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Research Article

Perfect Equilibria in Replies in Multiplayer Bargaining

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Multiplayer bargaining is a game in which all possible divisions are equilibrium outcomes. This paper presents the classical subgame perfect equilibria strategies and analyses their weak robustness, namely, the use of weakly dominated strategies. The paper then develops a refined equilibrium concept, based on trembling hand perfection applied only on the replies, in order to overcome such weakness. Concluding that none of the classical equilibrium strategies survives the imposition of the extrarobustness and albeit using more complex strategies, the equilibrium outcomes do not change.

1. Introduction

In n-players bargaining, there is a divisible good to be shared among them. The division is obtained by the following procedure: at each moment a player proposes a division, and the other $n-1$ players vote in favor or against it. If all agree, the division is made accordingly; if at least one player votes against it, the game goes on to another round, with another player proposing and a new suffrage taking place. The game ends when a proposal is accepted by all. At each round, the good in question loses value by δ.

The better known result on multiplayer bargaining is that all divisions are *Subgame Perfect Nash Equilibria* (SPNE) outcomes of the game, meaning that all divisions can be agreed on in equilibria. Crucial to obtain this result is the existence of a credible and painful threat for deviators of the “right” track. Reference [1] provides an ingenious mechanism, creating a strategy in which at least one player is unsatisfied with a deviation proposal. For this strategy, they used a state variable and if the proponent does not propose as implied by the state, the state changes to a new one in which one player receives everything. Players do not want to deviate because in the punishment state they will receive nothing. For this strategy to be an equilibrium, the discount value cannot be very small; namely, with 3 players $\delta > 1/2$. Reference [2] noted that an equilibrium for all divisions possibilities could be extended to $\delta \leq 1/2$. This strategy also uses a state variable and punishment threats that attribute everything to one player only; the main difference is in the repliers’ actions, with players accepting only if the proposition is equal to the state—any difference, even if awards all repliers, is rejected. The belief players have that the proposition will be rejected renders them indifferent between accepting and rejecting the offer, and they thus opt for refusing it.

Of notice is that all these equilibria do not depend on the replies and that it is unorthodox for players not to accept better proposals unless they are punished by doing so. This is a major shortcoming of this equilibrium: players, without being punished by acting differently, choose to play a dominated strategy. This is an evident weakness of the equilibria concept used; players choose weakly dominated strategies. In *Haller*’s strategy, players in specific history states accept zero offerings because they do not expect to receive more in the future if they reject them. They are powerless to change the outcome; it is a resigned acceptance. In *Herrero*’s strategy, players propose divisions in which they receive zero. Again this is a hopeless proposition and only happens thanks to the belief that others will also follow a resigned action course; as players believe others will reject, they believe their own actions do not have any effect. The need of unanimity gives total power to all players in terms of rejecting a proposal, and other players’ actions will have no impact. This case, of the players’ actions having no effect on the outcome of the game, may result in the best and more accurate strategies not being played and originates nonsensible equilibria. Players only choose their best available actions in singleton information sets; if, for example, players knew what others had voted before them, making their information set at the moment of voting a singleton, then players knew that if they accepted a good proposition, then others could also do it. This conviction would make them vote in favor of the good division. This type of structure in games and the possible appearance of nonsensible equilibria are very well known and have been studied and solved by the use of refined equilibria notions.

In this work, we develop different equilibrium concepts to analyse the game, based on Selten's [3] perfect equilibria, and introduce the possibilities of small mistakes by the players on the replies. *Perfect Equilibrium in Replies* (PER) imposes that all players in all replies moments commit a minor mistake. The use of trembles involves some distortion of the game and should be used with parsimony. The reason for the SPNE not to work is the nonsingleton information set at the reply, and in order to introduce the minimum distortions possible, we only impose trembles on the replies.

When a perturbed game is played, if the strategy does not punish replies, as is the case in all the strategies already described, players will always accept propositions that give them more than what they receive in future when the proposition is refused (although this may seem obvious it is not what happens in *Haller's* equilibrium, in which better propositions are rejected in face of the expected rejection of the other replier). Thus, they accept better propositions even if the chance of others accepting it is very small. This property of the PER equilibrium strategies which simultaneously are independent of replies is the pivotal point to show that *Haller*'s strategy is not PER. In *Herrero*'s strategy, the equilibrium is supported by a punishing scheme in which a deviator is attributed zero, and he has no possibility of receiving more unless someone deviates in the meantime. But if any player can make a mistake, for example, accepting a different proposition, the deviator will never accept zero; he will wait for his proposal moment and hope for an opponent to make a mistake. The deviator will always refuse a zero proposition and the strategy is not PER.

There is no easy equilibrium solution that works for all points in the simplex. The main difficulty is with divisions in which one player is receiving zero; for these divisions to be a PER outcome, we will use a strategy with a punishment scheme that not only punishes deviators but also has a mechanism of awarding the well-behaved players. It is the chance of receiving this award that acts as an incentive for players to accept receiving or proposing for themselves zero. They are hoping that some player deviates and they receive the premium for the compliance. This strategy is naturally weakly dominated, but on the approximation games it is not.

In the rest of the paper, we will present in Section 2 notation; the equilibrium strategies of *Haller* and *Herrero* are presented and the new equilibrium concept is defined. In Section 3, proofs that the standard equilibrium strategies are not PER are given and a new strategy, that is, PER, is defined. Finally, in Section 4, a conclusion is provided.

2. Materials and Methods

The set of players is $I = \{1,2,3\}$. At the moment $t \in \mathbb{N}$, a proposal is one point of the unitary simplex $p^t = (p_1^t, p_2^t, p_3^t) \in \Delta$, with $\Delta = \{(x_1,x_2,x_3) : \sum_{i=1}^{3} x_i \le 1, x_i \ge 0\}$, and p_i^t is the part attributed to player i. The proponent at t is the player $i(t)$, with $i(t)$ the function that determines the proponent; it has a cycle of period 3, $i(t) : T \to I$ and $i(t) = \{i \in I : \exists m \in \mathbb{N}_0, t = i + 3m\}$. $t(i) : I \rightrightarrows T$ is the correspondence that defines the moments in which player i proposes; these moments are $t(i) = \{t \in T : \exists m \in \mathbb{N}_0, t = i + 3m\}$.

Player's $j \ne i(t)$ response to the proposal is an action taken on $\{0,1\}$, with a_j^t, the action of j at t, being 0 if j rejects the proposition received and 1 if the player accepts it. So, $a_i^t \in \{0,1\}$ if $i(t) \ne i$ or $a_i^t \in \Delta$ if $i(t) = i$. For the sake of simplicity, define the set of actions available for i at t by

$$A_i^t = \begin{cases} \{0,1\} & \text{if } i \ne i(t) \\ \Delta & \text{if } i = i(t). \end{cases} \quad (1)$$

The vector of all actions taken at moment t is $a^t = (a_1^t, a_2^t, a_3^t)$ and the space of all actions at t is $A_t = A_1^t \times A_2^t \times A_3^t = \{0,1\}^2 \times \Delta = \overline{\Delta}$.

For $t \ge 1$, a t-size history can be a history either after or before the proposition is done, and a distinction between these two cases is necessary; we therefore define a $(t-1,2)$-history in which $t-1$ propositions and voting have been done which is denoted by $h^{|t-1,2} = (a^1,\ldots,a^{t-1})$ and a $(t,1)$-history, when a proposition has already been done at t, but no replies have been received yet, $h^{|t,1} = (a^1,\ldots,a^{t-1},a_{i(t)}^t)$, in which, for all $1 \le k \le t-1$, $a^k \in \overline{\Delta}$ and $a_{i(t)}^t \in \Delta$; the space of $(t,2)$-stage histories is $H^{t,2} = \prod_{k=1}^{t} \overline{\Delta} = \overline{\Delta}^t$, and the space of all $(t,1)$-histories is $H^{t,1} = H^{(t-1),2} \times \Delta = \overline{\Delta}^{t-1} \times \Delta$. $H^{0,2}$ stands for $\emptyset$ the unique 0-stage history. The set of all histories is $H = \bigcup_{t=1}^{\infty}(H^{t,1} \cup H^{t,2})$. The stage history at moment t in history h is $h^{t,1}$ for the proposal and $h^{t,2}$ for the responses, $h^t = (h^{t,1}, h^{t,2})$. The length of a history, $\tau(h)$, is a function from the set of histories into the stage moment $\tau : H \mapsto \mathbb{N}_0 \times \{1,2\}$, with $\tau(h) = (t,k)$ $t \in \mathbb{N}_0$ being the moment of the history, and $k \in \{1,2\}$ whether the voting has already been made $k = 2$ or not $k = 1$. $t(h)$ is the moment of history h, so $\tau(h) = (t(h), k)$ and $i(h) = i(t(h))$ are the proponent at h. For a history h with $t(h) > t$, $h^{|t,k}$ is the history h until stage (t,k). h^+ and h^- are, respectively, the history h plus one more stage or without the last stage, and it will be used only when the marginal actions are obvious from the context. It is assumed that at stage (t,k), each player knows $h^{|t,k}$; that is, each player knows the actions that were played in all previous stages. $(h, \overline{h})$ is the history h followed by $\overline{h}$.

