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Research Article

Stability and Uniform Boundedness in Multidelay Functional Differential Equations of Third Order

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We consider a nonautonomous functional differential equation of the third order with multiple deviating arguments. Using the Lyapunov-Krasovskiì functional approach, we give certain sufficient conditions to guarantee the asymptotic stability and uniform boundedness of the solutions.

1. Introduction

Differential equations of third order are valuable tools in the modeling of many phenomena in various fields of science and engineering (Chlouverakis and Sprott [1], Cronin-Scanlon [2], Eichhorn et al. [3], Friedrichs [4], Linz [5], and Rauch [6]). In reality, the stability and boundedness of solutions of certain nonlinear differential equations of the third order have been received intensive attentions by authors (Ademola et al. [7], Afuwape and Castellanos [8], Chukwu [9], Ezeilo ([10, 11]), Hara [12], Mehri and Shadman [13], Ogundare and Okecha [14], Omeike [15], Reissig et al. [16], Swick [17], Tejumola ([18, 19]), Tunç [20–33], and Yoshizawa [34]).

In 2009, Omeike [15] considered the nonlinear differential equation of the third order with the constant delay $r(>0)$:

$$\dddot{x} + a(t)\ddot{x} + b(t)g(\dot{x}) + c(t)h(x(t-r)) = p(t), \quad (1)$$

and he discussed the stability and boundedness of solutions of this equation.

In this paper, instead of the above equation, we consider the nonautonomous differential equation of the third order with multiple deviating arguments:

$$\dddot{x} + a(t)\ddot{x} + nb(t)g(\dot{x}) + c(t)\sum_{i=1}^{k} h_i(x(t-r_i)) = p(t), \quad (2)$$

where r_i are certain positive constants, and $a(t), b(t), c(t), g(\dot{x}), h(x)$ and $p(t)$ are real valued and continuous functions in their respective arguments with $g(0) = h(0) = 0, k = n$. The existence and uniqueness of the solutions of (2) are also assumed.

The motivation for this paper is a result of the researches mentioned regarding ordinary differential equations of the third order. It follows that the equation discussed in [15] is a special case of (2). Our aim is to improve the results established in [15] from one deviating argument to the multiple deviating arguments for the asymptotic stability and uniform boundedness of solutions. This work contributes to and complements previously known results on the topic in the literature, and it may be useful for researchers working on the qualitative behaviors of solutions. It should be noted that in recent years scores of papers have been published on the qualitative behaviors of solutions (stability of solutions, boundedness of the solutions, existence of the periodic solutions, etc.) of the functional differential equations of the second order with multiple deviating arguments. However, very little attention was given to stability and boundedness of functional differential equations of the third order with multiple deviating arguments ([32]). Therefore, it is worth investigating the qualitative behaviors of solutions in multidelay functional differential equations of the third order. This case is the novelty of the present paper. It should also be noted that the results to be established here are different from those in Tunç [20–33] and the literature.

2. Main Results

Let $p(t) \equiv 0$.

Theorem 1. *One assumes that there exist positive constants $a, b, c, \rho_i, \alpha, \mu_i, \delta_1, \delta_2, \delta_5, \delta_6$, and L such that the following conditions hold:*

(i) $h(0) = g(0) = 0$, $h_i(x)/x \geq \mu_i$, $x \neq 0$, $h_i'(x) \leq \rho_i$, $g(y)/y \geq b > 0$, $y \neq 0$ $(i = 1, 2, \ldots, k)$,

(ii) $0 < \delta_1 \leq c(t) \leq b(t)$, $-L \leq b'(t) \leq c'(t) \leq 0$, $0 < a < a(t)$.

