}^2 \cdot \left|v_n^1\right|_{W^{1,4}}^2. \tag{107}$$

By interpolation inequality, there exists some $C > 0$, such that

$$\begin{aligned}
\left|v_n^1\right|_{L^4} &\le C\left|v_n^1\right|_H^{1/2}\left|v_n^1\right|_{H^1}^{1/2}, \\
\left|v_n^1\right|_{W^{1,4}} &\le C\left|v_n^1\right|_H^{1/4}\left|v_n^1\right|_{H^2}^{3/4}.
\end{aligned} \tag{108}$$

Then,

$$\begin{aligned}
k_1 &\le \varepsilon\left|v_n^1\right|_{H^2}^2 + C\left|v_n^1\right|_H^{3/2} \cdot \left|v_n^1\right|_{H^1} \cdot \left|v_n^1\right|_{H^2}^{3/2} \\
&\le \varepsilon\left|v_n^1\right|_{H^2}^2 + \varepsilon\left|v_n^1\right|_{H^2}^2 + C\left|v_n^1\right|_H^6\left|v_n^1\right|_{H^1}^4 \\
&\le 2\varepsilon\left|v_n^1\right|_{H^2}^2 + C_T\left|v_n^1\right|_{H^1}^4,
\end{aligned} \tag{109}$$

where the last inequality follows from (101). For k_2, we deduce that

$$\begin{aligned} k_2 &\le \varepsilon\left|v_n^1\right|_{H^2}^2 + C\int_D \left(v_n^1\right)^2\left(\partial_1 W_{A,1}^n\right)^2 dx \\ &\le \varepsilon\left|v_n^1\right|_{H^2}^2 + C\left|v_n^1\right|_{H}^2 \\ &\le \varepsilon\left|v_n^1\right|_{H^2}^2 + C_T. \end{aligned} \tag{110}$$

For k_3, we arrive at

$$k_3 \le C\int_D \left|\partial_1 v_n^1 \cdot A v_n^1\right| dx \le \varepsilon\left|v_n^1\right|_{H^2}^2 + C\left|v_n^1\right|_{H^1}^2. \tag{111}$$

For k_4, we obtain

$$k_4 \le C + \varepsilon\left|v_n^1\right|_{H^2}^2. \tag{112}$$

By (106) and (109)–(112),

$$J_1 \le 5\varepsilon\left|v_n^1\right|_{H^2}^2 + C_T\left|v_n^1\right|_{H^1}^4 + C\left|v_n^1\right|_{H^1}^2 + C_T. \tag{113}$$

Similarly, for J_4, we infer that

$$J_4 \le 5\varepsilon\left|v_n^2\right|_{H^2}^2 + C_T\left|v_n^2\right|_{H^1}^4 + C\left|v_n^2\right|_{H^1}^2 + C_T. \tag{114}$$

For J_2, we have

$$\begin{aligned} J_2 &= \left\langle v_n^2\partial_2 v_n^1, A v_n^1\right\rangle + \left\langle W_{A,2}^n\partial_2 v_n^1, A v_n^1\right\rangle \\ &\quad + \langle v_n^2\partial_2 W_{A,1}^n, A v_n^1\rangle + \langle W_{A,2}^n\partial_2 W_{A,1}^n, A v_n^1\rangle \\ &= l_1 + l_2 + l_3 + l_4. \end{aligned} \tag{115}$$

By interpolation inequality and (101), we deduce that

$$\begin{aligned} l_1 &\le \varepsilon\left|v_n^1\right|_{H^2}^2 + C\left|v_n^2\right|_{L^4}^2 \cdot \left|v_n^1\right|_{W^{1,4}}^2 \\ &\le \varepsilon\left|v_n^1\right|_{H^2}^2 + C\left|v_n^2\right|_{H} \cdot \left|v_n^2\right|_{H^1} \cdot \left|v_n^1\right|_{H}^{1/2} \cdot \left|v_n^1\right|_{H^2}^{3/2} \\ &\le 2\varepsilon\left|v_n^1\right|_{H^2}^2 + C_T\left|v_n^2\right|_{H^1}^4. \end{aligned} \tag{116}$$

For l_2, we have

$$l_2 \le C\int_D \left|\partial_2 v_n^1\right| \cdot \left|A v_n^1\right| dx \le \varepsilon\left|v_n^1\right|_{H^2}^2 + C\left|v_n^1\right|_{H^1}^2. \tag{117}$$

Similarly, for l_3,

$$l_3 \le C\int_D \left|v_n^2\right| \cdot \left|A v_n^1\right| dx \le \varepsilon\left|v_n^1\right|_{H^2}^2 + C_T. \tag{118}$$

As for l_4, we get

$$l_4 \le \varepsilon\left|v_n^1\right|_{H^2}^2 + C_T. \tag{119}$$

By (115)-(119), we arrive at

$$J_2 \le 5\varepsilon\left\|v_n^1\right\|_{H^2}^2 + C_T\left\|v_n^2\right\|_{H^1}^4 + C\left\|v_n^1\right\|_{H^1}^2 + C_T. \tag{120}$$

Analogously to J_2, we have

$$J_3 \le 5\varepsilon\left\|v_n^2\right\|_{H^2}^2 + C_T\left\|v_n^1\right\|_{H^1}^4 + C\left\|v_n^2\right\|_{H^1}^2 + C_T. \tag{121}$$

By (105) and the estimates of $J_1 - J_4$, see (113), (114), (120), and (121), we get that

$$\begin{aligned} &\left\langle B\left(v_n + W_A^n, v_n + W_A^n\right), A v_n\right\rangle \\ &\quad\le 10\varepsilon\left\|v_n\right\|_{H^2}^2 + C_T\left\|v_n\right\|_{H^1}^4 \\ &\qquad + C\left\|v_n\right\|_{H^1}^2 + C_T. \end{aligned} \tag{122}$$

For the first term on the right hand side of (104), we have

$$\begin{aligned} &\left|\left\langle f\left(v_n + W_A^n\right), A v_n\right\rangle\right| \\ &\quad\le \varepsilon\left\|v_n\right\|_{H^2}^2 + C\left\|v_n + W_A^n\right\|_{L^6}^6 \\ &\quad\le \varepsilon\left\|v_n\right\|_{H^2}^2 + C\left\|v_n\right\|_{L^6}^6 + C_T \\ &\quad\le \varepsilon\left\|v_n\right\|_{H^2}^2 + C_T\left\|v_n\right\|_{H^1}^2\left\|v_n\right\|_{H}^4 + C_T \\ &\quad\le \varepsilon\left\|v_n\right\|_{H^2}^2 + C_T\left(1 + \left\|v_n\right\|_{H^1}^2\right). \end{aligned} \tag{123}$$

By (104), (122), and (123),

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left\|v_n\right\|_{H^1}^2 + \left\|v_n\right\|_{H^2}^2 \\ &\quad\le 11\varepsilon\left\|v_n\right\|_{H^2}^2 + C_T\left(1 + \left\|v_n\right\|_{H^1}^2\right)\left\|v_n\right\|_{H^1}^2 + C_T. \end{aligned} \tag{124}$$

By the Gronwall inequality, we get

$$\begin{aligned} &\left\|v_n(t)\right\|_{H^1}^2 \\ &\quad\le \left(\left\|v_n(0)\right\|_{H^1}^2 + C_T\right)e^{C_T\int_0^t(1+\|v_n(s)\|_{H^1}^2 ds)} \\ &\quad\le \left(\left\|v_n(0)\right\|_{H^1}^2 + C_T\right)e^{C_T}. \end{aligned} \tag{125}$$

Let $n \to \infty$, by Fatou Lemma,

$$\|v(t)\|_{H^1}^2 \le \left(\|v(0)\|_{H^1}^2 + C_T\right)e^{C_T}. \tag{126}$$

□

5. Invariant Measures

5.1. Existence. In this section, we will establish the existence of invariant measure for (2). Analogously to [24], we extend the Wiener process $W(t)$ to $\mathbb{R}$ by setting

$$W(t) := W^1(t), \quad t \le 0, \tag{127}$$

where $W^1(t)$ is another H-valued Wiener process satisfying conditions in Lemma 2 and being independent of $W(t)$. For any $\tau \ge 0$, we consider the following equation:

$$\begin{aligned} &du_\tau + \left[A u_\tau + B\left(u_\tau, u_\tau\right) + f\left(u_\tau\right)\right]dt = dW, \\ &\qquad \text{on } [0,T]\times D, \ u_\tau(-\tau) = 0. \end{aligned} \tag{128}$$

By Theorem 7, we know that there exists unique solution. In order to obtain the invariant measure, we should show that the family of laws $\{\mathscr{L}(u_\tau(0))\}_{\tau\ge 0}$ is tight. Since $H^{1+\delta} \subset H^1$

is compact, for any $\delta > 0$, we only need to show that $\{\mathscr{L}(u_\tau(0))\}_{\tau\geq 0}$ is bounded in probability in $H^{1+\delta}$. As we know,

$$W_A(t) = \int_{-\infty}^{t} e^{-(t-s)A} dW(s), \quad t \in \mathbb{R} \tag{129}$$

is the mild solution of (8) with the following initial condition:

$$W_A(0) = \int_{-\infty}^{0} e^{sA} dW(s). \tag{130}$$

Making the classical change of variable $v_\tau(t) = u_\tau(t) - W_A(t)$, (128) is equivalent to

$$\begin{aligned} \frac{dv_\tau(t)}{dt} &= Av_\tau(t) + B(v_\tau(t) + W_A(t), v_\tau(t) + W_A(t)) \\ &\quad + f(v_\tau(t) + W_A(t)), \end{aligned} \tag{131}$$

with initial condition

$$v_\tau(-\tau) = -W_A(-\tau). \tag{132}$$

In order to get the invariant measure of (131), it is enough to show that $v_\tau(0)$ is bounded in probability in $H^{1+\delta}$, for some $\delta > 0$. That is what we have to do in Theorem 8 below.

Theorem 8. *With conditions in Lemma 2, when $\vartheta > 1/4$, there exists an invariant measure for* (2).