A *pure strategy for player i* is a function $s_i : H \to \{0,1\} \cup \Delta$ with $s_i(h) \in A_i^{t(h^+)}$ mapping histories into actions. The set of player i pure strategies is denoted by S_i, and $S = S_1 \times S_2 \times S_3$ is the joint pure strategy space. Every pure strategy $s = (s_1,s_2,s_3) \in S$ induces a path after the history h, $\varpi_s(h)$. At h, the action will be $s(h)$; then, if an agreement has not been reached, $s(h,s(h))$ is the action played, so we can define the future after h when s is the strategy as $\varpi_s(h) = \{h, s(h), s(h,s(h)), s(s(h,s(h))), \ldots\}$. The utility for a given strategy is $\Pi_i^t(s \mid h) = \sum_{\overline{h} \in \varpi_s(h)} \delta^{t(\overline{h})-t(h)} \pi(h,\overline{h})$, in which

$\pi(\tilde{h})$ is the value of the division agreed on at the last moment of $\tilde{h}$ and therefore is the product of the last moment actions $\pi(\tilde{h}) = \tilde{h}^{t,1}\tilde{h}_j^{t,2}\tilde{h}_k^{t,2}$, $k, j \notin -i(\tilde{h}^-)$ (the usual notation will be used, $-i = I \setminus \{i\}$).

In this chapter, we will present the classical equilibrium strategies in multiplayer bargaining. Reference [1] was the first (although he never publish his results, it is also attributed to Shaked the creation of such strategies, see, for example, [4] or [5]) to prove that all points in Δ are equilibria outcomes when $\delta > 1/2$. Later, Haller noted that if the repliers' strategies were stricter, the equilibria could extend to any δ. Due to the dynamic character of the game, the equilibrium concept used is the SPNE that we hereby define.

Definition 1. $s \in S$ is a SPNE if $\Pi_i^t(s \mid h) \geq \Pi_i^t(s_i', s_{-i} \mid h)$ $\forall h \in H$, $\forall i \in I$ and $\forall s_i' \in S_i$.

The utility function in the bargaining game can be written, as noted before, in the form $\Pi_i^t(s) = \sum_{\tau=1}^{\infty} \delta^\tau a_\tau$ with a_τ, the payments at $t + \tau$, which is either zero or the value of the agreed on division at $t + \tau$ and is bounded by 1. It is relatively straightforward to see that if two strategies share the same future path for a long period, their actualized payment will be similar; therefore, utility function is continuous at infinity and the one-shot deviation principle is valid. Therefore to prove that a given strategy is a SPNE, we need only to look for alternative strategies that are different on one information set. For this purpose, we define the one-shot deviation strategy.

Definition 2. The set of one shot deviation (OSD) strategies from s_i at h is $\mathrm{OSD}(s_i, h) = \{\gamma_i \in S_i : \gamma_i(h) \neq s_i(h)$ and $\gamma_i(h') = s_i'(h'), \forall h' \in H \setminus h\}$.

2.1. Haller Equilibrium Strategy. In this subsection, we will present the equilibrium defined by [2]; a proof that such strategy is a SPNE will be presented for completeness. In the proof, we are only looking for better pure strategies; if no pure strategy is better, then no mixed strategy can be better either. This strategy uses a state function $r(h) : H \to E$ that tracks for any history h if a player has deviated from the planned and induces the punishment for that player. There is a bond between the state and the division to be proposed under the strategy; for this reason, we use the same symbol for a state and the division associated with it. $E = \{e^0, e^1, e^2, e^3\}$ is set of states; e^0 is any point in Δ; e^i is the division in which player i receives 1; $e_k^i = \begin{cases} 1, & \text{if } k=i \\ 0, & \text{if } k \neq i \end{cases}$. At $h \in H^{t-1,2}$, if the player $i = i(t)$ does not propose $r(h)$, the state changes to $e^{i(t+1)}$, in which the player i receives nothing. The state at the initial moment $h = \emptyset$ is $r(h) = e^0$. Transition takes place immediately after the proposal and before the replies so for $\tau(h) = (t, 2)$, $r(h) = r(h^-)$. For $\tau(h) = (t, 1)$,

$$r(h) = \begin{cases} r(h) & \text{if } h^{t,1} = r(h) \\ e^{i(t+1)} & \text{if } h^{t,1} \neq r(h). \end{cases} \tag{2}$$

Now, we will present Haller's equilibrium strategy, that is summarized in Table 1.

Table 1: Haller's strategy.

	State	e^j
Player i	Proposal	e^j
	Accept p	$p = e^j$

Definition 3. In Haller's equilibrium strategy for h such that $\tau(h) = (t-1, 2)$, $s_{i(h)}(h) = r(h)$, so the proposition will always be equal to the state. For $\tau(h) = (t, 2)$, replier's $j \neq i(h)$ strategy is

$$s_j(h) = \begin{cases} 1 & seh^{t,1} = r(h^-) \\ 0 & seh^{t,1} \neq r(h^-). \end{cases} \tag{3}$$

Repliers accept the proposition if it is equal to the state and reject if it is different; note that for replier j, the share offered is as important to him as to others; what matters is that the proposition is equal to $r(h^-)$ so the share of all players is relevant.

Theorem 4. *Haller's strategy is a SPNE and any $e^0 \in \Delta$ is an equilibrium outcome.*

Proof. s is Haller's strategy with $r(\emptyset) = e^0$, for any but fixed $e^0 \in \Delta$. We will prove that there is no history h after which the player can change his strategy to $s_i' \in \mathrm{OSD}(s_i, h)$ and improve his payment. Let us start by noting that due to $r(h) = r(h^-)$ for $\tau(h) = (t, 2)$, $h^{t,2}$ has no influence on the state; whatever the responses are the state does not change.

For $\mathbf{i = i(t)}$, $\boldsymbol{\tau}\mathbf{(h) = (t-1, 2)}$, if all players play according to the strategy s, i proposes $r(h)$ and all others accept, $\Pi_i^t(s \mid h) = r_i(h)$. If $s_i' \in \mathrm{OSD}(s_i, h)$, then $p = s_i'(h) \neq s_i(h) = r(h)$; i made a different proposal; repliers j, k only accept if the proposal is $p = r(h^-)$; as $p \neq r(h)$, they reject it. The state after the deviated proposition changes to $r(h, p) = e^{i(t+1)}$ and i's payoff is $\Pi_i^t(s_i', s_{-i} \mid h) = \delta\Pi_i^{t+1}(s_i', s_{-i} \mid h^+) = \delta\Pi_i^{t+1}(s \mid h^+) = \delta e^{i(t+1)} = 0$. Clearly, $\Pi_i^t(s_i', s_{-i} \mid h) \leq \Pi_i^t(s \mid h)$ for any $\mathrm{OSD}(s_i, h)$; the proponent $i(t)$ has no advantage in altering his strategy.

For $\mathbf{j \neq i(t)}$ and $\boldsymbol{\tau}\mathbf{(h) = (t, 1)}$ we have two possibilities for the player to act differently from s, either to accept a proposal different from $r(h)$ or to reject the proposal of $r(h)$. When the proposal was equal to the state $\mathbf{h^{t,1} = r(h)}$, if all players act by s, the proposition is accepted and $\Pi_j^t(s \mid h) = r_j(h)$. If $s_j' \in \mathrm{OSD}(s_j, h)$, j refuses the proposition, $s_j'(h) = 0$; we can define the stage history $h^{t,2} = (s_j'(h), s_k(h)) = (0, 1)$ and $h^+ = (h, h^{t,2})$. The state does not change, as the proposition was done according to s, so $r(h^+) = r(h)$. j's refusal delays the agreement one period because after h^+ all players follow s and the agreement is $r(h^+) = r(h)$. $\Pi_j^t(s_j', s_{-j} \mid h) = \delta\Pi_j^{t+1}(s_j', s_{-j} \mid h^+) = \delta\Pi_j^{t+1}(s \mid h^+) = \delta r_j(h^+) = \delta r_j(h) \leq r_j(h)$, and we conclude that $\Pi^t(s_j', s_{-j} \mid h) \leq \Pi_j^t(s \mid h)$. When the proposal is not equal to the state $\mathbf{h^{t,1} \neq r(h)}$ if $-i(h)$ follow s the proposal is refused; the state has changed to $r(h^+) = r(h) = e^{i(t+1)}$, where $h^+ = (h, (0, 0))$ and $\Pi_j^t(s \mid h) = \delta\Pi_j^{t+1}(s \mid h^+) = e_j^{i(t+1)}$. If j follows $s_j' \in \mathrm{OSD}(s_j, h)$, accepting the proposition,

Table 2: Herrero's startegy.

