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Research Article

Persistence and Nonpersistence of a Nonautonomous Stochastic Mutualism System

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In this paper, a two-species nonautonomous stochastic mutualism system is investigated. The intrinsic growth rates of the two species at time t are estimated by $r_i(t) + \sigma_i(t)\dot{B}_i(t),\ i = 1, 2$, respectively. Viewing the different intensities of the noises $\sigma_i(t), i = 1, 2$ as two parameters at time t, we conclude that there exists a global positive solution and the pth moment of the solution is bounded. We also show that the system is permanent, including stochastic permanence, persistence in mean, and asymptotic boundedness in time average. Besides, we show that the large white noise will make the system nonpersistent. Finally, we establish sufficient criteria for the global attractivity of the system.

1. Introduction

For more than three decades, mutualism of multispecies has attracted the attention of both mathematicians and ecologists. By definition, in a mutualism of multispecies, the interaction is beneficial for the growth of other species. Lotka-Volterra mutualism systems have long been used as standard models to mathematically address questions related to this interaction. Among these, nonautonomous Lotka-Volterra mutualism models are studied by many authors, see [1–7] and references therein. The classical nonautonomous Lotka-Volterra mutualism system can be expressed as follows:

$$\dot{x}_i(t) = x_i(t)\left[r_i(t) - a_{ii}(t)x_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}(t)x_j(t)\right], \quad i = 1, 2, \ldots, n, \tag{1}$$

where $x_i(t),\ i = 1, 2, \ldots, n$ is the density of the ith population at time t, $r_i(t) > 0,\ i = 1, 2, \ldots, n$ is the intrinsic growth rate of the ith population at time t, $r_i(t)/a_{ii}(t) > 0,\ i = 1, 2, \ldots, n$ is the carrying capacity at time t, and coefficient $a_{ij}(t) > 0,\ i, j = 1, 2, \ldots, n$ describes the influence of the jth population upon the ith population at time t.

It is shown in [1] that if different conditions hold (see conditions (a)–(e) in [1]), then the solution of system (1) is bounded, permanent, extinct, and global attractive, respectively. However, when the intrinsic growth rate and coefficient $a_{ij}(t)$ are periodic, it is shown in [3] that there exists positive periodic solution and almost periodic solutions are obtained.

From another point of view, environmental noise always exists in real life. It is an interesting problem, both mathematically and biologically, to determine how the structure of the model changes under the effect of a fluctuating environment. Many authors studied the biological models with stochastic perturbation, see [8–12] and references therein. In [8] Ji et al. discussed the following two-species stochastic mutualism system

$$\begin{aligned} dx_1(t) &= x_1(t)\left[\left(r_1 - a_{11}x_1(t) + a_{12}x_2(t)\right)dt + \sigma_1 dB_1(t)\right], \\ dx_2(t) &= x_2(t)\left[\left(r_2 + a_{21}x_1(t) - a_{22}x_2(t)\right)dt + \sigma_2 dB_2(t)\right], \end{aligned} \tag{2}$$

where $B_i(t),\ i = 1, 2$ are mutually independent one dimensional standard Brownian motions with $B_i(0) = 0,\ i = 1, 2$, and $\sigma_i,\ i = 1, 2$ are the intensities of white noise. It is shown in [8] that if $a_{11}a_{22} > a_{12}a_{21}$ then there is a unique nonnegative solution of system (2). For small white noise there is a stationary distribution of (2) and it has ergodic property.

Biologically, this implies that with small perturbation of environment, the stability of the two species varies with the intensity of white noise, and both species will survive.

However, almost all known stochastic models assume that the growth rate and the carrying capacity of the population are independent of time t. In contrast, the natural growth rates of many populations vary with t in real situation, for example, due to the seasonality. As a matter of fact, nonautonomous stochastic population systems have recently been studied by many authors, for example, [13–17].

In this paper we consider the system

$$\begin{aligned} dx_1(t) &= x_1(t)\left[\left(r_1(t) - a_{11}(t)x_1(t) + a_{12}(t)x_2(t)\right)dt + \sigma_1(t)\,dB_1(t)\right],\\ dx_2(t) &= x_2(t)\left[\left(r_2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t)\right)dt + \sigma_2(t)\,dB_2(t)\right], \end{aligned} \tag{3}$$

where $r_i(t)$, $a_{ij}(t)$, $\sigma_i(t)$, $i, j = 1, 2$ are all continuous bounded nonnegative functions on $[0, +\infty)$. The objective of our study is to investigate the long-time behavior of system (3). As in [8], we mainly discuss when the system is persistent and when it is not under a fewer conditions. More specifically, we show that there is a positive solution of system (3) and its pth moment bounded in Section 2. In Section 3, we deduce the persistence of the system. If the white noise is not large such that $r_i^l - ((\sigma_i^u)^2/2) > 0$, $i = 1, 2$, we will prove that the solution of system (3) is a stochastic persistence. In addition, we show that every component of the solution is persistent in mean. We further deduce that every component of the solution of system (3) is an asymptotic boundedness in mean. In Section 4, we show that larger white noise will make system (3) nonpersistent. Finally, we study the global attractivity of system (3).

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Let R_+^2 be the positive cone of R^2, namely, $R_+^2 = \{x \in R^2 : x_i > 0,\ i = 1, 2\}$. If $x \in R^n$, its norm is denoted by $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. If $f(t)$ is a continuous bounded function on $[0, +\infty)$, we use the notation sup

$$f^u = \sup_{t\in[0,+\infty)} f(t), \qquad f^l = \min_{t\in[0,+\infty)} f(t). \tag{4}$$

2. Existence and Uniqueness of the Positive Solution

In population dynamics, the first concern is that the solution should be nonnegative. In order to do that a stochastic differential equation can have a unique global (i.e., no explosion at any finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (Mao [18]). However, the coefficients of system (3) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of system (3) may explode at a finite time. Following the way developed by Mao et al. [19], we show that there is a unique positive solution of (3).

Theorem 1. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$. Then, there is a unique positive solution $x(t) = (x_1(t), x_2(t))$ of system (3) on $t \geq 0$ for any given initial value $x(0) \in R_+^2$, and the solution will remain in R_+^2 with probability 1, namely, $x(t) \in R_+^2$ for all $t \geq 0$ almost surely.*

The proof of Theorem 1 is similar to [8]. But it is skilled in taking the value of ϵ. We show it here.

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial value $x(0) \in R_+^2$ there is an unique local solution $x(t) = (x_1(t), x_2(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $m_0 > 1$ be sufficiently large for every component of $x(0)$ lying within the interval $[1/m_0, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\tau_m = \inf\left\{t \in [0, \tau_e) : \min\{x_1(t), x_2(t)\} \leq \frac{1}{m} \text{ or } \max\{x_1(t), x_2(t)\} \geq m\right\}, \tag{5}$$

where throughout this paper we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty}\tau_m$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R_+^2$ a.s. for all $t \geq 0$. In other words, to complete the proof, all we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, there is a pair of constant $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon. \tag{6}$$

Hence, there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon \quad \forall m \geq m_1. \tag{7}$$