Proof. Multiplying (131) by v_τ and integrating on D, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_\tau(t)\|_H^2 + \|v_\tau(t)\|_{H^1}^2 \\ &\quad + \langle f(v_\tau(t) + W_A(t)), v_\tau(t)\rangle \\ &\quad = \langle B(v_\tau(t) + W_A(t), v_\tau(t) \\ &\qquad + W_A(t)), v_\tau(t)\rangle. \end{aligned} \tag{133}$$

For the third term on the left hand side of (133), we deduce that

$$\begin{aligned} &\langle f(v_\tau(t) + W_A(t)), v_\tau(t)\rangle \\ &\quad = \vartheta \left\langle |v_\tau(t) + W_A(t)|^2 (v_\tau(t) + W_A(t)), v_\tau(t) + W_A(t)\right\rangle \\ &\qquad - \vartheta \left\langle |v_\tau(t) + W_A(t)|^2 (v_\tau(t) + W_A(t)), W_A(t)\right\rangle \\ &\quad = \vartheta\|v_\tau(t) + W_A(t)\|_{L^4}^4 \\ &\qquad - \vartheta \left\langle \|v_\tau(t) + W_A(t)\|^2 (v_\tau(t) + W_A(t)), W_A(t)\right\rangle \\ &\quad \geq \vartheta[\|v_\tau(t)\|_{L^4} - \|W_A(t)\|_{L^4}]^4 \\ &\qquad - \vartheta \left\langle \|v_\tau(t) + W_A(t)\|^2 (v_\tau(t) + W_A(t)), W_A(t)\right\rangle \\ &\quad \geq \vartheta\|v_\tau(t)\|_{L^4}^4 - 4\vartheta\|v_\tau(t)\|_{L^4}^3\|W_A(t)\|_{L^4} \\ &\qquad - 4\vartheta\|v_\tau(t)\|_{L^4}^1\|W_A(t)\|_{L^4}^3 \\ &\qquad - \vartheta \left\langle \|v_\tau(t) + W_A(t)\|^2 (v_\tau(t) + W_A(t)), W_A(t)\right\rangle. \end{aligned} \tag{134}$$

Substituting (134) into (133), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_\tau(t)\|_H^2 + \|v_\tau(t)\|_{H^1}^2 + \vartheta\|v_\tau(t)\|_{L^4}^4 \\ &\quad \leq 4\vartheta\|v_\tau(t)\|_{L^4}^3\|W_A(t)\|_{L^4} \\ &\qquad + 4\vartheta\|v_\tau(t)\|_{L^4}\|W_A(t)\|_{L^4}^3 \\ &\qquad + \vartheta\left\langle \|v_\tau(t) + W_A(t)\|^2 (v_\tau(t) + W_A(t)), W_A(t)\right\rangle \\ &\qquad + \langle B(v_\tau(t) + W_A(t), v_\tau(t) + W_A(t)), v_\tau(t)\rangle. \end{aligned} \tag{135}$$

For the third term on the right hand side of (135), we get by the Young inequality that

$$\begin{aligned} &\vartheta\left\langle \|v_\tau(t) + W_A(t)\|^2 (v_\tau(t) + W_A(t)), W_A(t)\right\rangle \\ &\quad \leq \varepsilon\|v_\tau(t)\|_{L^4}^4 + C\|W_A(t)\|_{L^4}^4. \end{aligned} \tag{136}$$

For the last term on the right hand side of (135),

$$\begin{aligned} &\langle B(v_\tau(t) + W_A(t), v_\tau(t) + W_A(t)), v_\tau(t)\rangle \\ &\quad = \langle (v_\tau(t)\cdot\nabla) v_\tau(t), v_\tau(t)\rangle \\ &\qquad + \langle (W_A(t)\cdot\nabla) v_\tau(t), v_\tau(t)\rangle \\ &\qquad + \langle (v_\tau(t)\cdot\nabla) W_A(t), v_\tau(t)\rangle \\ &\qquad + \langle (W_A(t)\cdot\nabla) W_A(t), v_\tau(t)\rangle \\ &\quad = r_1 + r_2 + r_3 + r_4. \end{aligned} \tag{137}$$

Since $v_\tau(t)$ is vector field, we denote it by $v_\tau(t) = (v_\tau^1(t), v_\tau^2(t))$, where $v_\tau^i(t)$ is real valued function, $i = 1, 2$. For r_1, we have

$$\begin{aligned} r_1 &= \left\langle v_\tau^1(t)\partial_1 v_\tau^1(t) + v_\tau^2(t)\partial_2 v_\tau^1(t), v_\tau^1(t)\right\rangle \\ &\quad + \left\langle v_\tau^1(t)\partial_1 v_\tau^2(t) + v_\tau^2(t)\partial_2 v_\tau^2(t), v_\tau^2(t)\right\rangle \\ &= \left\langle v_\tau^2(t)\partial_2 v_\tau^1(t), v_\tau^1(t)\right\rangle \\ &\quad + \left\langle v_\tau^1(t)\partial_1 v_\tau^2(t), v_\tau^2(t)\right\rangle \\ &\leq -\frac{1}{2}\left\langle \partial_2 v_\tau^2(t), (v_\tau^1(t))^2\right\rangle \\ &\quad - \frac{1}{2}\left\langle \partial_1 v_\tau^1(t), (v_\tau^2(t))^2\right\rangle \\ &\leq \frac{1}{4}\left|\partial_1 v_\tau^1(t)\right|_H^2 + \frac{1}{4}\left|v_\tau^2(t)\right|_{L^4}^4 \\ &\quad + \frac{1}{4}\left|\partial_2 v_\tau^2(t)\right|_H^2 + \frac{1}{4}\left|v_\tau^1(t)\right|_{L^4}^4 \\ &\leq \frac{1}{4}\|v_\tau(t)\|_{H^1}^2 + \frac{1}{4}\|v_\tau(t)\|_{L^4}^4. \end{aligned} \tag{138}$$

Similarly for r_2,

$$\begin{aligned} r_2 &= \left\langle W_{A,1}(t)\partial_1 v_\tau^1(t) + W_{A,2}(t)\partial_2 v_\tau^1(t), v_\tau^1(t)\right\rangle \\ &\quad + \left\langle W_{A,1}(t)\partial_1 v_\tau^2(t) + W_{A,2}(t)\partial_2 v_\tau^2(t), v_\tau^2(t)\right\rangle \\ &= -\left\langle \partial_1 W_{A,1}(t), \left(v_\tau^1(t)\right)^2\right\rangle \\ &\quad - \left\langle \partial_2 W_{A,2}(t), \left(v_\tau^1(t)\right)^2\right\rangle \\ &\quad - \left\langle \partial_1 W_{A,1}(t), \left(v_\tau^2(t)\right)^2\right\rangle \\ &\quad - \left\langle \partial_2 W_{A,2}(t), \left(v_\tau^2(t)\right)^2\right\rangle \\ &\le \varepsilon\left\|v_\tau(t)\right\|_{L^4}^4 + C\left\|W_A(t)\right\|_{H^1}^2. \end{aligned} \tag{139}$$

Analogously to r_1, we deduce that

$$\begin{aligned} r_3 &= \left\langle v_\tau^1(t)\partial_1 W_{A,1}(t) + v_\tau^2(t)\partial_2 W_{A,1}(t), v_\tau^1(t)\right\rangle \\ &\quad + \langle v_\tau^1(t)\partial_1 W_{A,2}(t) + v_\tau^2(t)\partial_2 W_{A,2}(t), v_\tau^2(t)\rangle \\ &\le \varepsilon\left\|v_\tau(t)\right\|_{L^4}^4 + C\left\|W_A(t)\right\|_{H^1}^2. \end{aligned} \tag{140}$$

For r_4, we have

$$\begin{aligned} r_4 &= \left\langle W_{A,1}(t)\partial_1 W_{A,1}(t) + W_{A,2}(t)\partial_2 W_{A,1}(t), v_\tau^1(t)\right\rangle \\ &\quad + \left\langle W_{A,1}(t)\partial_1 W_{A,2}(t) + W_{A,2}(t)\partial_2 W_{A,2}(t), v_\tau^2(t)\right\rangle \\ &\le \varepsilon\left|v_\tau^1(t)\right|_H^2 + C\left|W_{A,1}(t)\partial_1 W_{A,1}(t)\right|_H^2 \\ &\quad + \varepsilon\left|v_\tau^1(t)\right|_H^2 + C\left|W_{A,2}(t)\partial_2 W_{A,1}(t)\right|_H^2 \\ &\quad + \varepsilon\left|v_\tau^2(t)\right|_H^2 + C\left|W_{A,1}(t)\partial_1 W_{A,2}(t)\right|_H^2 \\ &\quad + \varepsilon\left|v_\tau^2(t)\right|_H^2 + C\left|W_{A,2}(t)\partial_2 W_{A,2}(t)\right|_H^2 \\ &\le \varepsilon\left|v_\tau^1(t)\right|_H^2 + \left|W_{A,1}(t)\right|_{L^4}^2 \cdot \left|W_{A,1}(t)\right|_{W^{1,4}}^2 \\ &\quad + C\left|W_{A,2}(t)\right|_{L^4}^2 \cdot \left|W_{A,1}(t)\right|_{W^{1,4}}^2 \\ &\quad + \varepsilon\left|v_\tau^2(t)\right|_H^2 + C\left|W_{A,1}(t)\right|_{L^4}^2 \cdot \left|W_{A,2}(t)\right|_{W^{1,4}}^2 \\ &\quad + C\left|W_{A,2}(t)\right|_{L^4}^2 \cdot \left|W_{A,2}(t)\right|_{W^{1,4}}^2 \\ &\le \varepsilon\left\|v_\tau(t)\right\|_H^2 + C\left\|W_A(t)\right\|_{L^4}^2 \cdot \left\|W_A(t)\right\|_{W^{1,4}}^2. \end{aligned} \tag{141}$$

Since $\{A^{1/2}W_A(t)\}_{t\in\mathbb{R}}$ is a Gaussian process, we infer that

$$E\left(\left|A^{1/2}W_A(t)\right|^4\right) \le C\left[E\left(\left|A^{1/2}W_A(t)\right|^2\right)\right]^2. \tag{142}$$

Then, with the proof of Lemma 2, we know that $\|W_A(t)\|_{W^{1,4}}^2$ is continuous with respect to t. By (137)–(141), we have

$$\begin{aligned} &\langle B(v_\tau(t) + W_A(t), v_\tau(t) + W_A(t)), v_\tau(t)\rangle \\ &\quad \le \left(\frac{1}{4} + \varepsilon\right)\left\|v_\tau(t)\right\|_{H^1}^2 + \left(\frac{1}{4} + 2\varepsilon\right)\left\|v_\tau(t)\right\|_{L^4}^4 \\ &\quad\quad + C\left(\left\|W_A(t)\right\|_{H^1}^2 + \left\|W_A(t)\right\|_{L^4}^2\left\|W_A(t)\right\|_{W^{1,4}}^2\right). \end{aligned} \tag{143}$$

By (135), (136), and (143), we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|v_\tau(t)\right\|_H^2 + \left\|v_\tau(t)\right\|_{H^1}^2 + \vartheta\left\|v_\tau(t)\right\|_{L^4}^4 \\ &\quad \le \left(\frac{1}{4} + 3\varepsilon\right)\left\|v_\tau(t)\right\|_{L^4}^4 + \left(\frac{1}{4} + \varepsilon\right)\left\|v_\tau(t)\right\|_{H^1}^2 \\ &\quad\quad + C\left\|W_A(t)\right\|_{W^{1,4}}^4 + C. \end{aligned} \tag{144}$$