	State	e^j
Player i	Proposal	e^j
	Reply	$p_i \geq \delta e_i^j$

$s'_j(h) = 1$. The proposal will still be declined by the other player and there will be no change in state caused by j response, and $r(\overline{h}^+) = e^{i(t+1)}$, with $\overline{h}^+ = (h, (1,0))$. $\Pi_j^t(s'_j, s_{-j} \mid h) = \delta\Pi_j^{t+1}(s \mid h^+) = \delta e_j^{i(t+1)} = \delta\Pi_j^{t+1}(s \mid h^+) = \Pi_j^t(s \mid h)$. Player j does not improve by changing strategy. □

2.2. Herrero's Strategy. Being less general than Haller's strategy, Herrero had proposed an equilibrium strategy that is less fragile. In this case, the players' acceptance is not reduced to one division only; they *apparently* consider only their own share, and the acceptance rule has a threshold. The punishment scheme is activated if a player does not propose what he was supposed to. A state function defining the state at history h and which division should be proposed (again there is an identification between state and proposal), $r(h) : H \to E$, is updated after each proposal but before the replies, so $r(h) = r(h^-)$ when $\tau(h) = (t, 2)$. The states are again $E = \{e^0, e^1, e^2, e^3\}$, with e^i the division in which player i receives the totality; the initial state is $r(\emptyset) = e^0$.

Define $k(p,t)$ as the replier worst off in proposition p made at t (of smaller index if there is more than one), $k(p,t) = \min\{j \in I \setminus i(t) : p_j = \min_{k\in I\setminus i(t)} p_k\}$. The state is defined in the following way for $\tau(h) = (t, 1)$:

$$r(h) = \begin{cases} r(h^-) & \text{if } h^{t,1} = r(h^-) \\ e^k & \text{if } h^{t,1} \neq r(h^-). \end{cases} \tag{4}$$

Briefly, if the player made the expected proposal, $h^{t,1} = r(h^-)$, there is no state change; if he did not, then the strategy enters in a punishment scheme of $i(h)$ that gives everything to player $k = k(h^{t,1}, t)$. Herrero's strategy is resumed in Table 2 and formally defined subsequently.

Definition 5. The proponent always proposes $r(h)$, $s_{i(h)}(h) = r(h)$; the strategy for repliers $j \neq i(h)$ is

$$s_j(h) = \begin{cases} 1 & \text{if } h_j^{t,1} \geq \delta r(h)_j \\ 0 & \text{if } h_j^{t,1} < \delta r(h)_j. \end{cases} \tag{5}$$

Theorem 6. *For $\delta > 1/2$ Herrero's strategy is SPNE for any $e^o \in \Delta$.*

Proof. We will use the one-shot deviation principle once more. Let us start by seeing that at $h \in H^{t-1,2}$ the player $\mathbf{i} = \mathbf{i}(\mathbf{t})$ gains nothing to act differently from s; when all players act accordingly, i utility after $h \in H^{t-1,2}$ is $\Pi_i^t(s \mid h) = r(h)_i$. If i uses $s'_i \in$ OSD (s,h) and makes a different proposition, $p \neq r(h)$, there is immediately a change of state to $r(h^+) = e^k$, with $k = k(p,t) \neq i$. If $h^{++} = (h, p, r)$, where r is the reply to $h^{t,1}$, $r \in \{0,1\}^2$; if $\min r_j = 0$, at least one player refused the proposition and $\Pi_i^{t+1}(s'_i, s_{-i} \mid h) = \delta\Pi_i^{t+1}(s \mid h^{++}) = \delta r(h^{++})_i = \delta e_i^k = 0 \leq \Pi_i^t(s_i, s_{-i} \mid h)$. Then, the only way i can improve is when all players accept. After proposition $p \neq r(h)$, state becomes e^k, with k being the player receiving the minimum, according to s for k to accept $p_k = \min\{p_j, p_k\} \geq \delta$, and then $p_j \geq \delta$. The total amount given to the repliers, for both of them to accept the proposal, must be at least 2δ; as the total cannot be bigger than a unity, we conclude that $\delta \leq 1/2$, contradicting the initial hypothesis. So, both repliers cannot accept simultaneously the out of equilibrium proposition. For $\mathbf{j} \neq \mathbf{i}(\mathbf{t})$ and $\boldsymbol{\tau}(\mathbf{h}) = (\mathbf{t}, \mathbf{1})$, the payment for player j under s depends on the actions of the other replier k as well; if $h_\iota^{t,1} \geq \delta r(h)_\iota$, for $\iota = j, k$. all repliers will accept, $\min_{\iota\in -i(h)} s_\iota(h) = 1$; payment is immediate and equal to $h_j^{t,1} = \Pi_j^t(s \mid h)$; if any of the repliers rejects (due to his share being smaller than the established by the state), $\min_{\iota\in -i(h)} s_\iota(h) = 0$; the agreement is delayed one period, but the state is not changed; once the state does not depend on the replies, $h^+ = (h, (s_j(h), s_k(h))) \in H^{t,2}$, and $r(h^+) = r(h)$. In this case, $\Pi_j^t(s \mid h) = \delta\Pi_j^{t+1}(s \mid h^+) = \delta r(h^+)_j = \delta r(h)_j$. And we can conclude that $\Pi_j^t(s \mid h) \geq \delta r(h)_j$ independently of the proposition $h^{t,1}$. At this moment, there are two ways in which the players can act contrarily to the strategy s: to accept a proposal that should be refused or to reject one that should be accepted. In neither one does the player improve. If $\mathbf{s_j}(\mathbf{h}) = \mathbf{1}$, player j chooses $s'_j \in \text{OSD}(s_j, h)$; then $s'_j(h) = 0$ his payment is $\Pi_j^t(s'_j, s_{-j} \mid h) = \delta\Pi_j^{t+1}(s'_j, s_{-j} \mid h^+)$, with $h^+ = (h, (s'_j(h), s_k(h)))$, as $r(h^+) = r(h)$, the state does not depend on the replies; $\Pi_j^{t+1}(s'_j, s_{-j} \mid h^+) = \Pi_j^{t+1}(s_j, s_{-j} \mid h^+) = r(h^+)_j = r(h)_j$. j's rejection leads to $\Pi_j^t(s'_j, s_{-j} \mid h) = \delta r(h)_j$, $\Pi_j^t(s'_i, s_j \mid h) \leq \Pi_j^t(s \mid h)$. When $\mathbf{s_j}(\mathbf{h}) = \mathbf{0}$, then a strategy $s'_j \in \text{OSD}(s_j, h)$ has $s'_j(h) = 1$. If player k accepts, $s_k(h) = 1$, the agreement is immediate and the payment of j is $h_j^{t,1}$. It is smaller than $\delta r(h)_j$ because according to s_j a proposal should only be rejected, $s_j(h) = 0$, if $h^{t,1} < \delta r(h)_j$. If $s_k(h) = 0$, the agreement is postponed and j's payment is $\delta\Pi_j^{t+1}(s \mid h^+)$. We can therefore define the payment of j as

$$\begin{aligned}
&\Pi_j^t(s'_j, s_{-j} \mid h) \\
&\quad = s_k(h)\, h_j^{t,1} + (1 - s_k(h))\, \delta\Pi_j^{t+1}(s'_j, s_{-j} \mid h^+) \\
&\quad = s_k(h)\, h_j^{t,1} + (1 - s_k(h))\, \delta\Pi_j^{t+1}(\mathbf{s_j}, s_{-j} \mid h^+) \\
&\quad = s_k(h)\, h_j^{t,1} + (1 - s_k(h))\, \delta r(h^+)_j \\
&\quad \leq s_k(h)\, \delta r(h)_j + (1 - s_k(h))\, \delta r(\mathbf{h})_j = \delta r(h)_j = \Pi_j^t(s \mid h).
\end{aligned} \tag{6}$$

□

2.3. Perfect Equilibrium in Replies. In *Haller's* strategy, repliers, without being punished by acting differently, reject propositions that leave them better off; they are choosing weakly dominated strategies. At the moment of an answer, when player j rejects the proposition, whatever k does, the proposal will still be rejected, the agreement moment will be delayed, and j's action is, for the time being, useless. Then, he can either accept or reject that his payment does not change. Of course the game continues and the path after rejection is important, but at this moment the player's actions do not have any impact on the game.

When a replier believes the other is rejecting the proposal, he is indifferent between accepting and rejecting it. If both players think the same way, there may be a rejection of a good proposal to both. This problem is a known weakness of SPNE and was in the origin of the sequential and perfect equilibrium concepts; for example, [6, page 9] identifies the problem with the fact that not all information sets are singletons.

> *"(...) For a subgame perfect equilibrium to be sensible, it is necessary that this equilibrium prescribes at each information set which is singleton a choice which maximizes the expected payoff after that information set. Note that the restriction to singleton information sets is necessary to ensure that the expected payoff after the information set is well defined. This restriction, however, has the consequence that not all subgame perfect equilibria which satisfy this additional condition are sensible."*

So, if all information sets are singleton, the SPNE is sensible; if they are not, then there might be a problem in some equilibria strategies. If the information set is nonsingleton, a choice of an action that is not the best may happen; the use of the concept is, in this case, questionable. Haller's strategy clearly demonstrates that a refined equilibrium concept should be used in the multibargaining game.