We define

$$V(x) = a_{21}^u(x_1 - 1 - \log x_1) + a_{12}^u(x_2 - 1 - \log x_2). \tag{8}$$

By Itô's formula, we have

$$\begin{aligned} dV(x) = {} & \left\{a_{21}^u\left(1 - \frac{1}{x_1}\right)x_1\left[r_1(t) - a_{11}(t)x_1 + a_{12}(t)x_2\right]\right.\\ & + a_{12}^u\left(1 - \frac{1}{x_2}\right)x_2\left[r_2(t) + a_{21}(t)x_1 - a_{22}(t)x_2\right]\\ & \left. + \frac{1}{2}\left[a_{21}^u\sigma_1^2(t) + a_{12}^u\sigma_2^2(t)\right]\right\}dt \end{aligned}$$

$$\begin{aligned} &+ a_{21}^u \sigma_1(t)(x_1 - 1)\, dB_1(t) \\ &+ a_{12}^u \sigma_2(t)(x_2 - 1)\, dB_2(t) \\ := {}& LV dt + a_{21}^u \sigma_1(t)(x_1 - 1)\, dB_1(t) \\ &+ a_{12}^u \sigma_2(t)(x_2 - 1)\, dB_2(t), \end{aligned} \tag{9}$$

where

$$\begin{aligned} LV = {}& a_{21}^u \left(1 - \frac{1}{x_1}\right) x_1 \left[r_1(t) - a_{11}(t) x_1 + a_{12}(t) x_2\right] \\ &+ a_{12}^u \left(1 - \frac{1}{x_2}\right) x_2 \left[r_2(t) + a_{21}(t) x_1 - a_{22}(t) x_2\right] \\ &+ \frac{1}{2}\left[a_{21}^u \sigma_1^2(t) + a_{12}^u \sigma_2^2(t)\right] \\ \le {}& a_{21}^u \Big[(r_1^u + a_{11}^u) x_1 - a_{12}^l x_2 - a_{11}^l x_1^2 \\ &\quad + a_{12}^u x_1 x_2 - r_1^l + \frac{1}{2}(\sigma_1^u)^2 \Big] \\ &+ a_{12}^u \Big[(r_2^u + a_{22}^u) x_2 - a_{21}^l x_1 - a_{22}^l x_2^2 \\ &\quad + a_{21}^u x_1 x_2 - r_2^l + \frac{1}{2}(\sigma_2^u)^2 \Big]. \end{aligned} \tag{10}$$

According to Young inequality, note that $x_1 x_2 \le \epsilon x_1^2 + (1/4\epsilon) x_2^2$, where $a_{21}^u/2a_{22}^l < \epsilon < a_{11}^l/2a_{12}^u$, then,

$$\begin{aligned} LV \le {}& a_{21}^u \Big[(r_1^u + a_{11}^u) x_1 - a_{12}^l x_2 - a_{11}^l x_1^2 \\ &\quad + a_{12}^u \left(\epsilon x_1^2 + \frac{1}{4\epsilon} x_2^2\right) - r_1^l + \frac{1}{2}(\sigma_1^u)^2 \Big] \\ &+ a_{12}^u \Big[(r_2^u + a_{22}^u) x_2 - a_{21}^l x_1 - a_{22}^l x_2^2 \\ &\quad + a_{21}^u \left(\epsilon x_1^2 + \frac{1}{4\epsilon} x_2^2\right) - r_2^l + \frac{1}{2}(\sigma_2^u)^2 \Big] \\ = {}& -\left(a_{21}^u a_{11}^l - 2\epsilon a_{21}^u a_{12}^u\right) x_1^2 \\ &+ \left[a_{21}^u (r_1^u + a_{11}^u) - a_{12}^u a_{21}^l\right] x_1 \\ &- a_{21}^u r_1^l + \frac{1}{2} a_{21}^u (\sigma_1^u)^2 \\ &- \left(a_{12}^u a_{22}^l - \frac{1}{2\epsilon} a_{21}^u a_{12}^u\right) x_2^2 \\ &+ \left[a_{12}^u (r_2^u + a_{22}^u) - a_{21}^u a_{12}^l\right] x_2 \\ &- a_{12}^u r_2^l + \frac{1}{2} a_{12}^u (\sigma_2^u)^2 \\ \le {}& K. \end{aligned} \tag{11}$$

Since $a_{21}^u/2a_{22}^l < \epsilon < a_{11}^l/2a_{12}^u$, we obtain $-(a_{21}^u a_{11}^l - 2\epsilon a_{12}^u) < 0$ and $-(a_{12}^u a_{22}^l - (1/2\epsilon) a_{21}^u a_{12}^u) < 0$. Hence, K is a positive constant. Integrating both sides of (9) from 0 to $\tau_m \wedge T$, we therefore obtain

$$\begin{aligned} & V(x(\tau_m \wedge T)) - V(x(0)) \\ &\quad \le \int_0^{\tau_m \wedge T} K dt + \int_0^{\tau_m \wedge T} a_{21}^u \sigma_1(t)(x_1(t) - 1)\, dB_1(t) \\ &\qquad + \int_0^{\tau_m \wedge T} a_{12}^u \sigma_2(t)(x_2(t) - 1)\, dB_2(t). \end{aligned} \tag{12}$$

Whence, taking expectations yields

$$\begin{aligned} E\left[V(x(\tau_m \wedge T))\right] &\le V(x(0)) + KE(\tau_m \wedge T) \\ &\le V(x(0)) + KT. \end{aligned} \tag{13}$$

Set $\Omega_m = \{\tau_m \le T\}$ for $m \ge m_1$ and by (7), $P(\Omega_m) \ge \varepsilon$. Note that for every $\omega \in \Omega_m$, there is $x_1(\tau_m, \omega)$ or $x_2(\tau_m, \omega)$ equals either m or $1/m$, and therefore

$$\begin{aligned} & V(x(\tau_m, \omega)) \\ &\quad \ge \min\{a_{21}^u, a_{12}^u\}(m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 - \log\frac{1}{m}\right) \\ &\quad := h(m), \end{aligned} \tag{14}$$

where $\lim_{m\to\infty} h(m) = \infty$. It then follows from (13) that

$$E[V(x(0))] + KT \ge E\left[1_{\Omega_m} \cdot V(x(\tau_m, \omega))\right] \ge \epsilon h(m), \tag{15}$$

where 1_{Ω_m} is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction

$$\infty > V(x(0)) + KT = \infty, \tag{16}$$

so we must have $\tau_\infty = \infty$ a.s. This completes the proof of Theorem 1. □

Remark 2. By Theorem 1, we observe that for any given initial value $x(0) \in R_+^2$, there is a unique solution $x(t) = (x_1(t), x_2(t))$ of system (3) on $t \ge 0$ and the solution will remain in R_+^2 with probability 1, no matter how large the intensities of white noise are. So, under the same assumption there is an global unique positive solution of the corresponding deterministic system of system (3).

Next, we show that the pth moment of the solution of system (3) is bounded in time average.

Theorem 3. *Assume that* $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$. *Then there exists a positive constant* $K(p)$ *such that the solution* $x(t)$ *of system* (3) *has the following property:*

$$E\left[c_1 x_1^p(t) + c_2 x_2^p(t)\right] \le K(p), \quad \forall t \in [0,\infty), \; p > 1, \tag{17}$$

where c_1, c_2 *satisfy*

$$\frac{(a_{21}^u)^{p+1}}{a_{11}^l (a_{22}^l)^p} < \frac{c_1}{c_2} < \frac{a_{22}^l (a_{11}^l)^p}{(a_{12}^u)^{p+1}}. \tag{18}$$

Proof. By Itô's formula, we have

$$\begin{aligned} dx_1^p(t) &= p x_1^p(t)\left[\left(r_1(t) - a_{11}(t)x_1(t) + a_{12}(t)x_2(t)\right)dt + \sigma_1(t)\,dB_1(t)\right] \\ &\quad + \frac{1}{2}p(p-1)x_1^p(t)\sigma_1^2(t)\,dt \\ &= p\left[\left(r_1(t) + \frac{p-1}{2}\sigma_1^2(t)\right)x_1^p(t) - a_{11}(t)x_1^{p+1}(t) + a_{12}(t)x_1^p(t)x_2(t)\right]dt \\ &\quad + \sigma_1(t)\,p x_1^p(t)\,dB_1(t) \\ &= p\left[\alpha_1(t)x_1^p(t) - a_{11}(t)x_1^{p+1}(t) + a_{12}(t)x_1^p(t)x_2(t)\right]dt \\ &\quad + \sigma_1(t)\,p x_1^p(t)\,dB_1(t) \\ &\le p\left[\alpha_1^u x_1^p(t) - a_{11}^l x_1^{p+1}(t) + a_{12}^u x_1^p(t)x_2(t)\right]dt \\ &\quad + p\sigma_1^u x_1^p(t)\,dB_1(t), \end{aligned} \tag{19}$$