It is equivalent to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|v_\tau(t)\right\|_H^2 + \left(\frac{3}{4} - \varepsilon\right)\left\|v_\tau(t)\right\|_{H^1}^2 \\ &\quad + \left(\vartheta - \frac{1}{4} - 3\varepsilon\right)\left\|v_\tau(t)\right\|_{L^4}^4 \\ &\quad \le C\left(1 + \left\|W_A(t)\right\|_{W^{1,4}}^4\right). \end{aligned} \tag{145}$$

Since $\vartheta > 1/4$, let ε be small enough, such that

$$\frac{3}{4} - \varepsilon > 0; \qquad \vartheta - \frac{1}{4} - 3\varepsilon > 0. \tag{146}$$

Then, the above estimates can be changed into

$$\begin{aligned} &\frac{d}{dt}\left\|v_\tau(t)\right\|_H^2 + \alpha_1\left\|v_\tau(t)\right\|_{H^1}^2 + C_\nu\left\|v_\tau(t)\right\|_{L^4}^4 \\ &\quad \le C\left(1 + \left\|W_A(t)\right\|_{W^{1,4}}^4\right). \end{aligned} \tag{147}$$

By the Gronwall inequality, we get

$$\begin{aligned} \left\|v_\tau(t)\right\|_H^2 &\le \left\|W_A(-\tau)\right\|_H^2 e^{-\alpha_1(\tau+t)} \\ &\quad + C\int_{-\tau}^t \left(1 + \left\|W_A(s)\right\|_{W^{1,4}}^4\right) e^{\alpha_1(s-t)} ds \\ &\le \left\|W_A(-\tau)\right\|_H^2 e^{-\alpha_1(\tau+t)} \\ &\quad + C\int_{-\infty}^0 \left(1 + \left\|W_A(s)\right\|_{W^{1,4}}^4\right) e^{\alpha_1(s-t)} ds. \end{aligned} \tag{148}$$

Similarly to the argument of [26], we will prove that $\|W_A(t)\|_{W^{1,4}}$ has at most polynomial growth, when $t \to -\infty$ a.s. So, we conclude that

$$\sup_{0\le\tau;t\le T}\left\|v_\tau(t)\right\|_H^2 < \infty. \quad \text{a.s.} \tag{149}$$

Multiplying $e^{\delta t}$ on both sides of (147) and integrating with respect to t, we have

$$\begin{aligned}
&\int_{-\tau}^{t} e^{\delta s}\|v_\tau(s)\|_{H^1}^2 ds \\
&\quad \leq e^{-\delta\tau}\|W_A(-\tau)\|_H^2 + \alpha_1 \int_{-\tau}^{t} e^{\delta s}\|v_\tau(s)\|_H^2 ds \\
&\qquad + C\int_{-\tau}^{t}\left(1+\|W_A(t)\|_{W^{1,4}}^4\right)e^{\delta s}ds \\
&\quad \leq e^{-\delta\tau}\|W_A(-\tau)\|_H^2 \\
&\qquad + \alpha_1\int_{-\infty}^{0} e^{\delta s}\|v_\tau(s)\|_H^2 ds \\
&\qquad + C\int_{-\infty}^{0}\left(1+\|W_A(t)\|_{W^{1,4}}^4\right)e^{\delta s}ds.
\end{aligned} \tag{150}$$

As

$$\int_{-\infty}^{0}\left(1+\|W_A(t)\|_{W^{1,4}}^4\right)e^{\delta s}ds < \infty, \tag{151}$$

by (149), we have

$$\sup_{0\leq\tau;t\leq T}\int_{-\tau}^{t} e^{\delta s}\|v_\tau(s)\|_{H^1}^2 ds < \infty. \quad \text{a.s.} \tag{152}$$

By Theorem 7, we know that for problem (131) there exists unique mild solution, which has the following:

$$\begin{aligned}
v_\tau(0) &= e^{\tau A}W_A(-\tau) \\
&\quad + \int_{-\tau}^{0} e^{tA}B\left((v_\tau(t)+W_A(t), v_\tau(t)+W_A(t))\right)dt \\
&\quad + \int_{-\tau}^{0} e^{tA} f\left(v_\tau(t)+W_A(t)\right)dt.
\end{aligned} \tag{153}$$

Then, for any $\zeta \in (0,\theta)\cap(0,1/4)$, where the θ is the parameter in Lemma 2,

$$\begin{aligned}
&\left\|A^{(1+\zeta)/2}v_\tau(0)\right\|_H \\
&\leq \left\|e^{\tau A}A^{(1+\zeta)/2}W_A(-\lambda)\right\|_H \\
&\quad + \int_{-\tau}^{0}\left\|A^{(1+\zeta)/2}e^{tA}B((v_\tau(t)+W_A(t), v_\tau(t)+W_A(t))\right\|_H dt \\
&\quad + \int_{-\tau}^{0}\left\|A^{(1+\zeta)/2}e^{tA}f\left(v_\tau(t)+W_A(t)\right)\right\|_H dt.
\end{aligned} \tag{154}$$

Since

$$\begin{aligned}
&B\left((v_\tau(t)+W_A(t), v_\tau(t)+W_A(t))\right) \\
&\quad = \left(v_\tau(t)\cdot\nabla\right)v_\tau(t) + \left(v_\tau(t)\cdot\nabla\right)W_A(t) \\
&\qquad + \left(W_A(t)\cdot\nabla\right)v_\tau(t) + \left(W_A(t)\cdot\nabla\right)W_A(t),
\end{aligned} \tag{155}$$

then,

$$\begin{aligned}
&\left\|A^{(1+\zeta)/2}e^{tA}B\left((v_\tau(t)+W_A(t), v_\tau(t)+W_A(t))\right)\right\|_H \\
&\quad \leq \left\|A^{(1+\zeta)/2}e^{tA}\left[v_\tau(t)\cdot\nabla\right]v_\tau(t)\right\|_H \\
&\qquad + \left\|A^{(1+\zeta)/2}e^{tA}\left[v_\tau(t)\cdot\nabla\right]W_A(t)\right\|_H \\
&\qquad + \left\|A^{(1+\zeta)/2}e^{tA}\left[W_A(t)\cdot\nabla\right]v_\tau(t)\right\|_H \\
&\qquad + \left\|A^{(1+\zeta)/2}e^{tA}\left[W_A(t)\cdot\nabla\right]W_A(t)\right\|_H \\
&\quad = z_1+z_2+z_3+z_4.
\end{aligned} \tag{156}$$

For z_1, we have

$$\begin{aligned}
z_1 &\leq \left|A^{(1+\zeta)/2}e^{tA}\left[v_\tau^1(t)\,\partial_1 v_\tau^1(t)\right]\right|_H \\
&\quad + \left|A^{(1+\zeta)/2}e^{tA}\left[v_\tau^2(t)\,\partial_2 v_\tau^1(t)\right]\right|_H \\
&\quad + \left|A^{(1+\zeta)/2}e^{tA}\left[v_\tau^1(t)\,\partial_1 v_\tau^2(t)\right]\right|_H \\
&\quad + \left|A^{(1+\zeta)/2}e^{tA}\left[v_\tau^2(t)\,\partial_2 v_\tau^2(t)\right]\right|_H \\
&= z_{1,1}+z_{1,2}+z_{1,3}+z_{1,4}.
\end{aligned} \tag{157}$$

In the following, we use Theorem 6.13 in chapter two of [27] to estimate them respectively as follows:

$$\begin{aligned}
z_{1,1} &= \frac{1}{2}\left|e^{tA}A^{(1+\zeta)/2}\partial_1\left(v_\tau^1\right)^2\right|_H \\
&\leq \frac{1}{2}\left|e^{tA}A^{(3+2\zeta)/4}\left(v_\tau^1\right)^2\right|_{H^{1/2}} \\
&\leq C|t|^{-(3+2\zeta)/4}e^{\delta t}\left|\left(v_\tau^1\right)^2\right|_{H^{1/2}} \\
&\leq C|t|^{-(3+2\zeta)/4}e^{\delta t}\left|2v_\tau^1(t)\,A^{1/4}v_\tau^1(t)+R_4\right|_H,
\end{aligned} \tag{158}$$

the last inequality follows by Theorem A.8 in [25], where $\delta > 0, R_4 = A^{1/4}(v_\tau^1)^2 - 2v_\tau^1 A^{1/4}v_\tau^1$, and $|R_4|_H \leq C|A^{1/8}v_\tau^1(t)|_{L^4}^2 \leq C|v_\tau^1(t)|_{H^1}^2$. So, by Hölder inequality and interpolation inequality, we have

$$z_{1,1} \leq C|t|^{-(3+2\zeta)/4}e^{\delta t}\|v_\tau(t)\|_{H^1}^2. \tag{159}$$

For $z_{1,2}$, we have

$$\begin{aligned}
z_{1,2} &= \left|A^{(1+\zeta)/2}e^{tA}\left[v_\tau^2(t)\,\partial_2 v_\tau^1(t)\right]\right|_H \\
&\leq \left|A^{(1+\zeta)/2}e^{tA}\left[A^{1/4}\left(v_\tau^2(t)\,A^{1/4}v_\tau^1(t)\right)\right]\right|_H \\
&\quad + \left|A^{(1+\zeta)/2}e^{tA}\left[A^{1/4}v_\tau^1(t)\,A^{1/4}v_\tau^2(t)\right]\right|_H \\
&\quad + \left|A^{(1+\zeta)/2}e^{tA}R_5\right|_H,
\end{aligned} \tag{160}$$

where

$$\begin{aligned}
R_5 &= A^{1/4}\left[v_\tau^2(t)\,A^{1/4}v_\tau^1(t)\right] \\
&\quad - \left[A^{1/4}v_\tau^1(t), A^{1/4}v_\tau^2(t)\right] \\
&\quad - v_\tau^2(t)\,A^{1/2}v_\tau^1(t).
\end{aligned} \tag{161}$$

Analogously to estimating $z_{1,1}$, we have

$$\begin{aligned} z_{1,2} &\le C|t|^{-(3+2\zeta)/4} e^{\delta t} \left| v_\tau^2(t) \right|_{L^4} \left| v_\tau^1(t) \right|_{W^{1/2,4}} \\ &\quad + C|t|^{-(1+\zeta)/2} e^{\delta t} \left| v_\tau^1(t) \right|_{W^{1/2,4}} \left| v_\tau^2(t) \right|_{W^{1/2,4}} \\ &\le C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) e^{\delta t} \left\| v_\tau(t) \right\|_{W^{1/2,4}}^2 \\ &\le C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) e^{\delta t} \left\| v_\tau(t) \right\|_{H^1}^2. \end{aligned} \tag{162}$$

Similarly, we can get the same estimates for $z_{1,3}$ and $z_{1,4}$. Therefore,

$$z_1 \le C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) e^{\delta t} \left\| v_\tau(t) \right\|_{H^1}^2. \tag{163}$$