For the purpose of this paper, we propose using one concept in the vein of perfect equilibria of [3], different from SPNE, that try to overcome the described problem by adding small randomness to replier's actions. In this way, all players' actions are decisive in every moment and all their actions and choices do have an impact on the future payments. We adopt an equilibrium notion in which players only mistake in replies because it is at these moments that the information sets are nonsingleton. The proponent information set is a singleton, he always knows what the repliers have just done and all the previous history. His actions always impact on the outcome of the game and therefore SPNE is a sensible equilibrium for this case. In this way, in order to avoid unnecessary complications and the distortions that the trembling hand perfection requirement induces, we opted for introducing the minimum number of alterations to the approximating games, and therefore the concept of *Perfect Equilibrium in Replies* (PER) uses only trembles in the replies.

A mixed strategy for this game will be defined in terms of behavioral mixed strategies, meaning that to each h, the player will choose a probability distribution over the possibilities A_h available at the time. According to [7], to choose a mixed distribution at each h is equivalent to choosing a mixed strategy over all simple strategies; this result is Khun's theorem adaptation for the case of infinite extensive games with continuum space of actions. Denote by $\mathfrak{F}(X, \sigma_X)$ the set of probabilities measures over the set X with σ-algebra σ_X. At moment h, with A_h the actions available for the players, a *behavioral strategy at h* for each i is to pick a probability measure $\sigma_i(h) \in \mathfrak{F}(A_h, \mathfrak{B}(A_h))$ (for $A_h = \Delta$, we will use the Borelian σ-algebra). A *behavioral mixed strategy for player i*, σ_i is a behavioral mixed strategies for every history $\sigma_i(h)$, $\forall h \in H$; the set of all possible behavioral mixed strategy is Σ_i. A *behavioral mixed strategy* is $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, with $\sigma_i \in \Sigma_i$ for $i = 1, 2, 3$.

To define the payment function it is important to know the agreement distribution over $\overline{\Delta} = \Delta \times \{0, 1\}^2$, that is, to know what is the probability measure on $\mathfrak{B}(\overline{\Delta})$. For that purpose, we will define a measure based on the *behavioral mixed strategy*, σ. ${}_k\overline{\sigma}_h$ defines the probability over the future histories of dimension k after h; it is therefore defined on the σ-algebra $\mathfrak{B}(\overline{\Delta}^k)$.

${}_k\overline{\sigma}_h$ will be defined iteratively. We start by the probability measure of the histories ending on the period next to h. For that, for each $h \in H^{t,2}$, define ${}_1\sigma_h(O) = \sigma_h(O) = \sigma(O \mid h)$, with $O \in \mathfrak{B}(\overline{\Delta})$. If at h the proposal was accepted and $h^{t,2} = (1, 1)$, then no path was followed and in that case ${}_1\sigma_h(O) = 0$ for any $O \in \mathfrak{B}(\overline{\Delta})$. Then, define the measure over future histories of size 2, for $O \in \mathfrak{B}(\overline{\Delta}^2)$, like

$$
{}_2\overline{\sigma}_h(O) = \int_{\overline{h} \in \overline{\Delta}} \sigma_{(h,\overline{h})}\left(O_{|\overline{h}}\right) \partial\left({}_1\overline{\sigma}_h\right), \tag{7}
$$

in which $O_{|\overline{h}}$ is the projection of $O \subseteq \overline{\Delta}^2$ on the last coordinate $O_{|\overline{h}} = \{\widetilde{h} \in \overline{\Delta} : (\overline{h}, \widetilde{h}) \in O\}$. Using the same idea, it is possible to define, recursively, ${}_{k+1}\overline{\sigma}_h$ as the measure among the histories with duration $k + 1$ superior to h when σ is the played strategy and $O \in \mathfrak{B}(\overline{\Delta}^k)$:

$$
{}_{k+1}\overline{\sigma}_h(O) = \int_{\overline{h} \in \overline{\Delta}^k} \sigma_{(h,\overline{h})}\left(O_{|\overline{h}}\right) \partial\left({}_k\overline{\sigma}_h\right). \tag{8}
$$

For $h \in H^{\bar{t},2}$, if i is the proponent and j and k are the repliers, the immediate payment at time $\bar{t}$ is $\pi(h) = h^{\bar{t},1} h_j^{\bar{t},2} h_k^{\bar{t},2}$; if both repliers accept, $\pi(h) = h^{\bar{t},1}$; if either rejects $\pi(h) = \overline{0}$. $\pi(h)$ is clearly continuous in h. The payment at $t = t(h)$, under the mixed strategy σ, can be defined as

$$
\Pi_i^t(\sigma \mid h) = \sum_k \delta^k \int_{\overline{h} \in \overline{\Delta}^k} \pi\left(h, \overline{h}\right) \partial\left({}_k\overline{\sigma}_h\right). \tag{9}
$$

The expected payment is a discounted sum of a stream of expected values received at each moment when σ is played; at h player i expects to receive $\int_{\overline{h} \in \overline{\Delta}^k} \pi(h, \overline{h}) \partial({}_k\sigma_h)$ in the moment $t(h) + k$.

The questions raised by [5, page 250] justify the use of an agent strategic form of the game in both *Trembling Hand* definitions. The PER is almost a direct translation of Selten's perfect equilibrium for the multiplayer bargaining. Reference [3] defines a sequence of approximating games, and to each of these games, each action has a positive minimum probability of being played. For the game in appreciation that means for each history $h \in H^{t,1}$, the minimum for each reply is $\epsilon_0^h > 0$ and $\epsilon_1^h > 0$, in the approximating games. A strategy in the approximating game σ^ϵ must have at any history $h \in H^{t,1}$ a positive probability attributed to both possibilities of reply, $\sigma^\epsilon(1 \mid h) \geq \epsilon_1^h$ and $\sigma^\epsilon(0 \mid h) \geq \epsilon_0^h$. However, to impose only this restriction, on the approximating game, destroys an important characteristic of the game, namely, the symmetry of it. For this reason, to keep the symmetric nature of the game, we will assume equal restrictions at all moments. That is, at replies, the minimum imposed in each approximating game is always the same regardless of the moment or the player, $\epsilon_k^h = \epsilon_k$, for $k = 0, 1$. Therefore, we use approximation games in which both actions at the moment of replies are played with at least ϵ_0 and ϵ_1 probability, for the rejecting and accepting action, respectively. For a strategy to be PER, it must be an accumulation point of the equilibrium strategy of one sequence of approximation games, when $\epsilon \downarrow \overline{0}$, with $\epsilon = (\epsilon_0, \epsilon_1)$.

Definition 7. For a given $\epsilon \in [0,1]^2$, let $\Sigma_i^\epsilon = \{\sigma_i \in \Sigma_i : \sigma_i(k \mid h) \geq \epsilon_k, \forall h \in H^1, k \in \{0,1\}\}$ be the strategy space. σ is *Perfect Equilibria in Replies* if there is one sequence of $\epsilon \downarrow \overline{0}$ and $\{\sigma^\epsilon\}_\epsilon$ such that $\sigma_i^\epsilon \in \Sigma_i^\epsilon$; $\sigma_{i(h)}^\epsilon(k \mid h) = \sigma_{i(h)}(k \mid h)$, for all $h \in H^{t,2}$ and all t; σ is an accumulation point of the sequence of $\{\sigma^\epsilon\}_{\{\epsilon \downarrow \overline{0}\}}$; σ^ϵ is a best reply at all histories h in the set Σ_i^ϵ; that is

$$\Pi_i^t\left(\sigma^\epsilon \mid h\right) \geq \Pi_i^t\left(\sigma_i'^\epsilon, \sigma_{-i}^\epsilon \mid h\right), \quad \forall \sigma_i'^\epsilon \in \Sigma_i^\epsilon \cap \mathrm{OSD}_i\left(\sigma^\epsilon, h\right). \tag{10}$$

3. Results and Discussion

3.1. Perfect Equilibrium in Replies and Classical Strategies. One property common to all equilibria strategies presented in Section 2 is that replies do not play a role in the future of the game. In case of rejection of a proposal, for what will be the future path of the game, it does not matter who rejected it. In this type of strategies, defined as *Reply Independent*, when PER is in use, as there are no future consequences of accepting or rejecting proposals, and there is always the possibility that the other player accepts, when a player is receiving zero, then those propositions that leave him better off should be accepted. The next result will prove this, but first we formally define a *Reply Independent* strategy as a strategy where the same action is taken for two histories with the same propositions (but possibly with different replies).

Definition 8. The strategy σ is *Reply Independent* if for any h and $\widetilde{h}$ with $\tau(h) = \tau(\widetilde{h})$ and $h^{t,1} = \widetilde{h}^{t,1}$, $\forall t \leq t(h)$, $\sigma(h) = \sigma(\widetilde{h})$. $\Sigma_p \subset \Sigma$ is the set of all Reply Independent strategies.