where $\alpha_1(t) = r_1(t) + ((p-1)/2)\sigma_1^2(t)$, and

$$\begin{aligned} dx_2^p(t) &= p x_2^p(t)\left[\left(r_2(t) - a_{22}(t)x_2(t) + a_{21}(t)x_1(t)\right)dt + \sigma_2(t)\,dB_2(t)\right] \\ &\quad + \frac{1}{2}p(p-1)x_2^p(t)\sigma_2^2(t)\,dt \\ &= p\left[\left(r_2(t) + \frac{p-1}{2}\sigma_2^2(t)\right)x_2^p(t) - a_{22}(t)x_2^{p+1}(t) + a_{21}(t)x_2^p(t)x_1(t)\right]dt \\ &\quad + \sigma_2(t)\,p x_2^p(t)\,dB_2(t) \\ &= p\left[\alpha_2(t)x_2^p(t) - a_{22}(t)x_2^{p+1}(t) + a_{21}(t)x_2^p(t)x_1(t)\right]dt + \sigma_2(t)\,p x_2^p(t)\,dB_2(t) \\ &\le p\left[\alpha_2^u x_2^p(t) - a_{22}^l x_2^{p+1}(t) + a_{21}^u x_2^p(t)x_1(t)\right]dt \\ &\quad + p\sigma_2^u x_2^p(t)\,dB_2(t), \end{aligned} \tag{20}$$

where $\alpha_2(t) = r_2(t) + ((p-1)/2)\sigma_2^2(t)$. According to Young inequality, we obtain

$$\begin{aligned} x_1^p(t)x_2(t) &\le \epsilon_1 x_1^{p+1}(t) + \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p\left(\frac{1}{\epsilon_1}\right)^p x_2^{p+1}(t), \\ \epsilon_1 &= \frac{p a_{11}^l}{(p+1)a_{12}^u}, \\ x_2^p(t)x_1(t) &\le \epsilon_2 x_2^{p+1}(t) + \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p\left(\frac{1}{\epsilon_2}\right)^p x_1^{p+1}(t), \\ \epsilon_2 &= \frac{p a_{22}^l}{(p+1)a_{21}^u}. \end{aligned} \tag{21}$$

Thus, we have

$$\begin{aligned} dx_1^p(t) &\le p\left[\alpha_1^u x_1^p(t) - a_{11}^l x_1^{p+1}(t) + a_{12}^u \epsilon_1 x_1^{p+1}(t) + a_{12}^u \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p\left(\frac{1}{\epsilon_1}\right)^p x_2^{p+1}(t)\right]dt \\ &\quad + p\sigma_1^u x_1^p(t)\,dB_1(t), \\ dx_2^p(t) &\le p\left[\alpha_2^u x_2^p(t) - a_{22}^l x_2^{p+1}(t) + a_{21}^u \epsilon_2 x_2^{p+1}(t) + a_{21}^u \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p\left(\frac{1}{\epsilon_2}\right)^p x_1^{p+1}(t)\right]dt \\ &\quad + p\sigma_2^u x_2^p(t)\,dB_2(t). \end{aligned} \tag{22}$$

Since $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, there exist two positive constants c_1, c_2 which satisfy

$$\frac{(a_{21}^u)^{p+1}}{a_{11}^l (a_{22}^l)^p} < \frac{c_1}{c_2} < \frac{a_{22}^l (a_{11}^l)^p}{(a_{12}^u)^{p+1}}. \tag{23}$$

Therefore,

$$\begin{aligned} &d\left(c_1 x_1^p(t) + c_2 x_2^p(t)\right) \\ &\quad \le -p\left[\left(c_1 a_{11}^l - c_1 a_{12}^u \epsilon_1 - c_2 a_{21}^u \frac{p^p}{(p+1)^{p+1}\epsilon_2^p}\right)x_1^{p+1}(t) \right. \\ &\qquad + \left(c_2 a_{22}^l - c_1 a_{21}^u \epsilon_2 - c_1 a_{12}^u \frac{p^p}{(p+1)^{p+1}\epsilon_1^p}\right)x_2^{p+1}(t) \\ &\qquad \left. - \sum_{i=1}^{2} c_i \alpha_i^u x_i^p(t)\right]dt + \sum_{i=1}^{2} c_i p \sigma_i^u x_i^p(t)\,dB_i(t). \end{aligned} \tag{24}$$

From (23) and the values of ϵ_1, ϵ_2, we obtain

$$\frac{a_{21}^u\left(p^p/(p+1)^{p+1}\epsilon_2^p\right)}{a_{11}^l - a_{12}^u \epsilon_1} < \frac{c_1}{c_2} < \frac{a_{22}^l - a_{21}^u \epsilon_2}{a_{12}^u\left(p^p/(p+1)^{p+1}\epsilon_1^p\right)}, \tag{25}$$

which implies that $c_1a_{11}^l - c_1a_{12}^u\epsilon_1 - c_2a_{21}^u(p^p/(p+1)^{p+1}\epsilon_2^p) > 0$ and $c_2a_{22}^l - c_1a_{21}^u\epsilon_2 - c_1a_{12}^u(p^p/((p+1)^{p+1}\epsilon_1^p)) > 0$. Let

$$\alpha = \max\{\alpha_1^u, \alpha_2^u\},$$
$$\beta = \min\left\{c_1^{-(p+1)/p}\left[c_1a_{11}^l - c_1a_{12}^u\epsilon_1 - c_2a_{21}^u\frac{p^p}{(p+1)^{p+1}\epsilon_2^p}\right],\right.$$
$$\left. c_2^{-(p+1)/p}\left[c_2a_{22}^l - c_1a_{21}^u\epsilon_2 - c_1a_{12}^u\frac{p^p}{(p+1)^{p+1}\epsilon_1^p}\right]\right\}, \tag{26}$$

then we have

$$d\left(c_1x_1^p(t) + c_2x_2^p(t)\right) \le p\left[\alpha\left(\sum_{i=1}^{2}c_ix_i^p(t)\right) - \beta\left(\sum_{i=1}^{2}c_i^{1+(1/p)}x_i^{p+1}\right)\right]dt + \sum_{i=1}^{2}c_ip\sigma_i^ux_i^p(t)\,dB_i(t). \tag{27}$$

Hence, we get

$$\begin{aligned}\frac{dE\left[c_1x_1^p(t) + c_2x_2^p(t)\right]}{dt} &\le p\alpha E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] - p\beta E\left[c_1^{1+(1/p)}x_1^{p+1}(t) + c_2^{1+(1/p)}x_2^{p+1}(t)\right]\\ &\le p\alpha E\left[c_1x_1^p(t)+c_2x_2^p(t)\right] - p\beta\left\{\left[E\left(c_1x_1^p(t)\right)\right]^{1+(1/p)} + \left[E\left(c_2x_2^p(t)\right)\right]^{1+(1/p)}\right\}\\ &\le p\alpha E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] - p\beta\cdot 2^{-1/p}\left[E\left(c_1x_1^p(t) + c_2x_2^p(t)\right)\right]^{1+(1/p)}.\end{aligned} \tag{28}$$

By the comparison theorem, we get

$$\limsup_{t\to\infty} E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le 2\left(\frac{\alpha}{\beta}\right)^p := M(p), \tag{29}$$

which implies that there is a $T_0 > 0$, such that

$$E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le 2M(p), \quad \forall t > T_0. \tag{30}$$

Besides, note that $E[c_1x_1^p(t) + c_2x_2^p(t)]$ is continuous, then there is a $\widetilde{M}(p) > 0$ such that

$$E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le \widetilde{M}(p), \quad \forall t \in [0, T_0]. \tag{31}$$

Let $K(p) = \max\{2M(p), \widetilde{M}(p)\}$, then

$$E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le K(p), \quad \forall t \in [0, \infty). \tag{32}$$

□

3. Persistence

Theorem 1 shows that the solution of system (3) will remain in the positive cone R_+^2 if $a_{11}^la_{22}^l > a_{12}^ua_{21}^u$. Studying a population system, we pay more attention on whether the system is persistent. In this section, we first show that the solution is a stochastic permanence. Next we show that the solution is persistent in time average. Moreover, we show that the solution $x(t)$ of system (3) is an asymptotic boundedness in time average.

3.1. Stochastic Permanence. Let $y(t)$ be the solution of a randomized nonautonomous competitive equation:

$$dy_i(t) = y_i(t)\left[\left(b_i(t) - \sum_{j=1}^{n}a_{ij}(t)\,y_j(t)\right)dt + \sigma_i(t)\,dB_i(t)\right], \quad i = 1, 2, \ldots, n, \tag{33}$$

where $B_i(t)$, $i = 1, 2, \ldots, n$, are independent standard Brownian motions, $y(0) = y_0 > 0$ while y_0 is independent of $B(t)$, and $b_i(t)$, $a_{ij}(t)$, $\sigma_i(t)$ are all continuous bounded nonnegative functions on $[0, +\infty)$.