Analogously to estimating z_1, we can get for z_2, z_3, and z_4 that

$$\begin{aligned} z_2 &\le C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) \\ &\qquad \times e^{\delta t}\left(\left\|W_A(t)\right\|_{H^1}^2 + \left\|v_\tau(t)\right\|_{H^1}^2\right), \\ z_3 &\le C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) \\ &\qquad \times e^{\delta t}\left(\left\|W_A(t)\right\|_{H^1}^2 + \left\|v_\tau(t)\right\|_{H^1}^2\right), \\ z_4 &\le C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) \\ &\qquad \times e^{\delta t}\left\|W_A(t)\right\|_{H^1}^2. \end{aligned} \tag{164}$$

So, by (163)–(164) and (156), we get

$$\begin{aligned} &\left\| A^{(1+\zeta)/2} e^{tA} B\left(\left(v_\tau(t) + W_A(t), v_\tau(t) + W_A(t)\right)\right) \right\|_H \\ &\quad \le C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) \\ &\qquad \times e^{\delta t}\left(\left\|W_A(t)\right\|_{H^1}^2 + \left\|v_\tau(t)\right\|_{H^1}^2\right). \end{aligned} \tag{165}$$

For the third term on the right hand side of (154), we obtain

$$\begin{aligned} &\left\| A^{(1+\zeta)/2} e^{tA} f\left(v_\tau(t) + W_A(t)\right) \right\|_H \\ &\quad \le C|t|^{-(1+\zeta)/2} e^{\delta t}\left(\left\|v_\tau(t)\right\|_{L^6}^3 + \left\|W_A(t)\right\|_{L^6}^3\right) \\ &\quad \le C|t|^{-(1+\zeta)/2} \\ &\qquad \times e^{\delta t}\left(\left\|v_\tau(t)\right\|_H \cdot \left\|v_\tau(t)\right\|_{H^1}^2 + \left\|W_A(t)\right\|_{H^1}^3\right) \\ &\quad \le C|t|^{-(1+\zeta)/2} e^{\delta t} \\ &\qquad \times e^{\delta t}\left(\left\|v_\tau(t)\right\|_{H^1}^2 + \left\|W_A(t)\right\|_{H^1}^2\right), \end{aligned} \tag{166}$$

since $\|v_\tau(t)\|_H$ and $e^{\delta t}\|W_A(t)\|_{H^1}^2$ are bounded for $t, \tau \in (-\infty, T]$, the last inequality follows. For the first term on the right hand side of (154), we have

$$\left\| e^{\tau A} A^{(1+\zeta)/2} W_A(-\tau) \right\|_H \le e^{-\delta\tau} \left\| A^{(1+\zeta)/2} W_A(-\tau) \right\|_H. \tag{167}$$

Similar to [26], we can prove that $\|A^{(1+\zeta)/2} W_A(-\tau)\|_H$ has at most polynomial growth when $\tau \to \infty$. For the reader convenience, we sketch a proof. By Lemma 2, we know that $W(t) - W(s)$ is a $D(A^{\theta/2})$ valued Brownian motion, for $s \le t \le 0$. So, by the law of iterated logarithm, we have

$$w_n := \sup_{n \le s \le t \le n+1} \frac{\|W(t) - W(s)\|_{H^\theta}}{|t-s|^{1/2+(\zeta-\theta)/4}} < \infty, \quad \text{a.s. } n \in \mathbb{Z}. \tag{168}$$

Obviously, w_n is a i.i.d sequence. By the law of large numbers, there exists an integer-valued random variable $n_0(w) > 0$, when $n \ge n_0(w)$, we have

$$\frac{w_{-n}}{n} \le \frac{w_{-n} + \cdots + w_{-1}}{n} \le E w_0 + 1 < \infty. \tag{169}$$

This implies that

$$w_{-n} \le C_0(w)\, n, \tag{170}$$

for all $n > 0$. In other words,

$$\|W(t) - W(s)\|_{H^\theta} \le C_0(w)\, |[s]| \cdot |t-s|^{1/2+(\zeta-\theta)/4}, \tag{171}$$

when $s \le t \le [s]+1$. By the law of iterated logarithm, we have

$$\|W(t)\|_{H^\theta} \le C_1(w)\, |t|, \qquad t \in (-\infty, 0], \tag{172}$$

for some positive random variable. By Theorem 5.14 in [23], we know that

$$W_A(t) = \int_{-\infty}^{t} A e^{-(t-s)A} \left(W(t) - W(s)\right) ds. \tag{173}$$

So, we have that

$$\begin{aligned} &\left\| A^{(1+\zeta)/2} W_A(t) \right\|_H \\ &\quad \le \int_{-\infty}^{t} \left\| A^{1+1/2+\zeta/2} e^{-(t-s)A} \left(W(t) - W(s)\right) \right\|_H ds \\ &\quad = \int_{-\infty}^{t} \left\| A^{1+1/2+(\zeta-\theta)/2} e^{-(t-s)A} \left[A^{\theta/2}\left(W(t) - W(s)\right)\right] \right\|_H ds \\ &\quad \le \int_{-\infty}^{t} \frac{e^{-\delta(t-s)}}{|t-s|^{1+1/2+(\zeta-\theta)/2}} \|W(t) - W(s)\|_{H^\theta} ds \\ &\quad \le \int_{[t]-1}^{t} \frac{e^{-\delta(t-s)}}{|t-s|^{1+(\zeta-\theta)/4}} \cdot \frac{\|W(t) - W(s)\|_{H^\theta}}{|t-s|^{1/2+(\zeta-\theta)/4}} \\ &\qquad + \int_{-\infty}^{[t]-1} \frac{e^{-\delta(t-s)}}{|t-s|^{1+(\zeta-\theta)/4}} \cdot \frac{C_1(w)\left(|t| + |s|\right)}{|t-s|^{1/2+(\zeta-\theta)/4}} \\ &\quad \le \int_{[t]-1}^{t} \frac{e^{-\delta(t-s)}}{|t-s|^{1+(\zeta-\theta)/4}} \cdot C_0(w)\, |[s]|\, ds \\ &\qquad + \int_{-\infty}^{[t]-1} e^{-\delta(t-s)} C_1(w)\left(|t| + |s|\right) \\ &\quad \le \left(C_0(w) + C_1(w)\right)\left(|t| + 1\right), \end{aligned} \tag{174}$$

since $s \le [t] - 1$, the fourth inequality follows. By (167) and (174), we know that

$$\sup_{\tau \ge 0} \left\| e^{\tau A} A^{(1+\zeta)/2} W_A(-\tau) \right\|_H < \infty, \quad \text{a.s.} \tag{175}$$

If we let $\zeta = 1/2 < \theta$, repeating the argument of (174), we can see that $\|W_A(t)\|_{W^{1,4}}$ also has at most polynomial growth, when $t \rightarrow -\infty$ a.s., since we have the Sobolev embedding $H^{3/2} \subset W^{1,4}$. Consider the second term on the right hand side of (154), by (165),

$$\begin{aligned}
&\int_{-\tau}^{0} \left\|A^{(1+\zeta)/2} e^{tA} B\left(\left(v_\tau(t) + W_A(t), v_\tau(t) + W_A(t)\right)\right)\right\|_H dt \\
&\leq \int_{-\tau}^{0} C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) \\
&\quad \times e^{\delta t}\left(\|W_A(t)\|_{H^1}^2 + \|v_\tau(t)\|_{H^1}^2\right) dt \\
&\leq \int_{-1}^{0} C\left(|t|^{-(3+2\zeta)/4} + |t|^{-(1+\zeta)/2}\right) \\
&\quad \times e^{\delta t}\left(\|W_A(t)\|_{H^1}^2 + \|v_\tau(t)\|_{H^1}^2\right) dt \\
&\quad + \int_{-\infty}^{-1} C e^{\delta t}\left(\|W_A(t)\|_{H^1}^2 + \|v_\tau(t)\|_{H^1}^2\right) dt < \infty,
\end{aligned} \tag{176}$$

where the last inequality follows by (152). Analogously, we can prove that

$$\begin{aligned}
&\int_{-\tau}^{0} \left\|A^{(1+\zeta)/2} e^{tA} f\left(v_\tau(t) + W_A(t)\right)\right\|_H dt \\
&\quad \leq \int_{-\tau}^{0} C\left(|t|^{-(1+\zeta)/2}\right) \\
&\qquad \times e^{\delta t}\left(\|v_\tau(t)\|_H \|v_\tau(t)\|_{H^1}^2 + \|W_A(t)\|_{H^1}^3\right) \\
&\quad < \infty,
\end{aligned} \tag{177}$$

where we used (149) and (152) for the last inequality. By (154) and (175)–(177), we get

$$\left\|A^{(1+\zeta)/2} v_\tau(0)\right\|_H \leq \xi(w), \quad \text{a.s.}, \tag{178}$$

for some positive random variable $\xi(w)$. As $H^{1+\delta} \subset H^1$ is compact, by Prohorov Theorem, we know that the family of laws for $(v_\tau(0))_{\tau\geq 0}$ taking values in H^1 is tight. Since $v_\tau(0) = u_\tau(0) - W_A(0)$, then so does the law of $(u_\tau(0))_{\tau\geq 0}$ taking values in the same space. For $t \geq 0$, set

$$\left(P_t f\right)(x) = Ef\left(u(t, .; 0, x)\right), \tag{179}$$

where $f \in C_b(H_0^1)$. Following the arguments in [24], for all $t_0 < s < t$ and all $u_{t_0} \in H_0^1$, by proving

$$E\left(f\left(u\left(t; t_0, u_{t_0}\right)\right) \mid \mathscr{F}_s\right) = P_{t-s}\left(u\left(s; t_0, u_{t_0}\right)\right), \tag{180}$$

we can show that u is a Markov process. Here, $\mathscr{F}_s$ is the σ-algebra generated by $W(r)$ for $r \leq s$. So, $(P_t)_{t\geq 0}$ is the Markov semigroup. Define a dual semigroup P_t^* in the space $P(H_0^1)$ of probability measures on H_0^1 as follows:

$$\int_{H_0^1} fd\left(P_t^* \mu\right) = \int_{H_0^1} P_t f d\mu. \tag{181}$$

Let μ_τ be the law of $u_\tau(0)$, which is the solution of (2) with initial condition $u(-\tau) = 0$. Then, we have

$$\begin{aligned}
\mu_\tau(f) &= Ef\left(u_{-\tau}(0)\right) = Ef\left(u(\tau, \cdot; 0, 0)\right) \\
&= \left(P_\tau f\right)(0) = \int_{H_0^1} P_\tau f d\delta_0 \\
&= \int_{H_0^1} fd\left(P_\tau^* \delta_0\right),
\end{aligned} \tag{182}$$

where we use the fact that $u(\tau, \cdot; 0, 0)$ and $u_\tau(0)$ have the same law, the second equality follows. Therefore,

$$P_{\tau_1}^* \mu_\tau = \mu_{\tau+\tau_1}. \tag{183}$$

Since $(\mu_\tau)_{\tau\geq 0}$ is tight, then by Prokhorov theorem, we know that $(\mu_\tau)_{\tau\geq 0}$ is relatively compact. We can choose a subsequence of $(\mu_\tau)_{\tau\geq 0}$ denoted by $(\mu_{\tau_n})_{n\in\mathbb{N}}$ such that for $\mu \in P(H^\sigma)$,

$$\begin{aligned}
&\int_{H_0^1} \left(P_t f\right)(x) \mu(dx) \\
&\quad = \lim_{n\to\infty} \int_{H_0^1} \left(P_t f\right)(x) \mu_{\tau_n}(dx) \\
&\quad = \lim_{n\to\infty} \int_{H_0^1} f(x) P_t^* \mu_{\tau_n}(dx) \\
&\quad = \lim_{n\to\infty} \int_{H_0^1} f(x) \mu_{\tau_n+t}(dx) \\
&\quad = \int_{H_0^1} f(x) \mu(dx).
\end{aligned} \tag{184}$$