If a strategy is *Reply Independent*, when a proposal is rejected, the payment is always the same no matter what the concrete reply vector $r \in R$ is, with $R = \{(0,0),(0,1),(1,0)\}$ the set of responses, where a proposition is rejected. So, $\Pi_i^{t+1}(\sigma \mid h, r) = \Pi_i^{t+1}(\sigma \mid h, r')$, $\forall r, r' \in R$. We can then define, for a *Reply Independent* strategy, the future payment after a proposal being refused $p_i^\sigma(h) = \delta\Pi_i^{t+1}(\sigma \mid h, r)$, $\forall r \in R$, with $\tau(h) = (t, 1)$. If a strategy is simple, it is possible after a history $h^0 \in H^{t,2}$ to determine the sequence of future propositions. There is a $p^0 \in \Delta$ such that $\sigma_{i(h^0)}(p^0 \mid h^0) = 1$. If the proposition is rejected, whatever is the $r \in R$, as σ is *Reply Independent*, with, abusing slightly on notation, $h^1 = (h, p^0, r)$ there is a $p^1 \in \Delta$ such that $\sigma_{i(h^1)}(p^1 \mid h^1) = 1$. Following on this way it is possible to define the sequence of propositions after h^0 as $P^\sigma(h^0) = \{p^0, p^1, \ldots\}$. We can now show that, under certain conditions, if a strategy is PER and *Reply Independent*, then better proposals are always accepted.

Theorem 9. *If a simple strategy σ is PER, Reply Independent and for $h \in H^{t,1}$, $P^\sigma(h, r) = \{p^0, p^1, \ldots\}$, $\forall r \in R$. If, for the replier $i \neq i(h)$, $p_i^k = 0$, $\forall k$, then $\sigma_i(1 \mid h) = 1$, when $h_i^{t,1} > 0 = p_i^\sigma(h)$; for the other replier $j \neq i(h)$, $\sigma_j(1 \mid h) = 0$ if $h_j^{t,1} < p_j^\sigma(h)$ and $\sigma_j(1 \mid h) = 1$ if $h_j^{t,1} > p_j^\sigma(h)$.*

Proof. By definition of a PER strategy, $\sigma_{i(h^t)}^\epsilon(p^t \mid h^t) = 1$, for any t. Therefore, the proposition after h, when σ^ϵ is being played, is the same as when it is σ, by PER $\sigma_{i(h)}^\epsilon(h) = \sigma_{i(h)}(h)$. As $p_i^k = 0$, whatever the moment the agreement is reached, player i gains zero. If $q_{t+k}^{\sigma^\epsilon}$ is the probability, an agrement is reached at $t+k$; when the strategy played is σ^ϵ, the payment of i is $\Pi_i^t(\sigma^\epsilon \mid h) = \sum_k \delta^k q_{t+k}^{\sigma^\epsilon} p_i^k = 0$. If $h_i^{t,1} > 0$, then the payment of player i in case of rejection is $\Pi_i^t(\sigma^\epsilon \mid h) = 0$, if he accepts, his payment is $\Pi_i^t(\sigma^\epsilon \mid h) \geq h_i^{t,1}\sigma_j^\epsilon(1 \mid h) > 0$. The player is better accepting the proposition, and therefore $\sigma_i^\epsilon(1 \mid h) = 1 - \epsilon_0$.

The other replier j has also two possibilities after h, consider the strategy in which at the moment h he always accepts $\sigma_j^a \in \mathrm{OSD}(\sigma_j^\epsilon, h)$ with $\sigma_j^a(1 \mid h) = 1$ and the strategy in which he always rejects $\sigma_j^r \in \mathrm{OSD}(\sigma_j^\epsilon, h)$ with $\sigma_j^r(0 \mid h) = 1$. The payment at each of these possibilities is

$$\begin{aligned}
&\Pi_j^t\left(\sigma_j^a, \sigma_{-j}^\epsilon \mid h\right) \\
&\quad = \sigma_i^\epsilon\left(1 \mid h\right) h_j^{t,1} + \delta\sigma_i^\epsilon\left(0 \mid h\right) \Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 1, 0\right) \\
&\quad = \left(1 - \epsilon_0\right) h_j^{t,1} + \delta\epsilon_0 \Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 1, 0\right), \\
&\Pi_j^t\left(\sigma_j^r, \sigma_{-j}^\epsilon \mid h\right) \\
&\quad = \delta\sigma_i^\epsilon\left(1 \mid h\right) \Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 0, 1\right) + \delta\sigma_i^\epsilon\left(0 \mid h\right) \\
&\qquad \times \Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 0, 0\right) \\
&\quad = \delta\left(1 - \epsilon_0\right) \Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 0, 1\right) + \delta\epsilon_0 \Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 0, 0\right).
\end{aligned} \tag{11}$$

And the difference between the payments of the two strategies, $\Pi_j^t(\sigma_j^a, \sigma_{-j}^\epsilon \mid h) - \Pi_j^t(\sigma_j^r, \sigma_{-j}^\epsilon \mid h)$, is equal to

$$\begin{aligned}
&\left(1 - \epsilon_0\right)\left[h_j^{t,1} - \delta\Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 0, 1\right)\right] \\
&\quad + \epsilon_0\delta\left[\Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 1, 0\right) - \Pi_j^{t+1}\left(\sigma^\epsilon \mid h, 0, 0\right)\right].
\end{aligned} \tag{12}$$

Clearly, $\Pi_j^t(\sigma_j^a, \sigma_{-j}^\epsilon \mid h) - \Pi_j^t(\sigma_j^r, \sigma_{-j}^\epsilon \mid h) \rightarrow h_k^{t,1} - \delta\Pi_k^{t+1}(\sigma \mid h, 0, 1)$, when $\epsilon \rightarrow \overline{0}$. So, for small values of ϵ, $\Pi_j^t(\sigma_j^a, \sigma_{-j}^\epsilon \mid h, \widetilde{d}) > \Pi_j^t(\sigma_j^r, \sigma_{-j}^\epsilon \mid h)$ if $h_j^{t,1} > \delta\Pi_j^{t+1}(\sigma \mid h, 0, 1) = p_j^\sigma(h)$, and therefore $\sigma_j^\epsilon(1 \mid h) = 1 - \epsilon_0$; if $h_j^{t,1} < p_j^\sigma(h)$, then $\sigma_j^\epsilon(1 \mid h) = \epsilon_1$. Take limits to $\sigma_j^\epsilon(1 \mid h)$ and the conclusions are immediate. □

An immediate consequence of the previous result is that Haller's strategy is not PER equilibria since repliers only accept a unique proposal and for that reason it cannot sustain the hypothesis of small errors. Without penalizing the answers it was relatively clear this would happen.

Corollary 10. *Haller's strategy is not PER strategy.*

Herrero's strategy is different; it respects the previous result, but it still maintains a shortcoming; not all the played strategies are nondominated; for instance when a player proposes a division that attributes him zero, he is playing a weakly dominated strategy. The next corollary shows that *Herrero*'s, strategy is not a PER equilibrium.

Corollary 11. *Herrero's strategy is not a PER.*

Proof. To prove that Herrero's strategy is not a PER, we will find a history moment $h \in H$ at which the strategy is not compatible with Definition 7. Take, for instance, a history $h \in H^{t,2}$ with state $r(h) = e^2$ and proponent $i(h) = 1$. So, player 1 is the proponent at a history with state e^2, receiving a payoff of zero $e_1^2 = 0$ if accepted. By definition of Herrero's strategy we know it is simple, reply independent, and $P^\sigma(h) = \{e^2, e^2, \ldots\}$, no matter what the replies are the future propositions will be always e^2. By an argument equal to the one at Theorem 9 we know that $\Pi_1^t(\sigma^\epsilon h, e^2) = 0$. However, if player 1 uses a strategy $\sigma_1^p \in \mathrm{OSD}(\sigma_1^\epsilon, h)$, in which he proposes $p \in \Delta$ with $p_1 > 0$, the payment of player 1 is $\Pi_1^t(\sigma^\epsilon h, p) \geq p_1\sigma_2^\epsilon(1h, p)\sigma_3^\epsilon(1h, p) \geq p_1\epsilon_1^2 > 0$. At h to propose e^2 is not the best option and clearly there is no approximating strategy σ^ϵ with $\sigma^\epsilon(h) = \sigma(h)$ that is the best reply at h. □

3.2. New Equilibrium Strategy. The next strategy will use an out of equilibrium incentive mechanism for players that follows it, and establish that all possible divisions in Δ are PER outcomes.

For that strategy, consider the set of states $E = \bigcup_{i \neq j}^3 \{e^i, e^{ij}\}$, where $e^i \in \Delta$ are as previously defined and the new states $e^{ij} \in \Delta$ are such that $e_i^{ij} = \gamma_1$, $e_j^{ij} = \gamma_2$, and $e_k^{ij} = 0$, for $k \notin \{i, j\}$; for example, $e^{31} = (\gamma_2, 0, \gamma_1)$. For each history h, there is a state $r(h) \in E$. The strategy for $h \in H^{t-1,2}$ is for the proponent to always propose a division equal to the state $s_{i(h)}(h) = r(h)$; for $h \in H^{t,1}$ and for $j \neq i(h)$ the player accepts if the proposal was equal to the state and rejects otherwise.