Lemma 4 (see [15]). *Assume that $b_i^l - ((\sigma_i^u)^2/2) > 0$, $i = 1, 2, \ldots, n$, then for any given initial value $y(0) \in R_+^n$, the solution $y(t)$ of (36) has the properties*

$$\limsup_{t\to\infty} E\left(\frac{1}{|y(t)|^\theta}\right) \le H, \tag{34}$$

where H is a constant, θ is an arbitrary positive constant satisfying

$$\theta \max_{1\le i\le n}(\sigma_i^u)^2 < 2\min_{1\le i\le n}\left(b_i(t) - \frac{\sigma_i^2(t)}{2}\right)^l. \tag{35}$$

Let $N(t)$ be the solution of a randomized nonautonomous logistic equation

$$dN(t) = N(t)\left[(a(t) - b(t)N(t))\,dt + \alpha(t)\,dB(t)\right], \tag{36}$$

where $B(t)$ is a 1-dimensional standard Brownian motion, $N(0) = N_0 > 0$, and N_0 is independent of $B(t)$.

Lemma 5 (see [13]). *Assume that $a(t)$, $b(t)$, and $\alpha(t)$ are bounded continuous functions defined on $[0, \infty)$, $a(t) > 0$ and $b(t) > 0$. Then there exists a unique continuous positive* *solution of* (36) *for any initial value* $N(0) = N_0 > 0$, *which is global and represented by*

$$N(t) = \exp\left\{\int_0^t \left[a(s) - \left(\frac{\alpha^2(s)}{2}\right)\right] ds + \alpha(s)\, dB(s)\right\} \times \left(\left(\frac{1}{N_0}\right) + \int_0^t b(s) \exp\left\{\int_0^s \left[a(\tau) - \left(\frac{\alpha^2(\tau)}{2}\right)\right] d\tau + \alpha(\tau)\, dB(\tau)\right\} ds\right)^{-1}, \quad t \geq 0. \tag{37}$$

From Lemma 4 we have the following.

Lemma 6. *Assume that* $a^l - ((\alpha^u)^2/2) > 0$, *then for any given initial value* $N(0) \in R_+$, *the solution* $N(t)$ *of* (36) *has the properties*

$$\limsup_{t\to\infty} E\left(\frac{1}{N^\theta(t)}\right) \leq H, \tag{38}$$

where H *is a constant,* θ *is positive constant satisfying*

$$\theta(\alpha^u)^2 < 2\left[a^l - \frac{(\alpha^u)^2}{2}\right]. \tag{39}$$

Let $\phi(t) = (\phi_1(t), \phi_2(t))^T$ be the solution of

$$d\phi_i(t) = \phi_i(t)\left[(r_i(t) - a_{ii}(t)\phi_i(t))\, dt + \sigma_i(t)\, dB_i(t)\right], \quad i = 1, 2, \tag{40}$$

where $B_i(t)$, $i = 1, 2$, are independent standard Brownian motions, $\phi(0) = \phi_0 > 0$, and $\phi_0 \in R_+^2$, $r_i(t), a_{ii}(t), \sigma_i(t)$, $i = 1, 2$ are all continuous bounded nonnegative functions on $[0, +\infty)$. From Lemma 4 it is easy to know the following.

Lemma 7. *Assume that* $\overline{r}_i^l = r_i^l - ((\sigma_i^u)^2/2) > 0$, $i = 1, 2$, *then for any given initial value* $\phi(0) \in R_+^2$, *the solution* $\phi(t)$ *of* (40) *has the properties*

$$\limsup_{t\to\infty} E\left(\frac{1}{\phi_i^\theta(t)}\right) \leq H_i, \quad i = 1, 2, \tag{41}$$

where H_i, $i = 1, 2$ *are two constants,* θ *is positive constant satisfying*

$$\theta(\sigma_i^u)^2 < 2\overline{r}_i^l, \quad i = 1, 2. \tag{42}$$

Lemma 8. *Assume that* $\overline{r}_i^l > 0$, $i = 1, 2$, *then for any given initial value* $x_0 \in R_+^2$, *the solution* $x(t)$ *of system* (3) *has the properties*

$$x_i(t) \geq \phi_i(t), \quad i = 1, 2, \tag{43}$$

$$\limsup_{t\to\infty} E\left(\frac{1}{x_i^\theta(t)}\right) \leq H_i, \quad i = 1, 2, \tag{44}$$

where H_i, $i = 1, 2$ *are two constants,* θ *is positive constant satisfying*

$$\theta(\sigma_i^u)^2 < 2\overline{r}_i^l, \quad i = 1, 2. \tag{45}$$

Proof. Equation (43) follows directly from the classical comparison theorem of stochastic differential equations (see [20]). Thus, we obtain

$$\limsup_{t\to\infty} E\left(\frac{1}{x_i^\theta(t)}\right) \leq \limsup_{t\to\infty} E\left(\frac{1}{\phi_i^\theta(t)}\right) \leq H_i, \quad i = 1, 2. \tag{46}$$

□

Definition 9. System (3) is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there exists a pair of positive constants $\delta = \delta(\epsilon)$ and $M = M(\epsilon)$ such that for any initial value $x_0 \in R_+^2$, the solution obeys

$$\begin{gathered} \liminf_{t\to\infty} P\{x_i(t) \leq M(\epsilon)\} \geq 1 - \epsilon, \\ \liminf_{t\to\infty} P\{x_i(t) \geq \delta(\epsilon)\} \geq 1 - \epsilon, \quad i = 1, 2. \end{gathered} \tag{47}$$

Theorem 10. *Assume that* $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, $\overline{r}_i^l > 0$, $i = 1, 2$, *then system* (3) *is stochastically permanent.*

The proof is a simple application of the Chebyshev inequality, we omit it.

3.2. Persistence in Time Average. Theorem 10 shows that if the white noise is not large, the solution of system (3) is survive with large probability. In this part, we show $x(t)$ is persistence in mean.

Lemma 11. *Assume that* $\overline{r}_i^l > 0$, $i = 1, 2$, *then for any given initial value* $\phi(0) \in R_+^2$, *the solution* $\phi(t)$ *of* (40) *has the properties*

$$z_i(t)\, e^{-[\max_{0\leq s\leq t}\int_0^s \sigma_i(\tau)dB_i(\tau) - \int_0^t \sigma_i(\tau)dB_i(\tau)]} \leq \phi_i(t) \leq z_i(t)\, e^{-[\min_{0\leq s\leq t}\int_0^s \sigma_i(\tau)dB_i(\tau) - \int_0^t \sigma_i(\tau)dB_i(\tau)]}, \quad i = 1, 2, \tag{48}$$

where $z(t) = (z_1(t), z_2(t))$ *is the solution of*

$$\begin{gathered} dz_i(t) = z_i(t)\left[r_i(t) - \frac{\sigma_i^2(t)}{2} - a_{ii}^u z_i(t)\right] dt, \\ z_i(0) = \phi_i(0), \; i = 1, 2. \end{gathered} \tag{49}$$

Proof. From Lemma 5, we know

$$\begin{aligned}
\frac{1}{\phi_i(t)} &= \frac{1}{\phi_i(0)} e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds+\sigma_i(s)dB_i(s)} \\
&\quad + a_{ii}^u \int_0^t e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds+\sigma_i(s)dB_i(s)} \\
&\quad \times e^{+\int_0^s [r_i(\tau)-(\sigma_i^2(\tau)/2)]d\tau+\sigma_i(\tau)dB_i(\tau)} ds \\
&= e^{-\int_0^t \sigma_i(s)dB_i(s)} \Bigg[\frac{1}{\phi_i(0)} e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds} \\
&\quad + a_{ii}^u \int_0^t e^{-\int_s^t [r_i(\tau)-(\sigma_i^2(\tau)/2)]d\tau} \\
&\quad \times e^{\int_0^s \sigma_i(\tau)dB_i(\tau)} ds \Bigg] \\
&\le e^{-\int_0^t \sigma_i(s)dB_i(s)} \Bigg[\frac{1}{\phi_i(0)} e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds} \\
&\quad + a_{ii}^u e^{\max_{0\le s\le t}(\int_0^s \sigma_i(\tau)dB_i(\tau))} \\
&\quad \times \int_0^t e^{-\int_s^t [r_i(\tau)-(\sigma_i^2(\tau)/2)]d\tau} ds \Bigg] \\
&\le \frac{e^{\max_{0\le s\le t}[\int_0^s \sigma_i(\tau)dB_i(\tau)]-\int_0^t \sigma_i(s)dB_i(s)}}{z_i(t)}.
\end{aligned} \tag{50}$$