□

5.2. Uniqueness. The main result of this part is as follows.

Theorem 9. *Assume $\theta > 1/2$ in Lemma 2 and $\vartheta > 1/4$; then,*

(i) *the stochastic Burgers equation* (2) *has a unique invariant measure μ;*

(ii) *for all $u_0 \in H_0^1$ φ, $H_0^1 \rightarrow \mathbb{R}$, such that $\int_{H_0^1} |\varphi| d\mu < \infty$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi\left(u(t; u_0)\right) dt = \int_{H_0^1} \varphi d\mu \quad a.s.; \tag{185}$$

(iii) *for every Borel measure μ^* on H_0^1, one has that*

$$\left\|P_t^* \mu^* - \mu\right\|_{TV} \longrightarrow 0 \quad as\ t \longrightarrow \infty, \tag{186}$$

where $\|\cdot\|_{TV}$ stands for the total variation of a measure. In particularly, one has that

$$P_t^* \mu^*(B) \longrightarrow \mu(B), \quad as\ t \longrightarrow \infty, \tag{187}$$

for every Borel set $B \in \mathscr{B}(H_0^1)$(the Borel σ-algebra of H_0^1).

In order to prove Theorem 9, we only need Theorem 10 below, see [28, Theorem 4.2.1]. We define $P(t,x,\cdot), t>0, x\in H_0^1$, to be the transition probability measure that is,

$$P(t,x,B)=P_t^*\delta_x(B)=P(u(t;x)\in B) \tag{188}$$

for $B\in\mathscr{B}(H_0^1)$.

Theorem 10. *Assume that the probability measures $P(t,x,\cdot), t>0, x\in H_0^1$, are all equivalent, in the sense that they are mutually absolutely continuous. Then, Theorem 9 holds true.*

In the following, we will prove the irreducibility and the strong Feller property in H_0^1 to get the equivalence of the measure $P(t,x,\cdot)$. For the two notations, we outline them below. For $y\in H_0^1, \varepsilon>0$, let

$$B(y,\varepsilon)=\left\{x\in H_0^1;\|x-y\|_{H^1}<\varepsilon\right\}. \tag{189}$$

(I) For any $x,y\in H_0^1$, such that for all $\varepsilon>0$,

$$P(t,x,B(y,\varepsilon))>0 \tag{190}$$

for each $t>0$.

(S) For all $O\in\mathscr{B}(H_0^1)$, every $t>0$, and all $x_n,x\in H_0^1$ such that $x_n\to x$ in H_0^1, it holds that

$$P(t,x_n,O)\longrightarrow P(t,x,O). \tag{191}$$

Before checking the condition (I), we need Lemma 11 below. For $x\in H_0^1$ and $\phi:[0,T]\to H_0^1$, set

$$u(t,x,\phi)=v(t,x,\phi)+\phi(t), \tag{192}$$

where $v(t,x,\phi)$ is solution of the following equation:

$$\frac{dv}{dt}+Av+B(v+\phi,v+\phi)+f(v+\phi)=0, \tag{193}$$

for $t\in[0,T]$, with initial condition $v(0)=x$. As it is proved in previously this equation has a unique solution as follows:

$$v\in C\left([0,T];H_0^1\right), \tag{194}$$

when $x\in H_0^1$ and $\phi\in C([0,T];H_0^1)$.

Lemma 11. *Define $\Psi(\phi)=u(\cdot,x,\phi)$; then,*

(i) *the mapping*

$$\Psi: C_0\left([0,T];H^{3/2}\right)\longrightarrow C\left([0,T];H_0^1\right) \tag{195}$$

is continuous, where $C_0([0,T];B):=\{h\in C([0,T];B); h(0)=0\}$ for Banach space B;

(ii) *for every $x,y\in H^{3/2}$ and $T>0$ there exists $\overline{z}\in C_0([0,T];H^{3/2})$ such that $u(T,x,\overline{z})=y$.*

Proof. (i) is proved by (A.30) in the Appendix. To prove (ii), let $x,y,\in H^{3/2}$ and $T>0$, define $\overline{u}$ as

$$\begin{gathered}\overline{u}(t)=e^{-tA}x,\quad t\in[0,t_0],\\ \overline{u}(t)=e^{-(T-t)A}y,\quad t\in[t_1,T],\\ \overline{u}(t)=\overline{u}(t_0)+\frac{t-t_0}{t_1-t_0}(\overline{u}(t_1)-\overline{u}(t_0)),\\ t\in(t_0,t_1).\end{gathered} \tag{196}$$

Obviously, $\overline{u}(t)\in C([0,T];H^{3/2})$. Define $\overline{v}$ as the solution of the following equation:

$$\frac{d}{dt}\overline{v}+A\overline{v}+B(\overline{u},\overline{u})+f(\overline{u})=0, \tag{197}$$

with initial condition $\overline{v}(0)=x$; then $\overline{v}\in C([0,T];H^{3/2})$. Set $\overline{z}=\overline{u}-\overline{v}$; then it satisfies all the requirements of the lemma. □

Proposition 12. *With conditions in Theorem 9, the irreducibility property (I) is satisfied.*

Proof. Let $x\in H^{3/2}$ and $\overline{z}$ be the same as (ii) in Lemma 11. By the above lemma, we have that for $\varepsilon>0$, we can find $\delta>0$, such that

$$\|z-\overline{z}\|_{C_0([0,T];H^{3/2})}<\delta \tag{198}$$

implies that

$$\|u(\cdot,x,z)-u(\cdot,x,\overline{z})\|_{C([0,T];H^1)}<\varepsilon. \tag{199}$$

If $\theta>1/2$ in Lemma 2, and denote z and $\overline{z}$ the corresponding Ornstein-Uhlenbeck process satisfying conditions in the lemma, then $z,\overline{z}\in C([0,T];H^{3/2})$. Choose $\delta_1>0$ such that $\delta_1<\delta$ and

$$z\in U_{\delta_1}=:\left\{z\in C_0\left([0,T];H^{3/2}\right);\|z-\overline{z}\|_{C([0,T];H^{3/2})}<\delta_1\right\}. \tag{200}$$

Then, for $z\in U_{\delta_1}$, we have that

$$\|u(T,,x,z)-y\|_{H^1}<\varepsilon. \tag{201}$$

Recall now that the solution u of the stochastic Burgers equation is equal to $\Psi(z)$, z being the Ornstein-Uhlenbeck process. Then, it remains to show that

$$P\left\{z(\cdot,w)\in U_{\delta_1}\right\}>0. \tag{202}$$

But this is obviously true. So far, we have proved that for for all $t>0$, for all $x,y\in H^{3/2}$, for all $\varepsilon>0$,

$$P(t,x,B(y,\varepsilon))>0. \tag{203}$$

Next, we will prove for all $x_0 \in H_0^1, y_0 \in H^{3/2}$, the above inequality also holds. Indeed, for $0 < h < t$, by Chapman-Kolmogorov equation, we have

$$\begin{aligned} &P(t, x_0, B(y_0, \varepsilon)) \\ &= \int_{H_0^1} P(t-h, x_0, dy) P(h, y, B(y_0, \varepsilon)) \\ &= \int_{H^{3/2}} P(t-h, x_0, dy) P(h, y, B(y_0, \varepsilon)) > 0. \end{aligned} \tag{204}$$

Since $P(t-h, x_0, H^{3/2}) = 1$, we will extend (204) to the case for all $x_0 \in H_0^1, y_0 \in H_0^1$. If this is not true, there exists $t_0 > 0, x_0, y_0 \in H_0^1, \varepsilon > 0$ such that

$$P(t_0, x_0, B(y_0, \varepsilon)) = 0. \tag{205}$$

Then, we can choose $y_1 \in H^{3/2}, \varepsilon_1 > 0$ such that $B(y_1, \varepsilon_1) \subset B(y_0, \varepsilon)$. By (204), we have

$$P(t_0, x_0, B(y_1, \varepsilon_1)) > 0, \tag{206}$$

which is contrary to (205). □

In this part, it is time to check the condition (S).

We will first obtain the strong Feller property in H_0^1 for modified Burgers equation (208) below, then let $R \to \infty$ to check the condition (S).

Fix $R > 0$, let $K_R : [0, \infty[\to [0, \infty[$ satisfy $K_R \in C^1(\mathbb{R}_+)$ such that $|K_R| \le 1, |K_R'| \le 2$ and

$$\begin{aligned} K_R &= 1, \quad \text{if } x < R, \\ K_R &= 0, \quad \text{if } x \ge R+1. \end{aligned} \tag{207}$$

Consider the following equation:

$$\begin{aligned} &du_R(t) + Au_R(t)\,dt \\ &\quad + K_R\left(\|u_R(t)\|_{H^1}^2\right) B(u_R(t), u_R(t))\,dt \\ &\quad + K_R\left(\|u_R(t)\|_{H^1}^2\right) f(u_R)(t) = dW(t). \end{aligned} \tag{208}$$

Proposition 13. *There exists a unique mild solution $u_R(\cdot, w) \in C([0,T]; H_0^1)$ for* (208) *which is Markov process with the Feller property in H_0^1, that is for every $R > 0$, $t > 0$, there exists a constant $L = L(t, R) > 0$ such that*

$$\left|P_t^{(R)}\phi(x) - P_t^{(R)}\phi(y)\right| \le L\|x - y\|_{H^1} \tag{209}$$

holds for all $x, y \in H_0^1$, and all $\phi \in C_b(H_0^1) \le 1$, where $P_t^{(R)}\phi(x) := \int_{H_0^1}\phi(y)P_R(t, x, dy), P_R(t, x, \cdot)$ is the transition probabilities corresponding to (204).