Consider

$$s_j(h) = \begin{cases} 1, & \text{if } h^{t,1} = r(h^-) \\ 0, & \text{if } h^{t,1} \neq r(h^-). \end{cases} \tag{13}$$

To define the state transition, we need to use a function from history to the subsets of players $g(h) : H^2 \rightarrow 2^I$ that tracks which players moved as defined in s at the last moment $h^t = (h^{t,1}, h^{t,2})$.

Consider

$$\begin{aligned} g(h) = \Big\{ i \in I : \left(i \neq i(h) \text{ and } s_i\left(h^{|t,1}\right) = h_i^{t,2}\right) \\ \text{or } \left(i = i(h) \text{ and } s_i\left(h^{|t-1,2}\right) = h^{t,1}\right) \Big\}. \end{aligned} \tag{14}$$

When all players follow s, the agreement is immediate; the proponent plays $r(h)$ and both repliers accept it; so, if h was not an ending history, some of the players did not play according to the strategy and either the proponent or at least one replier deviated. Therefore, there is an impossibility of $g(h) = I$ in a nonending history h. That is, a history with $h^{t,2} \neq (1, 1)$ must have $g(h) \neq I$.

At each history, it is possible to define an order of the players determined by the next moment each player will propose. Define for each moment t and for each player i, $t_i = \min\{\tilde{t} : \tilde{t} > t \text{ and } \tilde{t} \in t(i)\}$, and we say i proposes before j at t, $i \prec_t j$, if $t_i < t_j$. Take $\overline{g}(h)$ to be a vector with the same elements of $g(h)$ ordered by $\prec_{t(h)}$. One example, if $g(h) = \{1, 3\}$ and $t(h) = 4$ the next proponent is player 2, and then player 3 followed by 1, so $3 \prec_4 1$ and $\overline{g}(h) = (3, 1)$.

Transition occurs only after the voting stage; so, if $\tau(h) = (t, 1)$, $r(h) = r(h^-)$. For $h = (t, 2)$,

$$r(h) = \begin{cases} r(h^-), & \text{if } g(h) = \emptyset \\ e^{\overline{g}(h)}, & \text{if } g(h) \neq \emptyset. \end{cases} \tag{15}$$

Players that did not follow the strategy are punished by receiving zero in the next state. A player's willing to accept (or propose) 0 is based on the possibility of other players making a mistake, and in that case, the well-behaved player receives a premium.

For s to be a PER, there must exist a sequence of approximating strategies s^ϵ, with $s^\epsilon \xrightarrow{\epsilon \downarrow \overline{0}} s$. This strategy is a mixed strategy in replies, with both possibilities assuming a positive and equal probability; that is, we are assuming a sequence where $\epsilon = \{\epsilon_0, \epsilon_1\}$ with $\epsilon_0 = \epsilon_1$. So, to ease the notation, from now on we will consider that $\epsilon \in [0, 1]$ is the minimum at the approximation game for both options at the reply moment.

The strategy s^ϵ is similar to s; the action that coincides with s is played with probability $1-\epsilon$ and the one that does not is played only with ϵ probability; so, for $j \neq i(h)$ and $h \in H^{t,1}$,

$$s_j^\epsilon(1 \mid h) = \begin{cases} 1-\epsilon, & \text{if } h^{t,1} = r(h^-) \\ \epsilon, & \text{if } h^{t,1} \neq r(h^-). \end{cases} \tag{16}$$

For $h \in H^{t-1,2}$, we assume, according to the definition of PER, that $s_{i(h)}^\epsilon(r(h) \mid h) = s_{i(h)}(r(h) \mid h) = 1$, so $i(h)$ plays $r(h)$ with probability 1. It is clear that $s^\epsilon \rightarrow s$, and for s to be a PER, it only needs to be proved that s^ϵ is a best reply at all histories $h \in H$.

Before calculating the payment under s^ϵ, some facts about this strategy, which facilitate this job, should be noted. The

Table 3: Π_e^1 parcels.

		Player 2	
		Accept	Reject
Player 3	Accept	e_1	$\Pi^2_{e^{31}}$
	Reject	$\Pi^2_{e^{21}}$	$\Pi^2_{e^{1}}$

strategy, as function of h, only depends on the state of history $e = r(h)$, so the action taken at h is solely determined by the state e, and the strategy could be defined as $s^\epsilon(a \mid e) = s^\epsilon(a \mid h)$ for $a \in A_h$. The state $r(h)$ is determined by the previous state $r(h^-)$, the action taken $s^\epsilon(ah)$, and the proponent at h, $i(h)$. So, for two different histories h and $\widetilde{h}$, if they share the proponent $i(h) = i(\widetilde{h})$ and the state $r(h) = r(\widetilde{h})$, then the future play will have the same distribution, that is, ${}_k s^\epsilon_h = {}_k s^\epsilon_{\widetilde{h}}$ for all $k \in \mathbb{N}$. For this reason the future payment is the same at h and at $\widetilde{h}$, $\Pi_i^{t(h)}(s^\epsilon \mid h) = \Pi_i^{t(\widetilde{h})}(s^\epsilon \mid \widetilde{h})$. Therefore, we can define classes of histories where the future payment is the same if the s^ϵ is played. For $e \in E$ and $i \in I$, define the classes $[e, i] = \{h \in H : r(h) = e \text{ and } i(h) = i\}$.

Without loss of generality, we will focus on player 1 and for notation simplicity, define $\Pi_1^{t(h)}(s^\epsilon \mid h) = \Pi_e^i$ if $h \in [e, i]$. When all players follow s^ϵ, 1 is the proponent and e is the state; 1's payment, Π_e^1, is composed of several parcels presented in Table 3.

The content in the table will be explained through the example of one cell. After player 1 proposes e, suppose player 2 accepts, as it should, and player 3 rejects; the proposition is rejected and agreement is delayed. The players that followed the strategy s were 1 and 2, and then $g(h) = (1, 2)$; as 1 was the proposer, next round proposer's is 2 so $\overline{g}(h) = (2, 1)$, and the new state will be e^{21}. 1's payment, which comes from future agreement, is $\delta\Pi^2_{e^{21}}$. All the possibilities are covered in the table. To obtain 1's expected payoff, we multiply by the respective probabilities:

$$\begin{aligned}
\Pi_e^1 &= e_1 s^\epsilon_{-1}\left(1, 1 \mid h^{t,1} = e\right) + \delta s^\epsilon_{-1}\left(0, 1 \mid h^{t,1} = e\right)\Pi^2_{e^{31}} \\
&\quad + \delta s^\epsilon_{-1}\left(1, 0 \mid h^{t,1} = e\right)\Pi^2_{e^{21}} + \delta s^\epsilon_{-1}\left(0, 0 \mid h^{t,1} = e\right)\Pi^2_{e^{1}} \\
&= e_1(1-\epsilon)^2 + \delta\epsilon(1-\epsilon)\Pi^2_{e^{31}} + \delta\epsilon(1-\epsilon)\Pi^2_{e^{21}} + \delta\epsilon^2\Pi^2_{e^{1}}.
\end{aligned} \tag{17}$$

For two different states e and $\widetilde{e}$, all but the first term on (17) are equal, so $\Pi_e^1 - \Pi_{\widetilde{e}}^1 = (e_1 - \widetilde{e}_1)(1-\epsilon)^2$. This equality simplifies extremely Π_e^k, for example, we use the fact that player 1 receives nothing in the states e^2, e^3, and $\overline{e}^{23}$ to state that $\Pi^1_{e^2} = \Pi^1_{e^3} = \Pi^1_{e^{23}}$. For now we will focus on the payment of player 1 when the state is e^2; later, based on this case, we prove that s^ϵ is a best reply for the remaining histories and players. Replacing e by e^2 and using relations like $\Pi^k_{e^{12}} = \Pi^k_{e^{13}}$, $\Pi^k_{e^{13}} = \Pi^k_{e^2} + \gamma_1(1-\epsilon)^2$ and $\Pi^k_{e^1} = \Pi^k_{e^2} + (1-\epsilon)^2$. The payoff of player 1, when 1, 2, and 3 are the proponents, is