Similarly, we have

$$\begin{aligned}
\frac{1}{\phi_i(t)} &\ge e^{-\int_0^t \sigma_i(s)dB_i(s)} \Bigg[\frac{1}{\phi_i(0)} e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds} \\
&\quad + a_{ii}^u e^{\min_{0\le s\le t}(\int_0^s \sigma_i(\tau)dB_i(\tau))} \\
&\quad \times \int_0^t e^{-\int_s^t [r_i(\tau)-(\sigma_i^2(\tau)/2)]d\tau} ds \Bigg] \\
&\ge \frac{e^{\min_{0\le s\le t}[\int_0^s \sigma_i(\tau)dB_i(\tau)]-\int_0^t \sigma_i(s)dB_i(s)}}{z_i(t)}.
\end{aligned} \tag{51}$$

□

Lemma 12. *Assume that $\overline{r}_i^l > 0$, $i = 1,2$, then for any given initial value $z(0) \in R_+^2$, the solution $z(t)$ of (49) has the following properties*

$$\begin{gathered}
\widetilde{z}_i(t) \le z_i(t) \le \widehat{z}_i(t), \\
\lim_{t\to\infty} \widetilde{z}_i(t) = \frac{\overline{r}_i^l}{a_{ii}^u}, \qquad \lim_{t\to\infty} \widehat{z}_i(t) = \frac{\overline{r}_i^u}{a_{ii}^u},
\end{gathered} \tag{52}$$

where $\widetilde{z}(t) = (\widetilde{z}_1(t), \widetilde{z}_2(t))$, $\widehat{z}(t) = (\widehat{z}_1(t), \widehat{z}_2(t))$ are the solutions of the two equations, respectively,

$$d\widetilde{z}_i(t) = \widetilde{z}_i(t)\left[\overline{r}_i^l - a_{ii}^u \widetilde{z}_i(t)\right]dt, \quad \widetilde{z}_i(0) = z_i(0), \ i = 1,2, \tag{53}$$

$$d\widehat{z}_i(t) = \widehat{z}_i(t)\left[\overline{r}_i^u - a_{ii}^u \widehat{z}_i(t)\right]dt, \quad \widehat{z}_i(0) = z_i(0), \ i = 1,2. \tag{54}$$

Proof. Let $\widetilde{z}(t) = (\widetilde{z}_1(t), \widetilde{z}_2(t))$, $\widehat{z}(t) = (\widehat{z}_1(t), \widehat{z}_2(t))$ are the solutions of SDE (53) and (54), respectively, with the positive initial value $z(0)$. By Lemma 5, we know

$$\begin{aligned}
\widetilde{z}_i(t) &= \frac{e^{\overline{r}_i^l t}}{1/\widetilde{z}_i(0) + \left(a_{ii}^u/\overline{r}_i^l\right)\left(e^{\overline{r}_i^l t} - 1\right)}, \\
\widehat{z}_i(t) &= \frac{e^{\overline{r}_i^u t}}{1/\widehat{z}_i(0) + \left(a_{ii}^u/\overline{r}_i^u\right)\left(e^{\overline{r}_i^u t} - 1\right)}.
\end{aligned} \tag{55}$$

Thus,

$$\lim_{t\to\infty} \widetilde{z}_i(t) = \frac{\overline{r}_i^l}{a_{ii}^u}, \qquad \lim_{t\to\infty} \widehat{z}_i(t) = \frac{\overline{r}_i^u}{a_{ii}^u}. \tag{56}$$

By the classical comparison theorem of ordinary differential equations, we know

$$\widetilde{z}_i(t) \le z_i(t) \le \widehat{z}_i(t). \tag{57}$$

□

Lemma 13. *Assume that $\overline{r}_i^l > 0$, $i = 1,2$, then for any given initial value $\phi(0) \in R_+^2$, the solution $\phi(t)$ of (40) has the properties*

$$\lim_{t\to\infty} \frac{\log \phi_i(t)}{t} = 0, \quad a.s. \tag{58}$$

Proof. By Lemma 12, we know

$$\begin{aligned}
&e^{-[\max_{0\le s\le t}\int_0^s \sigma_i(\tau)dB_i(\tau)-\int_0^t \sigma_i(\tau)dB_i(\tau)]} \\
&\quad \le \frac{\phi_i(t)}{z_i(t)} \le e^{-[\min_{0\le s\le t}\int_0^s \sigma_i(\tau)dB_i(\tau)-\int_0^t \sigma_i(\tau)dB_i(\tau)]}.
\end{aligned} \tag{59}$$

So, we have

$$\begin{aligned}
&\int_0^t \sigma_i(\tau)dB_i(\tau) - \max_{0\le s\le t}\int_0^s \sigma_i(\tau)dB_i(\tau) \\
&\quad \le \log \phi_i(t) - \log z_i(t) \\
&\quad \le \int_0^t \sigma_i(\tau)dB_i(\tau) \\
&\qquad - \min_{0\le s\le t}\int_0^s \sigma_i(\tau)dB_i(\tau).
\end{aligned} \tag{60}$$

Let $M_i(t) = \int_0^t \sigma_i(\tau)dB_i(\tau)$, then

$$\langle M_i, M_i \rangle_t = \int_0^t \sigma_i^2(\tau)d\tau. \tag{61}$$

Since $\sigma_i(t)$, $i = 1, 2$ are bounded, then

$$\lim_{t\to\infty} \frac{\langle M_i, M_i \rangle_t}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \sigma_i^2(\tau)\,d\tau < \infty, \quad \text{a.s.} \tag{62}$$

By the strong law of large numbers, we know

$$\lim_{t\to\infty} \frac{M_i(t)}{t} = \lim_{t\to\infty} \frac{\int_0^t \sigma_i(\tau)\,dB_i(\tau)}{t} = 0, \quad \text{a.s.} \tag{63}$$

Thus,

$$\lim_{t\to\infty} \max_{0\le s\le t} \left| \frac{M_i(s)}{t} \right| = 0, \quad \text{a.s.} \tag{64}$$

Then from (60) we obtain

$$\lim_{t\to\infty} \frac{\log \phi_i(t)}{t} = 0, \quad \text{a.s.} \tag{65}$$

□

Lemma 14. *Assume that $\bar{r}_i^l > 0$, $i = 1, 2$, then for any given initial value $\phi(0) \in R_+^2$, the solution $\phi(t)$ of (40) has the properties*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \phi_i(s)\,ds \ge \frac{\bar{r}_i^l}{a_{ii}^u}, \quad \text{a.s.} \tag{66}$$

Proof. By Itô's formula, we have

$$d\log\phi_i(t) = \left[r_i(t) - \frac{\sigma_i^2(t)}{2} - a_{ii}^u \phi_i(t) \right] dt + \sigma_i(t)\,dB_i(t). \tag{67}$$

Integrating both sides of this equation from 0 to t yields

$$\frac{\log\phi_i(t)}{t} - \frac{\log\phi_i(0)}{t} = \frac{\int_0^t \left[r_i(s) - \left(\sigma_i^2(s)/2\right) \right] ds}{t} - \frac{a_{ii}^u \int_0^t \phi_i(s)\,ds}{t} + \frac{\int_0^t \sigma_i(s)\,dB_i(s)}{t}. \tag{68}$$

By Lemma 13, we know that

$$\lim_{t\to\infty} \frac{\int_0^t \sigma_i(s)\,dB_i(s)}{t} = \lim_{t\to\infty} \frac{\log\phi_i(t)}{t} = 0, \quad \text{a.s.} \tag{69}$$

Hence,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \phi_i(s)\,ds = \frac{1}{a_{ii}^u} \liminf_{t\to\infty} \frac{1}{t}\int_0^t \left[r_i(s) - \frac{\sigma_i^2(s)}{2} \right] ds \ge \frac{\bar{r}_i^l}{a_{ii}^u}, \quad \text{a.s.} \tag{70}$$

□

Definition 15. System (3) is said to be persistent in time average if

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds > 0, \quad i = 1, 2. \tag{71}$$

Theorem 16. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$ and $\bar{r}_i^l > 0$, $i = 1, 2$, then the solution $x(t)$ of system (3) with any initial value $x(0) \in R_+^2$ has the following property:*

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \ge \frac{\bar{r}_i^l}{a_{ii}^u}, \\ &\liminf_{t\to\infty} \frac{\log x_i(t)}{t} \ge 0, \quad a.s., \end{aligned} \tag{72}$$

and so system (3) is persistent in time average.