Proof. The proof of existence and uniqueness is similar to Section 2. Let $\phi_1 = \phi_2$ in (A.28), by the Gronwall inequality, we know that u_R is Lipschitz continuous with respect to initial value. Using the method in Proposition 4.3.3 in [24], we can prove that the solution is a Markov process. To prove the Fell property, we first consider the following Galerkin approximations of (208). Let P_n be the orthogonal projection in H defined as $P_n x = \sum_{j=1}^n \langle x, e_j\rangle e_j, x \in H$. Clearly, $H_n := P_n H$ for every n. Consider the equation in H_n as follows:

$$\begin{aligned} &du_n^{(R)}(t) + Au_n^{(R)}(t)\,dt \\ &\quad + K_R\left(\left\|u_n^{(R)}(t)\right\|_{H^1}^2\right) P_n B\left(u_n^{(R)}(t), u_n^{(R)}(t)\right) \\ &\quad + K_R\left(\left\|u_n^{(R)}(t)\right\|_{H^1}^2\right) f\left(u_n^{(R)}\right)(t) = dW(t), \end{aligned} \tag{210}$$

with initial condition $u_n^{(R)}(0) = P_n u_0$. This is a finite-dimensional equation with globally Lipschitz nonlinear functions, so it has a unique progressively measurable solution with P-a.e. trajectory $u_n^{(R)}(\cdot, w) \in C([0,T]; H_n)$, which is also a Markov process in H_n with associated semigroup $P_{n,t}^{(R)}$ defined as

$$P_{n,t}^{(R)}\phi(x) = E\phi\left(u_n^{(R)}(t; x)\right), \tag{211}$$

for all $x \in H_n$ and $\phi \in C_b(H_n)$. For every $R > 0, t > 0$, we can prove that there exists a constant $L = L(t, R) > 0$ such that

$$\left|P_{n,t}^{(R)}\phi(x) - P_{n,t}^{(R)}\phi(y)\right| \le L\|x - y\|_{H^1} \tag{212}$$

hold for all $n \in \mathbb{N}, x, y \in H_n$, and all $\phi \in C_b(H_n)$ with $\|\phi\|_{H^1} \le 1$. Indeed, the following remarkable formula holds true for the differential in x of $P_{n,t}^{(R)}\phi$ [29]:

$$\begin{aligned} &D_x P_{n,t}^{(R)}\phi(x)\cdot h \\ &= \frac{1}{t}E\left(\phi\left(u_n^{(R)}(t;x)\right)\int_0^t\left\langle\left(P_nQQ^*P_n\right)^{-1/2}D_x u_n^{(R)}(s;x)\right.\right. \\ &\qquad\qquad \left.\left.\cdot h, d\beta_n(s)\right\rangle\right), \end{aligned} \tag{213}$$

for all $h \in H_n$, where β_n is a n-dimensional standard Wiener process with incremental covariance P_nQ and Q is the covariance operator of $W(t)$. Obviously, Q is nonnegative, adjoint, Hilbert-Schmidt operator with inverse. Since the eigenvalues α_n of the Stokes operator A, in 2-space dimension, behave like n, let $\theta = 1/2 + \varepsilon$ for some $\varepsilon > 0$, in Lemma 2, we have $D(A) \subset \mathscr{R}(Q) \subset D(A^{3/4})$, where $\mathscr{R}(Q)$ is the image of Q. Therefore,

$$\begin{aligned} &\left|D_x P_{n,t}^{(R)}\phi(x)\cdot h\right| \\ &\quad \le \frac{1}{t}E\left(\int_0^t\left\|\left(P_nQQ^*P_n\right)^{-1/2}D_x u_n^{(R)}(s;x)\cdot h\right\|_H^2 ds\right)^{1/2}. \end{aligned} \tag{214}$$

Since for $y \in H_n$

$$\begin{aligned} &\left\|\left(P_nQQ^*P_n\right)^{-1/2}y\right\|_H^2 \\ &\quad = \left\langle\left(P_nQQ^*P_n\right)^{-1}y, y\right\rangle \\ &\quad = \left\langle\left(AP_nQQ^*P_nA\right)^{-1}Ay, Ay\right\rangle \le C\|y\|_{H^2}^2, \end{aligned} \tag{215}$$

it follows that

$$\begin{aligned}
&\left|D_x P_{n,t}^{(R)}\phi(x)\cdot h\right| \\
&\quad \le \frac{1}{t} CE\left(\int_0^t \left\|D_x u_n^{(R)}(s;x)\cdot h\right\|_{H^2}^2 ds\right)^{1/2} \\
&\quad \le \frac{1}{t} C(R)\|h\|_{H^1},
\end{aligned} \tag{216}$$

where the last inequality follows by the Estimate 4 of the Appendix (note that $C(R)$ is independent of $x \in H_n$ and $n \in \mathbb{N}$). Indeed, $u_n^{(R)}(t,x)$ is given by $v_n(t,x) + P_n z(t)$, where z is the Ornstein-Uhlenbeck process, and v_n is the solution of (A.2). Therefore,

$$\begin{aligned}
&\left|P_{n,t}^{(R)}\phi(x) - P_{n,t}^{(R)}\phi(y)\right| \\
&\quad \le \sup_{\|h\|_{H^1}\le 1, k\in H_n} \left|D_x P_{n,t}^{(R)}\phi(k)\cdot h\right| \cdot \|x-y\|_{H^1} \\
&\quad \le \frac{1}{t} C(R)\|x-y\|_{H^1}.
\end{aligned} \tag{217}$$

In the following step, we will let $n \to \infty$ to get the Fell property for (208). Let $x \in H_0^1$ and $\phi \in C_b(H_0^1)$ be given. From the Appendix, Remark A.1, we know that $u_n^{(R)}(t)$ converges to $u^{(R)}(t)$ strongly in $L^2(0,T;H_0^1)$, p-a.s.. By the boundedness and continuous of ϕ as well as Lebesgue dominated convergence theorem, we have

$$E\int_0^T \left|\phi\left(u_n^{(R)}(,;x)\right) - \phi\left(u^{(R)}(,;x)\right)\right| dt \longrightarrow 0, \tag{218}$$

which implies that for some subsequence n_k,

$$E\phi\left(u_{n_k}^{(R)}(,;x)\right) \longrightarrow E\phi\left(u^{(R)}(,;x)\right), \tag{219}$$

for a.e. $t \in [0,T]$. Take $x, y \in H_0^1$, by the previous argument, we can find a subsequence n_k such that the previous almost sure convergence in $t \in [0,T]$ holds true both x and y.

Thus, from (212), we have

$$\left|P_t^{(R)}\phi(x) - P_t^{(R)}\phi(y)\right| \le L\|x-y\|_{H^1}, \tag{220}$$

for a.e. $t \in [0,T]$. As $u^{(R)}(t;x)$ has continuous trajectories with values in H_0^1, the above inequality holds for all $t \in [0,T]$. □

Proposition 14. *Under conditions of Theorem 9, (S) holds true.*

Proof. Take $t > 0, x_n, x \in H_0^1$ satisfying $x_n \to x$ in H^1. For every $R > 0$, we have that

$$\begin{aligned}
&\left\|P_R(t,x_n,\cdot) - P_R(t,x,\cdot)\right\|_{TV} \\
&\quad = \sup_{\|\phi\|_{C_b(H^1)}\le 1} \left|P_t^{(R)}\phi(x_n) - P_t^{(R)}\phi(x)\right| \\
&\quad \le L\|x_n - x\|_{H^1} \longrightarrow 0,
\end{aligned} \tag{221}$$

as $n \to \infty$ by Proposition 13. Then,

$$\begin{aligned}
&\left\|P_R(t,x_n,\cdot) - P(t,x_n,\cdot)\right\|_{TV} \\
&\quad + \left\|P_R(t,x,\cdot) - P(t,x,\cdot)\right\|_{TV} \\
&\quad = \sup_{\|\phi\|_{C_b(H^1)}\le 1} \left|P_t^{(R)}\phi(x_n) - P_t\phi(x_n)\right| \\
&\qquad + \sup_{\|\phi\|_{C_b(H^1)}\le 1} \left|P_t^{(R)}\phi(x) - P_t\phi(x)\right| \\
&\quad = \sup_{\|\phi\|_{C_b(H^1)}\le 1} \left|E\phi(u_R(t;x_n)) - E\phi(u(t;x_n))\right| \\
&\qquad + \sup_{\|\phi\|_{C_b(H^1)}\le 1} \left|E\phi(u_R(t;x)) - E\phi(u(t;x))\right| \\
&\quad \le 2\int_\Omega I_{\{\sup_{n\in\mathbb{N}}\|u(t;x_n)\|_{H^1} > R\}} P(dw) \\
&\qquad + 2\int_\Omega I_{\{\|u(t;x)\|_{H^1} > R\}} P(dw) \longrightarrow 0, \quad \text{as } R \to \infty,
\end{aligned} \tag{222}$$

where the inequality follows by the consistency of $u(t;x)$ and $u^{(R)}(t;x)$, when $\|u(t;x)\|_{H^1} \le R$, and the limit follows by (A.21). Therefore,

$$\begin{aligned}
&\left\|P(t,x_n,\cdot) - P(t,x,\cdot)\right\|_{TV} \\
&\quad \le \left\|P(t,x_n,\cdot) - P_R(t,x_n,\cdot)\right\|_{TV} \\
&\qquad + \left\|P_R(t,x_n,\cdot) - P_R(t,x,\cdot)\right\|_{TV} \\
&\qquad + \left\|P_R(t,x,\cdot) - P(t,x,\cdot)\right\|_{TV} \longrightarrow 0,
\end{aligned} \tag{223}$$

as $n \to \infty$. □

6. Example

Our theory can be applied to stochastic reaction diffusion equations or stochastic real valued Ginzburg Landau equation in high dimensions as follows:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u + |u|^2 u - u = dW, \quad \text{on } [0,T]\times D, \\
&u(t,x) = 0, \quad t\in[0,T], x \in \partial D, \\
&u(0,x) = u_0(x), \quad x \in D,
\end{aligned} \tag{224}$$

where $u(t,x) = (u^1(t,x), u^2(t,x))$ is the velocity field, Δ denotes the Laplace operator, W stands for the Q-Wiener process, and D is a regular bounded open domain of $\mathbb{R}^2$.

Appendix

Fix $R > 0$ and let $K_R : [0,\infty[\to [0,\infty[$ satisfy $K_R \in C^1(\mathbb{R}_+)$ such that $|K_R| \le 1, |K_R'| \le 2$ and

$$\begin{aligned}
K_R(x) &= 1, \quad \text{if } x < R, \\
K_R(x) &= 0, \quad \text{if } x \ge R+1.
\end{aligned} \tag{A.1}$$

Consider the following equation:

$$\frac{dv_n}{dt} + Av_n + K_R\left(\|v_n + P_n\phi\|^2_{H^1}\right) \times P_nB\left(v_n + P_n\phi, v_n + P_n\phi\right) + K_R\left(\|v_n + P_n\phi\|^2_{H^1}\right) f\left(v_n + P_n\phi\right) = 0, \tag{A.2}$$

where $\phi \in C([0,T]; H^{3/2})$.