$$\begin{aligned}
\Pi^1_{e^2} &= \delta\epsilon(1-\epsilon)\Pi^2_{e^{31}} + \delta\epsilon(1-\epsilon)\Pi^2_{e^{21}} + \delta\epsilon^2\Pi^2_{e^1} \\
&= \delta\epsilon\left[2(1-\epsilon)\Pi^2_{e^{31}} + \epsilon\Pi^2_{e^1}\right] \\
&= \delta\epsilon\left[2(1-\epsilon)\left(\Pi^2_{e^2} + (1-\epsilon)^2\gamma_2\right) + \epsilon\left(\Pi^2_{e^2} + (1-\epsilon)^2\right)\right] \\
&= \delta\epsilon\left[(2-\epsilon)\Pi^2_{e^2} + 2(1-\epsilon)^3\gamma_2 + \epsilon(1-\epsilon)\right], \\
\Pi^2_{e^2} &= e_1^2(1-\epsilon)^2 + \delta\epsilon(1-\epsilon)\Pi^3_{e^{12}} \\
&\quad + \delta\epsilon(1-\epsilon)\Pi^3_{e^{32}} + \delta\epsilon^2\Pi^3_{e^2} \\
&= \delta\epsilon(1-\epsilon)\Pi^3_{e^{12}} + \delta\epsilon(1-\epsilon)\Pi^3_{e^{32}} + \delta\epsilon^2\Pi^3_{e^2} \\
&= \delta\epsilon\left[(1-\epsilon)\Pi^3_{e^{12}} + (1-\epsilon)\Pi^3_{e^{32}} + \epsilon\Pi^3_{e^2}\right] \\
&= \delta\epsilon\left[(1-\epsilon)\left(\Pi^3_{e^2} + \gamma_1(1-\epsilon)^2\right) + (1-\epsilon)\Pi^3_{e^2} + \epsilon\Pi^3_{e^2}\right] \\
&= \delta\epsilon\left[(2-\epsilon)\Pi^3_{e^2} + \gamma_1(1-\epsilon)^3\right], \\
\Pi^3_{e^2} &= \delta\epsilon(1-\epsilon)\Pi^1_{e^{13}} + \delta\epsilon(1-\epsilon)\Pi^1_{e^{23}} + \delta\epsilon^2\Pi^1_{e^3} \\
&= \delta\epsilon\left[(1-\epsilon)\left(\Pi^1_{e^2} + \gamma_1(1-\epsilon)^2\right) + \Pi^1_{e^2}\right] \\
&= \delta\epsilon\left[(2-\epsilon)\Pi^1_{e^2} + (1-\epsilon)^3\gamma_1\right].
\end{aligned} \tag{18}$$

We get the following system of equations:

$$\begin{aligned}
\Pi^1_{e^2} &= \delta\epsilon(2-\epsilon)\Pi^2_{e^2} + \delta\epsilon(1-\epsilon)\left[2(1-\epsilon)^2\gamma_2 + \epsilon\right] \\
&= \xi_1\beta_1 + \xi_0\Pi^2_{e^2}, \\
\Pi^2_{e^2} &= \delta\epsilon(2-\epsilon)\Pi^3_{e^2} + \delta\epsilon(1-\epsilon)^3\gamma_1 \\
&= \xi_1\beta_2 + \xi_0\Pi^3_{e^2}, \\
\Pi^3_{e^2} &= \delta\epsilon(2-\epsilon)\Pi^1_{e^2} + \delta\epsilon(1-\epsilon)^3\gamma_1 \\
&= \xi_1\beta_2 + \xi_0\Pi^1_{e^1},
\end{aligned} \tag{19}$$

with $\xi_0 = \delta\epsilon(2-\epsilon)$, $\xi_1 = \delta\epsilon(1-\epsilon)$ and $\beta_1 = 2(1-\epsilon)^2\gamma_2 + \epsilon$, $\beta_2 = (1-\epsilon)^2\gamma_1$.

Solving the system, we get the values of $\Pi^k_{e^2}$, for $k = 1, 2, 3$, and calculate the following limits for later use:

$$\begin{aligned}
\Pi^1_{e^2} &= \frac{\xi_1}{1-\xi_0^3}\left(\beta_1 + \xi_0\beta_2 + \xi_0^2\beta_2\right) & \lim_{\epsilon\downarrow 0}\frac{\Pi^1_{e^2}}{\epsilon} &= 2\delta\gamma_2, \\
\Pi^2_{e^2} &= \frac{\xi_1}{1-\xi_0^3}\left(\beta_2 + \xi_0\beta_2 + \xi_0^2\beta_1\right) & \lim_{\epsilon\downarrow 0}\frac{\Pi^2_{e^2}}{\epsilon} &= \delta\gamma_1, \\
\Pi^3_{e^2} &= \frac{\xi_1}{1-\xi_0^3}\left(\beta_2 + \xi_0\beta_1 + \xi_0^2\beta_2\right) & \lim_{\epsilon\downarrow 0}\frac{\Pi^3_{e^2}}{\epsilon} &= \delta\gamma_1.
\end{aligned} \tag{20}$$

To analyse the best reply of player 1, in state e^2, when s^ϵ is being played, we consider the strategies $s_1^a, s_1^r \in \mathrm{OSD}(s_1^\epsilon, h)$

in which $s_1^a(1 \mid h) = 1$, $s_1^r(0 \mid h) = 1$. Now, we will consider all the possibilities and prove that in the approximating game the actions defined in s^ϵ are in fact the best.

Player 3 was the proponent and proposed $r(h^-) = e^2$; the payment for player 1 in each of his actions is $\Pi_1^t(s_1^a, s_{-1}^\epsilon \mid h) = 0\cdot(1-\epsilon)+\delta\epsilon\Pi_{e^{13}}^1 = \delta\epsilon\Pi_{e^{13}}^1$ and $\Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h) = (1-\epsilon)\delta\Pi_{e^{23}}^1 + \epsilon\delta\Pi_{e^3}^1$. And the difference between the two payoffs is

$$\begin{aligned}
&\Pi_1^t\left(s_1^a, s_{-1}^\epsilon \mid h\right) - \Pi_1^t\left(s_1^r, s_{-1}^\epsilon \mid h\right) \\
&\quad = \delta\epsilon\Pi_{e^{13}}^1 - \left[(1-\epsilon)\,\delta\Pi_{e^{23}}^1 + \epsilon\delta\Pi_{e^{31}}^1\right] \\
&\quad = \delta\epsilon\left[\Pi_{e^3}^1 + \gamma_1(1-\epsilon)^2\right] - (1-\epsilon)\,\delta\Pi_{e^3}^1 - \epsilon\delta\Pi_{e^3}^1 \\
&\quad = (1-\epsilon)\,\epsilon\delta\left(\gamma_1(1-\epsilon) - \frac{\Pi_{e^2}^1}{\epsilon}\right).
\end{aligned} \tag{21}$$

As $\Pi_{e^2}^1/\epsilon \to 2\delta\gamma_2$, if $\gamma_1 > 2\delta\gamma_2$, the inequality $\Pi_1^t(s_1^a, s_{-1}^\epsilon \mid h) \geq \Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h)$ is verified for small values of ϵ. And the acceptance of the proposition should happen with the maximum probability; that is, $s_1^\epsilon(1 \mid h) = 1-\epsilon$.

If in the state e^2 player 3 made a proposition $e \neq e^2$, player 1 payment in case of acceptance is $\Pi_1^t(s_1^a, s_{-1}^\epsilon \mid h) = e_1 s_2^\epsilon(1 \mid h) + \delta s_2^\epsilon(0 \mid h)\Pi_{e^2}^1 \leq \epsilon + \delta(1-\epsilon)\Pi_{e^2}^1$ or in case of rejection is $\Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h) = \delta s_2^\epsilon(1 \mid h)\Pi_{e^1}^1 + \delta s_2^\epsilon(0 \mid h)\Pi_{e^{12}}^1$. As $s_2^\epsilon(1 \mid h) \to 0$, $\Pi_{e^2}^1 \to 0$ and $\Pi_{e^{12}}^1 \to \gamma_1$, $\Pi_1^t(s_1^r, s_{-1}^\epsilon h) - \Pi_1^t(s_1^a, s_{-1}^\epsilon \mid h) \to \delta\gamma_1 > 0$, for small ϵ, the best option for player 1 is to reject the proposal, and $s_1^\epsilon(0 \mid h) = 1-\epsilon$.

When 2 was the proponent and proposed $r(h^-) = e^2$, the payment for player 1 in each of his actions is $\Pi_1^t(s_1^a, s_{-1}^\epsilon \mid h) = 0\cdot(1-\epsilon)+\delta\epsilon\Pi_{e^{12}}^3 = \delta\epsilon\Pi_{e^{13}}^1$ and $\Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h) = (1-\epsilon)\delta\Pi_{e^{32}}^3 + \epsilon\delta\Pi_{e^2}^3$. The difference between the two payoffs is

$$\begin{aligned}
&\Pi_1^t\left(s_1^a, s_{-1}^\epsilon \mid h\right) - \Pi_1^t\left(s_1^r, s_{-1}^\epsilon \mid h\right) \\
&\quad = \delta\epsilon\Pi_{e^{12}}^3 - \left[(1-\epsilon)\,\delta\Pi_{e^{32}}^3 + \epsilon\delta\Pi_{e^2}^3\right] \\
&\quad = \delta\epsilon\left[\Pi_{e^2}^3 + \gamma_1(1-\epsilon)^2\right] - \delta\Pi_{e^2}^3 \\
&\quad = \delta\epsilon\,(1-\epsilon)\left(\gamma_1\,(1-\epsilon) - \frac{\Pi_{e^2}^3}{\epsilon}\right).
\end{aligned} \tag{22}$$

As seen in (20), $\Pi_{e^2}^3/\epsilon \to \delta\gamma_1$, and $[\Pi_1^t(s_1^a, s_{-1}^\epsilon \mid h) - \Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h)]/\epsilon \to (1-\delta)\gamma_1 > 0$; the necessary inequality is verified, for small values of ϵ.