Proof. By Lemma 8, we know that

$$x_i(t) \ge \phi_i(t) \quad i = 1, 2, \tag{73}$$

where $\phi(t) = (\phi_1(t), \phi_2(t))$ is the solution of system (40). Moreover, by Lemma 14 we know that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \ge \liminf_{t\to\infty} \frac{1}{t}\int_0^t \phi_i(s)\,ds \ge \frac{\bar{r}_i^l}{a_{ii}^u}, \quad \text{a.s.} \tag{74}$$

Hence, by Lemma 13 we know that

$$\liminf_{t\to\infty} \frac{\log x_i(t)}{t} \ge \liminf_{t\to\infty} \frac{\log\phi_i(t)}{t} = 0, \quad \text{a.s.} \tag{75}$$

□

3.3. Asymptotic Boundedness of Integral Average. Theorem 16 shows that every component of the solution $x(t)$ of system (3) will survive forever in time average, if the white noise is not large. In this part, we further deduce that every component of $x(t)$ of system (3) will be an asymptotic boundedness in time average. Before we give the result, we do some preparation work.

Lemma 17. *Let $f \in C[[0,\infty) \times \Omega, (0,\infty)]$, $F(t) \in ((0,\infty) \times \Omega, R)$. If there exist positive constants λ_0 and λ such that*

$$\log f(t) \ge \lambda t - \lambda_0 \int_0^t f(s)\,ds + F(t), \quad t \ge 0,\ a.s., \tag{76}$$

and $\lim_{t\to\infty}(F(t)/t) = 0$ a.s., then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,ds \ge \frac{\lambda}{\lambda_0}, \quad a.s. \tag{77}$$

Proof. The proof is similar to the proof of Lemma in [21]. Let

$$\varphi(t) = \int_0^t f(s)\,ds. \tag{78}$$

Since $f \in C[[0,\infty) \times \Omega, (0,\infty)]$, $\varphi(t)$ is differentiable on $[0,\infty)$ and

$$\frac{d\varphi(t)}{dt} = f(t) > 0, \quad \text{for } t \ge 0. \tag{79}$$

Substituting $d\varphi(t)/dt$ and $\varphi(t)$ into (76), we obtain the following:

$$\log \frac{d\varphi(t)}{dt} \geq \lambda t - \lambda_0 \varphi(t) + F(t), \tag{80}$$

thus

$$e^{\lambda_0 \varphi(t)} \frac{d\varphi(t)}{dt} \geq e^{\lambda t + F(t)}, \quad \text{for } t \geq 0. \tag{81}$$

Note that $\lim_{t\to\infty}(F(t)/t) = 0$ a.s., then for $0 < \varepsilon < \min\{1, \lambda\}$, $\exists T = T(\omega) > 0$ and $\Omega_\varepsilon \subset \Omega$ such that $P(\Omega_\varepsilon) > 1-\varepsilon$ and $F(t) \geq -\varepsilon t,\ t \geq T,\ \omega \in \Omega_\varepsilon$. Then we have

$$e^{\lambda_0 \varphi(t)} \frac{d\varphi(t)}{dt} \geq e^{(\lambda-\varepsilon)t}, \quad \text{for } t \geq T,\ \omega \in \Omega_\varepsilon. \tag{82}$$

Integrating inequality (82) from 0 to t results in the following:

$$\lambda_0^{-1}\left[e^{\lambda_0\varphi(t)} - e^{\lambda_0\varphi(T)}\right] \geq (\lambda-\varepsilon)^{-1}\left[e^{(\lambda-\varepsilon)t} - e^{(\lambda-\varepsilon)T}\right]. \tag{83}$$

This inequality can be rewritten into

$$e^{\lambda_0\varphi(t)} \geq e^{\lambda_0\varphi(T)} + \lambda_0(\lambda-\varepsilon)^{-1}\left[e^{(\lambda-\varepsilon)t} - e^{(\lambda-\varepsilon)T}\right]. \tag{84}$$

Taking the logarithm of both sides and dividing both sides by $t(> 0)$ yields

$$\frac{\varphi(t)}{t} \geq \lambda_0^{-1} \frac{\log\left\{e^{\lambda_0\varphi(T)} + \lambda_0(\lambda-\varepsilon)^{-1}\left[e^{(\lambda-\varepsilon)t - e^{(\lambda-\varepsilon)T}}\right]\right\}}{t}. \tag{85}$$

Then,

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} \geq \frac{\lambda-\varepsilon}{\lambda_0}, \quad \omega \in \Omega_\varepsilon. \tag{86}$$

Letting $\varepsilon \to \infty$ yields

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,ds \geq \frac{\lambda}{\lambda_0}, \quad \text{a.s.} \tag{87}$$

This finishes the proof of the Lemma. □

Theorem 18. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$ and $\bar{r}_i^l > 0$, $i = 1, 2$, then the solution $x(t)$ of system (3) with any initial value $x(0) \in R_+^2$ has the property*

$$\tilde{x}_i^* \leq \lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \leq \hat{x}_i^*, \quad i = 1, 2, \ a.s., \tag{88}$$

where

$$\begin{aligned} \tilde{x}_1^* &= \frac{a_{22}^u \bar{r}_1^l + a_{12}^l \bar{r}_2^l}{a_{11}^u a_{22}^u - a_{12}^l a_{21}^l}, \qquad \tilde{x}_2^* = \frac{a_{11}^u \bar{r}_2^l + a_{21}^l \bar{r}_1^l}{a_{11}^u a_{22}^u - a_{12}^l a_{21}^l}, \\ \hat{x}_1^* &= \frac{a_{22}^l \bar{r}_1^u + a_{12}^u \bar{r}_2^u}{a_{11}^l a_{22}^l - a_{12}^u a_{21}^u}, \qquad \hat{x}_2^* = \frac{a_{11}^l \bar{r}_2^u + a_{21}^u \bar{r}_1^u}{a_{11}^l a_{22}^l - a_{12}^u a_{21}^u}. \end{aligned} \tag{89}$$

Proof. To prove the results, we only need to prove

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \geq \tilde{x}_i^*, \quad i = 1, 2, \text{ a.s.} \tag{90}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \leq \hat{x}_i^*, \quad i = 1, 2, \text{ a.s.} \tag{91}$$

By Itô's formula, we have

$$\begin{aligned} d\log x_1(t) &= \left[r_1(t) - \frac{1}{2}\sigma_1^2(t) - a_{11}(t)x_1(t) + a_{12}(t)x_2(t)\right]dt \\ &\quad + \sigma_1(t)\,dB_1(t), \\ d\log x_2(t) &= \left[r_2(t) - \frac{1}{2}\sigma_2^2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t)\right]dt \\ &\quad + \sigma_2(t)\,dB_2(t). \end{aligned} \tag{92}$$

First, we prove (91). Integrating both sides of (92) from 0 to t yields

$$\begin{aligned} \log x_1(t) &= \log x_1(0) + \int_0^t \bar{r}_1(s)\,ds - \int_0^t a_{11}(s)x_1(s)\,ds \\ &\quad + \int_0^t a_{12}(s)x_2(s)\,ds + \int_0^t \sigma_1(s)\,dB_1(s), \\ \log x_2(t) &= \log x_2(0) + \int_0^t \bar{r}_2(s)\,ds - \int_0^t a_{22}(s)x_2(s)\,ds \\ &\quad + \int_0^t a_{21}(s)x_1(s)\,ds + \int_0^t \sigma_2(s)\,dB_2(s), \end{aligned} \tag{93}$$

where $\bar{r}_i(s) = r_i(s) - (1/2)\sigma_i^2(s),\ i = 1, 2$. Since $x_i(t) > 0,\ i = 1, 2$, hence

$$\begin{aligned} \log x_1(t) &\leq \log x_1(0) + \bar{r}_1^u t - a_{11}^l \int_0^t x_1(s)\,ds \\ &\quad + a_{12}^u \int_0^t x_2(s)\,ds + \int_0^t \sigma_1(s)\,dB_1(s), \\ \log x_2(t) &\leq \log x_2(0) + \bar{r}_2^u t - a_{22}^l \int_0^t x_2(s)\,ds \\ &\quad + a_{21}^u \int_0^t x_1(s)\,ds + \int_0^t \sigma_2(s)\,dB_2(s). \end{aligned} \tag{94}$$