Estimate 1. We have the following estimate in H for (A.2):

$$\|v_n\|_{C([0,T];H)} + \|v_n\|_{L^2([0,T];H^1)} \le C\left(\|x\|_H, \|\phi\|_{C([0,T];H^{3/2})}, T\right), \tag{A.3}$$

where $C(a,b,c)$ indicates a constant C depending on a,b,c. Analogously to the derivation of (147), we get

$$\frac{d}{dt}\|v_n\|^2_H + \|v_n\|^2_{H^1} + \|v_n\|^4_{L^4} \le C\left(\|\phi\|^4_{H^{3/2}} + 1\right). \tag{A.4}$$

Therefore, for all $t \in [0,T]$,

$$\|v_n(t)\|^2_H + \int_0^t \|v_n\|^2_{H^1} ds + \int_0^t \|v_n\|^4_{L^4} ds \le \|x\|^2_H + C\int_0^t \left(\|\phi(s)\|^4_{H^{3/2}} + 1\right), \tag{A.5}$$

Then, we get (A.3).

Estimate 2. We obtain the following estimate in H^1_0 for (A.2):

$$\|v_n(t)\|^2_{C([0,T];H^1_0)} + \int_0^T \|v_n(s)\|^2_{H^2} ds \le C\left(\|x\|_{H^1}, \|\phi\|_{C([0,T];H^{3/2})}, T\right). \tag{A.6}$$

Since we have

$$\frac{d}{dt}\|v_n(t)\|^2_{H^1} + \|v_n(t)\|^2_{H^2} + \left\langle f\left(v_n(t) + P_n\phi(t)\right), Av_n(t)\right\rangle = \left\langle B\left(v_n(t) + P_n\phi(t), v_n(t) + P_n\phi(t)\right), Av_n(t)\right\rangle, \tag{A.7}$$

the equation is equivalent to

$$\begin{aligned}\frac{d}{dt}\|v_n(t)\|^2_{H^1} &+ \|v_n(t)\|^2_{H^2} + \left\langle f\left(v_n(t) + P_n\phi(t)\right), A\left(v_n(t) + P_n\phi(t)\right)\right\rangle \\ &= \left\langle B\left(v_n(t) + P_n\phi(t), v_n(t) + P_n\phi(t)\right), Av_n(t)\right\rangle + \left\langle f\left(v_n(t) + P_n\phi(t)\right), AP_n\phi(t)\right\rangle.\end{aligned} \tag{A.8}$$

Denote by $u_n := v_n(t) + P_n\phi(t)$ and $u_n = (u_n^1, u_n^2)$; then

$$\begin{aligned}\left\langle |u_n|^2 u_n, Au_n\right\rangle = {} & 3\int_D \left(u_n^1\right)^2\left(\partial_1 u_n^1\right)^2 dx + 3\int_D \left(u_n^1\right)^2\left(\partial_2 u_n^1\right)^2 dx \\ &+ 3\int_D \left(u_n^2\right)^2\left(\partial_1 u_n^2\right)^2 dx + 3\int_D \left(u_n^2\right)^2\left(\partial_2 u_n^2\right)^2 dx \\ &+ \int_D \left(u_n^2\right)^2\left(\partial_1 u_n^1\right)^2 dx + \int_D \left(u_n^2\right)^2\left(\partial_2 u_n^1\right)^2 dx \\ &+ \int_D \left(u_n^1\right)^2\left(\partial_1 u_n^2\right)^2 dx + \int_D \left(u_n^1\right)^2\left(\partial_2 u_n^2\right)^2 dx \\ &+ 4\int_D \left(u_n^1\partial_1 u_n^1\right)\left(u_n^2\partial_1 u_n^2\right) dx + 4\int_D \left(u_n^1\partial_2 u_n^1\right)\left(u_n^2\partial_2 u_n^2\right) dx.\end{aligned} \tag{A.9}$$

As

$$\begin{aligned}4\int_D \left(u_n^1\partial_1 u_n^1\right)\left(u_n^2\partial_1 u_n^2\right) dx &\le 2\int_D \left(u_n^1\right)^2\left(\partial_1 u_n^1\right)^2 dx + 2\int_D \left(u_n^2\right)^2\left(\partial_1 u_n^2\right)^2 dx, \\ 4\int_D \left(u_n^1\partial_2 u_n^1\right)\left(u_n^2\partial_2 u_n^2\right) dx &\le 2\int_D \left(u_n^1\right)^2\left(\partial_2 u_n^1\right)^2 dx + 2\int_D \left(u_n^2\right)^2\left(\partial_2 u_n^2\right)^2 dx,\end{aligned} \tag{A.10}$$

so, we have that

$$\left\langle |u_n|^2 u_n, Au_n\right\rangle \ge \int_D \left(u_n^1\right)^2\left(\partial_1 u_n^1\right)^2 dx + \int_D \left(u_n^1\right)^2\left(\partial_2 u_n^1\right)^2 dx + \int_D \left(u_n^2\right)^2\left(\partial_1 u_n^2\right)^2 dx + \int_D \left(u_n^2\right)^2\left(\partial_2 u_n^2\right)^2 dx$$

$$+\int_D \left(u_n^2\right)^2\left(\partial_1 u_n^1\right)^2 dx$$
$$+\int_D \left(u_n^2\right)^2\left(\partial_2 u_n^1\right)^2 dx$$
$$+\int_D \left(u_n^1\right)^2\left(\partial_1 u_n^2\right)^2 dx$$
$$+\int_D \left(u_n^1\right)^2\left(\partial_2 u_n^2\right)^2 dx$$
$$=\int_D |u_n|^2|\nabla u_n|^2 dx. \tag{A.11}$$

For the first term on the right hand side of (A.3), we have

$$\begin{aligned}&\langle [(u_n\cdot\nabla)u_n], Av_n(t)\rangle\\ &\quad\le \|v_n(t)\|_{H^2}^2+\frac{1}{4}\int_D \|u_n\|^2\|\nabla u_n\|^2dx.\end{aligned} \tag{A.12}$$

Substitute (A.11) and (A.12) into (A.8), we get

$$\begin{aligned}&\frac{d}{dt}\|v_n(t)\|_{H^1}^2+\|v_n(t)\|_{H^2}^2\\ &\quad\le \langle f(v_n(t)+P_n\phi(t)), A\phi(t)\rangle\\ &\quad=\left\langle A^{1/4}f\left(v_n(t)+P_n\phi(t), A^{3/4}\phi(t)\right)\right\rangle .\end{aligned} \tag{A.13}$$

Denote

$$u_n(t)=v_n(t)+P_n\phi(t). \tag{A.14}$$

Then,

$$\begin{aligned}&\left\langle A^{1/4}f\left(v_n(t)+P_n\phi(t), A^{3/4}\phi(t)\right)\right\rangle\\ &\quad\le \|\phi(t)\|_{H^{3/2}}\cdot\left\|A^{1/4}\left(|u_n(t)|^2u_n(t)\right)\right\|_H\\ &\quad\le \|\phi(t)\|_{H^{3/2}}\\ &\qquad\cdot\left\|\left(A^{1/4}|u_n(t)|^2\right)u_n(t)+|u_n(t)|^2A^{1/4}u_n(t)+R\right\|_H\\ &\quad\le \|\phi(t)\|_{H^{3/2}}\\ &\qquad\cdot\left[\left\|\left(A^{1/4}|u_n(t)|^2\right)u_n(t)\right\|_H\right.\\ &\qquad\left.+\left\||u_n(t)|^2A^{1/4}u_n(t)\right\|_H+\|R\|_H\right]\\ &\quad=\|\phi(t)\|_{H^{3/2}}\cdot[I_1+I_2+I_3],\end{aligned} \tag{A.15}$$

where

$$\begin{aligned}R&=A^{1/4}\left(|u_n(t)|^2u_n(t)\right)\\ &\quad-\left(A^{1/4}|u_n(t)|^2\right)u_n(t)\\ &\quad-|u_n(t)|^2A^{1/4}u_n(t).\end{aligned} \tag{A.16}$$

For I_1, we have

$$I_1\le\left\|\left(u_n(t)A^{1/4}u_n(t)+R_1\right)u_n(t)\right\|_H, \tag{A.17}$$

where

$$R_1=A^{1/4}|u_n(t)|^2-2u_n(t)A^{1/4}u_n(t). \tag{A.18}$$

So,

$$\begin{aligned}I_1&\le C\left\||u_n(t)|^2A^{1/4}u_n(t)+R_1u_n(t)\right\|_H\\ &\le C\|u_n(t)\|_{L^8}^2\|u_n(t)\|_{H^{1/2,4}}\\ &\quad+\|u_n(t)\|_{L^4}\|R_1\|_{L^4}\\ &\le C\|u_n(t)\|_{L^8}^2\|u_n(t)\|_{H^{1/2,4}}\\ &\quad+\|u_n(t)\|_{L^4}\|u_n(t)\|_{H^{1/4,8}}\\ &\le C\|u_n(t)\|_{H^1}^3.\end{aligned} \tag{A.19}$$

Analogously, we can get the same estimate for I_2 and I_3. Take advantage of the estimates for I_1, I_2, and I_3, we have

$$\begin{aligned}&\frac{d}{dt}\|v_n(t)\|_{H^1}^2+\|v_n(t)\|_{H^2}^2\\ &\quad\le C\|\phi(t)\|_{H^{3/2}}\|u_n(t)\|_{H^1}^3\\ &\quad\le C\left(\|v_n(t)\|_{H^1}^3+\|\phi(t)\|_{H^{3/2}}^3\right).\end{aligned} \tag{A.20}$$

By the Gronwall inequality and (A.3), we get (A.6).

Remark A.1. It is standard to show that, for $x\in H_0^1$ and $\phi\in C([0,T];H^{3/2})$, there exists a subsequence which converges to some v, strongly in $L^2([0,T];H^1)$, weekly in $L^2([0,T];H^2)$, and weak star in $L^\infty([0,T];H^1)$. Therefore, we have

$$\begin{aligned}&\|v(t)\|_{C([0,T];H_0^1)}^2+\int_0^T\|v(s)\|_{H^2}^2ds\\ &\quad\le C\left(\|x\|_{H^1},\|\phi\|_{C([0,T];H^{3/2})},T\right).\end{aligned} \tag{A.21}$$

Estimate 3. We compare, only in the case $R=\infty$. Let v_n^1, v_n^2 be two solutions with the same initial condition $x\in H^1$ but with different functions ϕ_1,ϕ_2, there exists a constant $C(\|x\|_{H^1},\|\phi_1\|_{C([0,T];H^{3/2})},\|\phi_2\|_{C([0,T];H^{3/2})},T)$, such that

$$\begin{aligned}&\left\|v_n^1-v_n^2\right\|_{C([0,T];H_0^1)}\\ &\quad\le C\left(\|x\|_{H^1},\|\phi_1\|_{C([0,T];H^{3/2})},\|\phi_2\|_{C([0,T];H^{3/2})},T\right)\\ &\qquad\times\|\phi_1-\phi_2\|_{C([0,T];H^{3/2})},\end{aligned} \tag{A.22}$$

for every $n, x\in H^1, \phi_1,\phi_2, T$. We have

$$\begin{aligned}&\frac{dv_n^i}{dt}+Av_n^i+P_nB\left(v_n^i+P_n\phi_i, v_n^i+P_n\phi_i\right)\\ &\quad+\vartheta\left|v_n^i+P_n\phi_i\right|^2\left(v_n^i+P_n\phi_i\right)=0,\end{aligned} \tag{A.23}$$

with initial condition $v_n^i(0) = P_n x$, for $i = 1, 2$. Set $\eta_n = v_n^1 - v_n^2$, $\psi = \phi_1 - \phi_2$. Then,

$$\begin{aligned}&\frac{d\eta_n}{dt} + A\eta_n + P_n B\left(v_n^1 + P_n\phi_1, \eta_n + P_n\psi\right)\\&\quad + P_n B\left(\eta_n + P_n\psi, v_n^2 + P_n\phi_2\right)\\&\quad + \vartheta\left|v_n^1 + \phi_1\right|^2\left(v_n^1 + \phi_1\right)\\&\quad - \vartheta\left|v_n^2 + \phi_2\right|^2\left(v_n^2 + \phi_2\right) = 0.\end{aligned} \tag{A.24}$$