In the case player 2 made a proposition different from the state, it can be proved that player 1 is better off by rejecting the proposition; in the same way, we did when player 3 proposed a division different from the state. Nothing changes in the proof.

When player 1 is proposing, and state is e^2, consider two strategies $s_1^{nd}(e^2 \mid h) = 1$, the "nondeviating" strategy in which 1 always proposes e^2 after h and the "deviating" strategy with player always proposing e, $s_1^d(e \mid h) = 1$, different from e^2. For s to be PER, $\Pi_1^t(s_1^{nd}, s_{-1}^\epsilon \mid h) \geq \Pi_1^t(s_1^d, s_{-1}^\epsilon \mid h)$ for small values of ϵ:

$$\begin{aligned}
&\Pi_1^t\left(s_1^d, s_{-1}^\epsilon \mid h\right) \\
&\quad \leq 1\cdot\epsilon^2 + \delta\epsilon\,(1-\epsilon)\,\Pi_{e^3}^2 + \delta\epsilon\,(1-\epsilon)\,\Pi_{e^2}^2 \\
&\qquad + \delta(1-\epsilon)^2\Pi_{e^{23}}^2 \\
&\quad = \epsilon^2 + \delta\left(1-\epsilon^2\right)\Pi_{e^2}^2, \\
&\Pi_1^t\left(s_1^{nd}, s_{-1}^\epsilon \mid h\right) \\
&\qquad = \delta\epsilon\,(1-\epsilon)\,\Pi_{e^{31}}^2 + \delta\epsilon\,(1-\epsilon)\,\Pi_{e^{21}}^2 + \delta\epsilon^2\Pi_{e^1}^2 \\
&\qquad = 2\delta\epsilon\,(1-\epsilon)\,\Pi_{e^{31}}^2 + \delta\epsilon^2\Pi_{e^1}^2 \\
&\qquad = \delta\,(2-\epsilon)\,\epsilon\Pi_{e^2}^2 + 2\delta\epsilon(1-\epsilon)^3\gamma_2 + \delta\epsilon^2(1-\epsilon)^2 \\
&\qquad = \delta\,(2-\epsilon)\,\epsilon\Pi_{e^2}^2 + \delta\epsilon(1-\epsilon)^2\left[2\gamma_2\,(1-\epsilon) + \epsilon\right].
\end{aligned} \tag{23}$$

And the difference is

$$\begin{aligned}
&\Pi_1^t\left(s_1^{nd}, s_{-1}^\epsilon \mid h\right) - \Pi_1^t\left(s_1^d, s_{-1}^\epsilon \mid h\right) \\
&\quad \geq \delta\epsilon\left\{(2\epsilon-1)\,\frac{\Pi_{e^2}^2}{\epsilon} + (1-\epsilon)^2\left[2\gamma_2\,(1-\epsilon)+\epsilon\right] - \frac{\epsilon}{\delta}\right\}.
\end{aligned} \tag{24}$$

The expression inside the curly brackets, using again (20), converges to $-\delta\gamma_1 + 2\gamma_2$, and if $2\gamma_2 > \delta\gamma_1$, the necessary inequality is assured.

If $\gamma_1 = 2/3$ and $\gamma_2 = 1/3$, all the inequalities are verified. And we conclude that, for the player 1, when the other players follow s^ϵ, the best option at all the possible histories with the state e^2 is to follow it as well.

We will now see that for the other states $e \in E$, player 1 never improves his payment by deviating from strategy s^ϵ. First, when 1 is the proponent, notice that for the proponent the expected payment of a deviation does not depend on the state; it is always equal no matter what the initial state was, $\Pi_1^t(s_1', s_{-1}^\epsilon \mid e) = \Pi_1^t(s_1', s_{-1}^\epsilon \mid e^2)$. Hence, if the proposition is equal to the state, as $\Pi_1^t(s^\epsilon \mid e) \geq \Pi_1^t(s^\epsilon \mid e^2)$, and if in e^2 deviating was not profitable in e, it is not as well, $\Pi_1^t(s^\epsilon \mid e) \geq \Pi_1^t(s^\epsilon \mid e^2) \geq \Pi_1^t(s_1', s_{-1}^\epsilon \mid e^2) = \Pi_1^t(s_1', s_{-1}^\epsilon \mid e)$.

When 1 is the replier and the proposition is not equal to the state e, 1's expected payment by rejecting the proposal is the same as when rejecting a proposition not equal to the state and the state was e^2. So if $r(h^-) = e$ and $h^{t,1} \neq e$ and $r(\tilde{h}^-) = e^2$ and $\tilde{h}^{t,1} \neq e^2$, with $s_1^r \in \mathrm{OSD}(s_1^\epsilon, h)$ and $\tilde{s}_1^r \in \mathrm{OSD}(s_1^\epsilon, \tilde{h})$, are the OSD strategies that reject the deviating proposition at h and $\tilde{h}$, respectively, $\Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h) = \Pi_1^t(\tilde{s}_1^r, s_{-1}^\epsilon \mid \tilde{h})$. The same is valid if the player accepts the deviating proposition; his payment is exactly the same in state e to what it was in state e^2. Defining $s_1^a \in \mathrm{OSD}(s_1^\epsilon, h)$ and $\tilde{s}_1^a \in \mathrm{OSD}(s_1^\epsilon, \tilde{h})$ as the OSD strategies that accept the deviating proposition at h and $\tilde{h}$, respectively, $\Pi_1^t(s_1^a, s_{-1}^\epsilon \mid h) = \Pi_1^t(\tilde{s}_1^a, s_{-1}^\epsilon \mid \tilde{h})$. Accordingly, if in $r(\tilde{h}) = e^2$ there was no advantage in accepting a deviating proposal, $\Pi_1^t(s_1^a, s_{-1}^\epsilon \mid \tilde{h}) \geq \Pi_1^t(s_1^r, s_{-1}^\epsilon \mid \tilde{h})$, in $r(h) = e$ there is no advantage also because the payments are equal in both states, $\Pi_1^t(s_1^a, s_{-1}^\epsilon \mid h) \geq \Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h)$.

The same reasoning can be applied to the histories in which the last proposition was equal to the state $r(h) = h^{t,1} = e$. The player's payoff by rejecting the proposition is equal to the payoff when he rejects $r(\widetilde{h}) = \widetilde{h}^{t,1} = e^2$. That is, the OSD strategies that reject the propositions, $s_1^r \in \mathrm{OSD}(s_1^\epsilon, h)$ and $\widetilde{s}_1^r \in \mathrm{OSD}(s_1^\epsilon, \widetilde{h})$, have the same payment $\Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h) = \Pi_1^t(\widetilde{s}_1^r, s_{-1}^\epsilon \mid \widetilde{h})$, as $\Pi_1^t(s^\epsilon \mid h) - \Pi_1^t(s^\epsilon \mid \widetilde{h}) = (e_1 - e_1^2)(1-\epsilon)^2$. Due to the state's definition, for any $e \in E$, $e_1 \geq e_1^2$; therefore, $\Pi_1^t(s^\epsilon \mid h) \geq \Pi_1^t(s^\epsilon \mid \widetilde{h})$, and we conclude that $\Pi_1^t(s^\epsilon \mid h) \geq \Pi_1^t(s^\epsilon \mid \widetilde{h}) \geq \Pi_1^t(\widetilde{s}_1^r, s_{-1}^\epsilon \mid \widetilde{h}) = \Pi_1^t(s_1^r, s_{-1}^\epsilon \mid h)$. Not to deviate is the best for player 1 when the proposition coincides with the state. This way 1 has no advantage in choosing a different strategy for any of the states in E.

Due to the symmetry of the strategies used in s^ϵ for a state to exist in which any player i had something to gain by deviating then there must also exist a state where 1 would gain by playing the same deviating strategy. As there is no such case, there is no player and no state in which there is a profitable deviation; for this reason, s^ϵ is a best reply to itself, and s is a PER.

4. Conclusion

This paper proposed a new equilibrium concept based on Selten's [3] perfect equilibrium but customized to the multiplayer bargaining game, the PER. It shows that none of the classical equilibria strategies fulfills the requirements to be PER. Builds a new strategy that, using an incentive mechanism to players that follow it, is PER. And in which all divisions in Δ are equilibrium outcomes of game.

It must be noted that in the multiplayer bargaining, strategies should be interpreted as the way to impose a division that was previously established, not as the way to reach the said bargaining division. So, what matters here is to find a strategy that makes the bargaining division (somehow agreed) binding for all players, that is, to find a strategy which does not allow players to diverge from the established path.

However, as all theoretical abstractions, this one is not without application potential. Therefore, although we might find numerous economic situations where multiplayer bargaining takes place, the agreement is obtained in the first period of time, so we do not witness the unroll of equilibrium strategies (besides that first moment). Those strategies are just the warranty the agreed on division is implementable.

While part of the economic theory focus on the first part of the bargaining process, obtaining the best bargain, this specific field of enquiry acts as a reminder that securing the outcome of the bargaining game as it enters its next stage is at least as important as part of the negotiation process.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this article.

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