So we have

$$\begin{aligned} a_{22}^{l}\log x_1(t)+a_{12}^{u}\log x_2(t) &\le a_{22}^{l}\left[\log x_1(0)+\int_0^t\sigma_1(s)\,dB_1(s)\right]+a_{22}^{l}\overline{r}_1^{u}t \\ &\quad+a_{12}^{u}\left[\log x_2(0)+\int_0^t\sigma_2(s)\,dB_2(s)\right]+a_{12}^{u}\overline{r}_2^{u}t \\ &\quad-\left(a_{11}^{l}a_{22}^{l}-a_{21}^{u}a_{12}^{u}\right)\int_0^t x_1(s)\,ds. \end{aligned} \tag{95}$$

By Theorem 16, we know that

$$\liminf_{t\to\infty}\frac{\log x_i(t)}{t}\ge 0,\quad i=1,2,\ \text{a.s.} \tag{96}$$

Obviously,

$$\lim_{t\to\infty}\frac{\log x_i(0)+\int_0^t\sigma_i(s)\,dB_i(s)}{t}=0,\quad i=1,2,\ \text{a.s.} \tag{97}$$

Hence, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds\le\frac{a_{22}^{l}\overline{r}_1^{u}+a_{12}^{u}\overline{r}_2^{u}}{a_{11}^{l}a_{22}^{l}-a_{21}^{u}a_{12}^{u}}\triangleq\widehat{x}_1^{*},\quad \text{a.s.} \tag{98}$$

Similarly, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds\le\frac{a_{11}^{l}\overline{r}_2^{u}+a_{12}^{u}\overline{r}_1^{u}}{a_{11}^{l}a_{22}^{l}-a_{21}^{u}a_{12}^{u}}\triangleq\widehat{x}_2^{*},\quad \text{a.s.} \tag{99}$$

Next, we prove that (90) is true. Taking integration both sides of (92) from 0 to t, we have

$$\begin{aligned} \log x_1(t) &\ge \log x_1(0)+\overline{r}_1^{l}t-a_{11}^{u}\int_0^t x_1(s)\,ds \\ &\quad+a_{12}^{l}\int_0^t x_2(s)\,ds+\int_0^t\sigma_1(s)\,dB_1(s), \\ \log x_2(t) &\ge \log x_2(0)+\overline{r}_2^{l}t-a_{22}^{u}\int_0^t x_2(s)\,ds \\ &\quad+a_{21}^{l}\int_0^t x_1(s)\,ds+\int_0^t\sigma_2(s)\,dB_2(s). \end{aligned} \tag{100}$$

By Theorem 16 we know that

$$\begin{aligned} \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds &\ge \frac{\overline{r}_1^{l}}{a_{11}^{u}}\triangleq M_1,\quad \text{a.s.}, \\ \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds &\ge \frac{\overline{r}_2^{l}}{a_{22}^{u}}\triangleq N_1,\quad \text{a.s.}, \end{aligned} \tag{101}$$

then for any $\varepsilon>0$, there is a $T(\omega)>0$ such that

$$\frac{1}{t}\int_0^t x_2(s)\,ds\ge N_1-\varepsilon, \tag{102}$$

for $t>T(\omega)$. It follows from (100) that, for $t>T(\omega)$,

$$\begin{aligned} \log x_1(t) &\ge \log x_1(0)+\overline{r}_1^{l}t-a_{11}^{u}\int_0^t x_1(s)\,ds \\ &\quad+a_{12}^{l}(N_1-\varepsilon)t+\int_0^t\sigma_1(s)\,dB_1(s) \\ &= \log x_1(0)-a_{11}^{u}\int_0^t x_1(s)\,ds \\ &\quad+\left[\overline{r}_1^{l}+a_{12}^{l}(N_1-\varepsilon)\right]t+\int_0^t\sigma_1(s)\,dB_1(s). \end{aligned} \tag{103}$$

From Lemma 17, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds\ge\frac{\overline{r}_1^{l}+a_{12}^{l}(N_1-\varepsilon)}{a_{11}^{u}}:=M_2>M_1. \tag{104}$$

Similarly, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds\ge\frac{\overline{r}_2^{l}+a_{21}^{l}(M_1-\varepsilon)}{a_{22}^{u}}:=N_2>N_1. \tag{105}$$

Continuing this process, we obtain two sequences M_n, N_n $(n=1,2,\ldots)$ such that

$$M_n=\frac{\overline{r}_1^{l}+a_{12}^{l}(N_{n-1}-\varepsilon)}{a_{11}^{u}}, \tag{106}$$

$$N_n=\frac{\overline{r}_2^{l}+a_{21}^{l}(M_{n-1}-\varepsilon)}{a_{22}^{u}}. \tag{107}$$

By induction, we can easily show that $M_{n+1}>M_n$, $N_{n+1}>N_n$, $n=1,2,\ldots$, that is, sequences $\{M_n, n=1,2,\ldots\}$ and $\{N_n, n=1,2,\ldots\}$ are nondecreasing. Moreover, note that (98) and (99), then the sequences $\{M_n, n=1,2,\ldots\}$ and $\{N_n, n=1,2,\ldots\}$, have upper bounds. Therefore, there are two positive M, N such that

$$\begin{gathered} \lim_{n\to\infty}M_n=M,\qquad \lim_{n\to\infty}N_n=N, \\ \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds\ge M,\qquad \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds\ge N, \end{gathered} \tag{108}$$

which together with (106) implies

$$\begin{aligned} a_{11}^{u}M-a_{12}^{l}N &= \overline{r}_1^{l}-\varepsilon a_{12}^{l}, \\ a_{22}^{u}N-a_{21}^{l}M &= \overline{r}_2^{l}-\varepsilon a_{21}^{l}. \end{aligned} \tag{109}$$

Letting $\varepsilon\to 0$ yields

$$\begin{aligned} M &= \frac{a_{22}^{u}\overline{r}_1^{l}+a_{12}^{l}\overline{r}_2^{l}}{a_{11}^{u}a_{22}^{u}-a_{12}^{l}a_{21}^{l}}\triangleq\widetilde{x}_1^{*}, \\ N &= \frac{a_{11}^{u}\overline{r}_1^{l}+a_{21}^{l}\overline{r}_2^{l}}{a_{11}^{u}a_{22}^{u}-a_{12}^{l}a_{21}^{l}}\triangleq\widetilde{x}_1^{*}. \end{aligned} \tag{110}$$

Hence,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds \geq \tilde{x}_i^*, \quad i=1,2, \text{ a.s.,} \tag{111}$$

which is as required. □

4. Nonpersistence

In this section, we discuss the dynamics of system (3) as the white noise is getting larger. We show that system (3) will be nonpersistent if the white noise is large, which does not happen in the deterministic system.

Definition 19. System (3) is said to be nonpersistent, if there are positive constants q_1, q_2 such that

$$\lim_{t\to\infty}\prod_{i=1}^{2} x_i^{q_i}(t) = 0 \quad \text{a.s.} \tag{112}$$

Theorem 20. *Assume that* $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$ *and* $a_{22}^l \overline{r}_1^u + a_{12}^u \overline{r}_2^u < 0$, *then system* (3) *is nonpersistent, where* $\overline{r}_i(s) = r_i(s) - (\sigma_i^2(s)/2)$, $i = 1,2$.