Take inner product in H with respect to $A\eta_n$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\eta_n\|_{H^1}^2 + \|\eta_n\|_{H^2}^2\\&\quad + \left\langle P_n B\left(\left(v_n^1 + P_n\phi_1\right), \left(\eta_n + P_n\psi\right)\right), A\eta_n\right\rangle\\&\quad + \left\langle P_n B\left(\eta_n + P_n\psi, v_n^2 + P_n\phi_2\right), A\eta_n\right\rangle\\&\quad + \vartheta\left\langle\left|v_n^1 + \phi_1\right|^2\left(v_n^1 + \phi_1\right)\right.\\&\qquad\left. -\left|v_n^2 + \phi_2\right|^2\left(v_n^2 + \phi_2\right), A\eta_n\right\rangle = 0.\end{aligned} \tag{A.25}$$

For the third term on the left hand side of (A.23), we have

$$\begin{aligned}&\left\langle P_n B\left(v_n^1 + P_n\phi_1, \eta_n + P_n\psi\right), A\eta_n\right\rangle\\&\quad \le \|\eta_n\|_{H^2}\|\eta_n + P_n\psi\|_{H^{1,4}}\left\|v_n^1 + P_n\phi_1\right\|_{L^4}\\&\quad \le \|\eta_n\|_{H^2}\left(\|\eta_n\|_{H^{3/2}} + \|\psi\|_{H^{3/2}}\right)\left(\left\|v_n^1 + \phi_1\right\|_{H^1}\right)\\&\quad \le \left\|v_n^1 + \phi_1\right\|_{H^1}\|\eta_n\|_{H^2}\left(\|\eta_n\|_{H^1}^{1/2}\|\eta_n\|_{H^2}^{1/2} + \|\psi\|_{H^{3/2}}\right)\\&\quad \le \varepsilon\|\eta_n\|_{H^2}^2 + C\left\|v_n^1 + \phi_1\right\|_{H^1}^4\|\eta_n\|_{H^1}^2\\&\qquad + C\left\|v_n^1 + \phi_1\right\|_{H^1}^2\|\psi\|_{H^{3/2}}^2.\end{aligned} \tag{A.26}$$

Similarly, we can get

$$\begin{aligned}&\left\langle P_n B\left(\eta_n + P_n\psi, v_n^2 + P_n\phi_2\right), A\eta_n\right\rangle\\&\qquad \le \varepsilon\|\eta_n\|_{H^2}^2 + C\left\|v_n^2\right\|_{H^2}^2\|\eta_n\|_{H^1}^2\\&\qquad\quad + C\left\|v_n^2\right\|_{H^2}^2\|\psi\|_{H^1}^2\\&\qquad\quad + C\|\phi_2\|_{H^{3/2}}^2\|\eta_n\|_{H^1}^2\\&\qquad\quad + C\|\phi_2\|_{H^{3/2}}^2\|\psi\|_{H^1}^2,\\&\vartheta\left\langle\left|v_n^1 + \phi_1\right|^2\left(v_n^1 + \phi_1\right) - \left|v_n^2 + \phi_2\right|^2\left(v_n^2 + \phi_2\right), A\eta_n\right\rangle\\&\quad \le \varepsilon\|\eta_n\|_{H^2}^2 + C\|\eta_n\|_{H^1}^2\left(\left\|v_n^1 + \phi_1\right\|_{H^1}^4 + \left\|v_n^2 + \phi_2\right\|_{H^1}^4\right)\\&\qquad + C\|\psi\|_{H^1}^2\left(\left\|v_n^1 + \phi_1\right\|_{H^1}^4 + \left\|v_n^2 + \phi_2\right\|_{H^1}^4\right).\end{aligned} \tag{A.27}$$

(A.28)

By (A.23)–(A.27), we have

$$\begin{aligned}&\frac{d}{dt}\|\eta_n\|_{H^1}^2 + \|\eta_n\|_{H^2}^2\\&\quad \le C\|\eta_n\|_{H^1}^2\left(\left\|v_n^1 + \phi_1\right\|_{H^1}^4 + \left\|v_n^2 + \phi_2\right\|_{H^1}^4\right.\\&\qquad\qquad\left. + \left\|v_n^1 + \phi_1\right\|_{H^1}^2 + \left\|v_n^2\right\|_{H^2}^2 + \|\phi_2\|_{H^{3/2}}^2\right)\\&\qquad + C\|\psi\|_{H^{3/2}}^2\left(\left\|v_n^1 + \phi_1\right\|_{H^1}^4 + \left\|v_n^2 + \phi_2\right\|_{H^1}^4\right.\\&\qquad\qquad\left. + \left\|v_n^1 + \phi_1\right\|_{H^1}^2 + \left\|v_n^2\right\|_{H^2}^2 + \|\phi_2\|_{H^{3/2}}^2\right).\end{aligned} \tag{A.29}$$

So, by the Gronwall inequality and (A.6), we get (A.21). By (A.6), we know that v_n^i converges week star to v^i in $C([0,T]; H_0^1)$, for $i = 1, 2$, we have

$$\begin{aligned}&\left\|v^1 - v^2\right\|_{C([0,T];H^1)}\\&\quad \le C\left(\|x\|_{H^1}, \|\phi_1\|_{C([0,T];H^{3/2})}, \|\phi_2\|_{C([0,T];H^{3/2})}, T\right)\\&\qquad \|\phi_1 - \phi_2\|_{C([0,T];H^{3/2})}.\end{aligned} \tag{A.30}$$

Estimate 4. Let us consider only the case $R \in (0, \infty)$, and denote by $v_n(t)$ the solution to (A.2). Let ξ_n be the differential mapping $x \to v_n$ in the direction h at point x, defined by, for given $x, h \in H$ as follows:

$$\xi_n(t) = D_x v_n(t; x)\cdot h. \tag{A.31}$$

Set also

$$u_n(t; x) = v_n(t, x) + P_n\phi(t), \tag{A.32}$$

so that ξ_n is also the differential of the mapping $x \to u_n(t; x)$ in the direction h at the point x. Thus, ξ_n satisfies

$$\begin{aligned}&\frac{d}{dt}\xi_n + A\xi_n\\&\quad = 2K_R'\left(\|u_n\|_{H^1}^2\right)\left\langle A^{1/2}u_n, A^{1/2}\xi_n\right\rangle B(u_n, u_n)\\&\qquad + K_R\left(\|u_n\|_{H^1}^2\right)\{B(u_n, \xi_n) + B(\xi_n, u_n)\}\\&\qquad + 2K_R'\left(\|u_n\|_{H^1}^2\right)\left\langle A^{1/2}u_n, A^{1/2}\xi_n\right\rangle u_n^3\\&\qquad + 3K_R\left(\|u_n\|_{H^1}^2\right)|u_n|^2\xi_n.\end{aligned} \tag{A.33}$$

So,

$$\begin{aligned}&\frac{d}{dt}\|\xi_n\|_{H^1}^2 + \|\xi_n\|_{H^2}^2\\&\quad = 2K_R'\left(\|u_n\|_H^2\right)\langle B(u_n, u_n), A\xi_n\rangle\\&\qquad + K_R\left(\|u_n\|_H^2\right)\langle B(u_n, \xi_n), A\xi_n\rangle\\&\qquad + K_R\left(\|u_n\|_H^2\right)\langle B(\xi_n, u_n), A\xi_n\rangle\\&\qquad + 2K_R'\left(\|u_n\|_H^2\right)\left\langle A^{1/2}u_n, A^{1/2}\xi_n\right\rangle\left\langle|u_n|^2 u_n, A\xi_n\right\rangle\\&\qquad + 3K_R\left(\|u_n\|_H^2\right)\left\langle|u_n|^2\xi_n, A\xi_n\right\rangle.\end{aligned} \tag{A.34}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\|\xi_n\|_{H^1}^2 + \|\xi_n\|_{H^2}^2 \\
&\leq 2K_R'\left(\|u_n\|_H^2\right)\|u_n\|_{H^1}\|\xi_n\|_{H^1}\|\xi_n\|_{H^2}\|u_n\|_{L^4}\|u_n\|_{H^{1,4}} \\
&\quad + K_R\left(\|u_n\|_{H^1}^2\right)\|\xi_n\|_{H^2}\|\xi_n\|_{H^{1,4}}\|u_n\|_{L^4} \\
&\quad + K_R\left(\|u_n\|_{H^1}^2\right)\|\xi_n\|_{H^2}\|\xi_n\|_{L^4}\|u_n\|_{H^{1,4}} \\
&\quad + 2K_R'\left(\|u_n\|_{H^1}^2\right)\|u_n\|_{H^1}\|\xi_n\|_{H^1}\|\xi_n\|_{H^2}\|u_n\|_{L^6}^3 \\
&\quad + 3K_R\left(\|u_n\|_{H^1}^2\right)\|\xi_n\|_{H^2}\|\xi_n\|_{L^4}\|u_n\|_{L^8}^2 \\
&\leq C(R)\|\xi_n\|_{H^1}\|\xi_n\|_{H^2}\left(\|v_n\|_{H^2} + \|\phi\|_{H^{3/2}}\right) \\
&\quad + C(R)\|\xi_n\|_{H^2}^{3/2}\|\xi_n\|_{H^1}^{1/2} \\
&\quad + C(R)\|\xi_n\|_{H^2}\|\xi_n\|_{H^1} \\
&\leq \varepsilon\|\xi_n\|_{H^2}^2 + C(R)\|\xi_n\|_{H^1}^2 \\
&\quad \times\left(1 + \|v_n\|_{H^2}^2 + \|\phi\|_{H^{3/2}}^2\right).
\end{aligned} \tag{A.35}$$

By the Gronwall inequality and (A.6), we have

$$\|\xi_n(t)\|_{H^1}^2 \leq C(R)\|h\|_{H^1}^2. \tag{A.36}$$

And therefore, using again the previous inequality,

$$\int_0^T \|\xi_n(t)\|_{H^2}^2 dt \leq C(R)\|h\|_{H^1}^2. \tag{A.37}$$

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