Proof. Since $x_i(t) > 0$, $i = 1,2$ and $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, from (93) we have

$$\begin{aligned}
&a_{22}^l \log x_1(t) + a_{12}^u \log x_2(t) \\
&\quad\leq \left\{a_{22}^l \overline{r}_1^u + a_{12}^u \overline{r}_2^u\right\} t - \left(a_{11}^l a_{22}^l - a_{21}^u a_{12}^u\right)\int_0^t x_1(s)\,ds \\
&\qquad + a_{22}^l\left[\log x_1(0) + \int_0^t \sigma_1(s)\,dB_1(s)\right] \\
&\qquad + a_{12}^u\left[\log x_2(0) + \int_0^t \sigma_2(s)\,dB_2(s)\right] \\
&\quad\leq K_1 t + a_{22}^l\left[\log x_1(0) + \int_0^t \sigma_1(s)\,dB_1(s)\right] \\
&\qquad + a_{12}^u\left[\log x_2(0) + \int_0^t \sigma_2(s)\,dB_2(s)\right],
\end{aligned} \tag{113}$$

where $K_1 = a_{22}^l \overline{r}_1^u + a_{12}^u \overline{r}_2^u$ which together with

$$\begin{aligned}
&\lim_{t\to\infty}\frac{a_{22}^l\left[\log x_1(0) + \int_0^t \sigma_1(s)\,dB_1(s)\right]}{t} \\
&\quad = \lim_{t\to\infty}\frac{a_{12}^u\left[\log x_2(0) + \int_0^t \sigma_2(s)\,dB_2(s)\right]}{t} = 0, \quad \text{a.s.,}
\end{aligned} \tag{114}$$

implies

$$\lim_{t\to\infty}\frac{1}{t}\left[a_{22}^l \log x_1(t) + a_{12}^u \log x_2(t)\right] \leq K_1, \quad \text{a.s.} \tag{115}$$

If $K_1 < 0$, then there must be

$$\lim_{t\to\infty} x_1^{a_{22}^l}(t)\, x_2^{a_{12}^u}(t) = 0, \quad \text{a.s.} \tag{116}$$

Hence, system (3) is nonpersistent. □

Theorem 21. *Assume that* $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$ *and* $(a_{21}^u \overline{r}_1^u + a_{11}^l \overline{r}_2^u) < 0$, *then system* (3) *is nonpersistent, where* $\overline{r}_i(s) = r_i(s) - (\sigma_i^2(s)/2)$, $i = 1,2$.

Here we omit the proof of Theorem 21 which is similar to the proof of Theorem 20.

Remark 22. If $(\sigma_i^l)^2 > 2r_i^u$, $i = 1,2$, then the conditions in Theorems 20 and 21 are obviously satisfied, respectively. That is to say, the large white noise will lead to the population system being non-persistent.

5. Global Attractivity

In this section, we turn to establishing sufficient criteria for the global attractivity of stochastic system (3).

Definition 23. Let $x(t)$, $y(t)$ be two arbitrary solutions of system (3) with initial values $x(0), y(0) \in R_+^2$, respectively. If

$$\lim_{t\to\infty}\left|x(t) - y(t)\right| = 0, \quad \text{a.s.,} \tag{117}$$

then we say system (3) is globally attractive.

Theorem 24. *Assume that* $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, *then system* (3) *is globally attractive.*

Proof. Let $x(t)$, $y(t)$ be two arbitrary solutions of system (3) with initial values $x(0), y(0) \in R_+^2$. By the Itô's formula, we have

$$\begin{aligned}
d\log x_i(t) &= \left[r_i(t) - \frac{1}{2}\sigma_i^2(t) - a_{ii}(t)\,x_i(t) + a_{ij}(t)\,x_j(t)\right]dt \\
&\quad + \sigma_i(t)\,dB_i(t), \quad i,j = 1,2,\ j \neq i, \\
d\log y_i(t) &= \left[r_i(t) - \frac{1}{2}\sigma_i^2(t) - a_{ii}(t)\,y_i(t) + a_{ij}(t)\,y_j(t)\right]dt \\
&\quad + \sigma_i(t)\,dB_i(t), \quad i,j = 1,2,\ j \neq i.
\end{aligned} \tag{118}$$

Then,

$$\begin{aligned}
&d\left(\log x_i(t) - \log y_i(t)\right) \\
&\quad = \left\{-a_{ii}(t)\left[x_i(t) - y_i(t)\right] + a_{ij}(t)\left[x_i(t) - y_i(t)\right]\right\}dt, \\
&\qquad\qquad i,j = 1,2,\ j \neq i.
\end{aligned} \tag{119}$$

Since $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, there exist two positive constants c_1, c_2 which satisfy

$$\frac{a_{21}^u}{a_{11}^l} < \frac{c_1}{c_2} < \frac{a_{22}^l}{a_{12}^u}. \tag{120}$$

Thus, $c_1 a_{11}^l - c_2 a_{21}^u > 0$, $c_2 a_{22}^l - c_1 a_{12}^u > 0$.

Consider a Lyapunov function $V(t)$ defined by

$$\begin{aligned} V(t) = {} & c_1 \left|\log x_1(t) - \log y_1(t)\right| \\ & + c_2 \left|\log x_2(t) - \log y_2(t)\right|, \quad t \geq 0. \end{aligned} \tag{121}$$

A direct calculation of the right differential $d^+V(t)$ of $V(t)$ along the ordinary differential equation (119) leads to

$$\begin{aligned} d^+V(t) = {} & c_1 \operatorname{sgn}\left(x_1(t) - y_1(t)\right) d\left[\log x_1(t) - \log y_1(t)\right] \\ & + c_2 \operatorname{sgn}\left(x_2(t) - y_2(t)\right) d\left[\log x_2(t) - \log y_2(t)\right] \\ = {} & c_1 \operatorname{sgn}\left(x_1(t) - y_1(t)\right) \\ & \times \left[-a_{11}(t)\left(x_1(t) - y_1(t)\right) dt \right. \\ & \left. + a_{12}(t)\left(x_2(t) - y_2(t)\right) dt\right] \\ & + c_2 \operatorname{sgn}\left(x_2(t) - y_2(t)\right) \\ & \times \left[a_{21}(t)\left(x_1(t) - y_1(t)\right) dt \right. \\ & \left. - a_{22}(t)\left(x_2(t) - y_2(t)\right) dt\right] \\ \leq {} & -c_1 a_{11}^l \left|x_1(t) - y_1(t)\right| dt \\ & + c_1 a_{12}^u \left|x_2(t) - y_2(t)\right| dt \\ & - c_2 a_{22}^l \left|x_2(t) - y_2(t)\right| dt \\ & + c_2 a_{21}^u \left|x_1(t) - y_1(t)\right| dt \\ = {} & -\left(c_1 a_{11}^l - c_2 a_{21}^u\right) \left|x_1(t) - y_1(t)\right| dt \\ & - \left(c_2 a_{22}^l - c_1 a_{12}^u\right) \left|x_2(t) - y_2(t)\right| dt \\ \leq {} & -\gamma \sum_{i=1}^{2} \left|x_i(t) - y_i(t)\right| dt, \end{aligned} \tag{122}$$

where $\gamma = \min\{c_1 a_{11}^l - c_2 a_{21}^u,\ c_2 a_{22}^l - c_1 a_{12}^u\}$. Integrating both sides of (122) form 0 to t, we have

$$V(t) + \gamma \int_0^t \sum_{i=1}^{2} \left|x_i(s) - y_i(s)\right| ds \leq V(0) < \infty. \tag{123}$$

Let $t \to \infty$, we obtain

$$\begin{aligned} \int_0^\infty \left|x(s) - y(s)\right| ds & \leq \int_0^\infty \sum_{i=1}^{2} \left|x_i(s) - y_i(s)\right| ds \\ & \leq \frac{V(0)}{\gamma} < \infty \quad \text{a.s.} \end{aligned} \tag{124}$$

Note that $u(t) = x(t) - y(t)$. Clearly, $u(t) \in C(R_+, R^2)$ a.s. It is straightforward to see from (124) that

$$\liminf_{t \to \infty} |u(t)| = 0 \quad \text{a.s.} \tag{125}$$

Next, we prove that

$$\lim_{t \to \infty} |u(t)| = 0 \quad \text{a.s.} \tag{126}$$

By Theorem 3 we obtain that the pth moment of the solution of system (3) is bounded, the following proof is similar to the proof of Theorem 6.2 in [15] and hence is omitted. □

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