If

$$\frac{b}{\rho_i} > \alpha > \frac{1}{a},$$
$$\frac{1}{2}a'(t) \leq \delta_2 < \delta_1 \left(nb - \alpha \sum_{i=1}^{k} \rho_i \right), \tag{3}$$
$$\sum_{i=1}^{k} r_i < \min \left\{ \frac{2\delta_5}{Lc(\alpha+2)}, \frac{\delta_6}{Lc\alpha} \right\},$$

then every solution $x \equiv x(t)$ of (2) is uniform bounded and satisfies

$$x(t) \longrightarrow 0, \quad \dot{x}(t) \longrightarrow 0, \quad \ddot{x}(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \tag{4}$$

Remark 2. It should be noted that it follows from (ii) that $b(t)$ and $c(t)$ are nonincreasing functions on $[0, \infty)$. Therefore, since these functions are continuous on this interval and bounded below by $\delta_1 > 0$, they are bounded on $[0, \infty)$ and the limit of each exists as $t \to \infty$. Since L in (ii) is an arbitrary selected bound, we can also assume the following estimates:

$$\begin{gathered} 0 < \delta_1 \leq c(t) \leq b(t) \leq L, \\ \lim_{t\to\infty} c(t) = c_0, \qquad \lim_{t\to\infty} b(t) = b_0, \\ \delta_1 \leq c_0 \leq b_0 \leq L. \end{gathered} \tag{5}$$

Proof. We write (2) in the system form as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -a(t)z - nb(t)g(y) - c(t)\sum_{i=1}^{k} h_i(x) \\ &\quad + c(t)\sum_{i=1}^{k} \int_{t-r_i}^{t} h_i'(x(s))\,y(s)\,ds. \end{aligned} \tag{6}$$

Define a Lyapunov-Krasovskiì functional ([35]) $V(\cdot) = V(t, x_t, y_t, z_t)$ by

$$\begin{aligned} V(\cdot) &= c(t)H(x) + n\alpha b(t)G(y) + \alpha c(t)y\sum_{i=1}^{k} h_i(x) \\ &\quad + \frac{1}{2}a(t)y^2 + \frac{1}{2}\alpha z^2 + yz \\ &\quad + \sum_{i=1}^{k} \lambda_i \int_{-r_i}^{0} \int_{t+s}^{t} y^2(\theta)\,d\theta\,ds, \end{aligned} \tag{7}$$

where $H(x) \equiv \int_0^x \sum_{i=1}^k h_i(s)ds$, $G(y) \equiv \int_0^y g(\xi)d\xi$, and λ_i are certain positive constants, which will be determined later in the proof.

This functional can be arranged as follows:

$$V(\cdot) = V_1 + \frac{1}{2}V_2 + \sum_{i=1}^{k} \lambda_i \int_{-r_i}^{0} \int_{t+s}^{t} y^2(\theta)\,d\theta\,ds, \tag{8}$$

where

$$\begin{gathered} V_1 = c(t)\left[H(x) + n\alpha\frac{b(t)}{c(t)}G(y) + \alpha y\sum_{i=1}^{k} h_i(x) \right], \\ V_2 = a(t)y^2 + \alpha z^2 + 2yz. \end{gathered} \tag{9}$$

Using the assumptions of Theorem 1, it follows that $\alpha a(t) \geq \alpha a > 1$ since $\alpha > a^{-1}$ and $\alpha a(t) - 1 > 0$.

Thus, there exist constants $\delta_1 > 0$ and $\delta_3 > 0$ such that

$$\begin{aligned} V_2 &= a(t)y^2 + \alpha z^2 + 2yz \\ &= a(t)\left[y + \frac{z}{2a(t)} \right]^2 + \frac{1}{4a(t)}[4\alpha a(t) - 1]z^2 \\ &\geq \frac{1}{2}\delta_3 y^2 + \frac{1}{2}\delta_3 z^2, \\ V_1 &\geq \delta_1 \left[H(x) + \frac{1}{2}n\alpha by^2 + \alpha y\sum_{i=1}^{k} h_i(x) \right] \end{aligned} \tag{10}$$

since $b(t)/c(t) \geq 1$, $c(t) \geq \delta_1 > 0$, and $g(y)/y \geq b > 0$ imply that $G(y) \geq (1/2)by^2$. Further, using the assumptions of Theorem 1 and $k = n$, it follows that

$$\begin{aligned}
&H(x)+\frac{1}{2}n\alpha by^2+\alpha y\sum_{i=1}^{k}h_i(x)\\
&\quad=\frac{1}{2}\left[2H(x)+n\alpha by^2+2\alpha y\sum_{i=1}^{k}h_i(x)\right]\\
&\quad=\frac{1}{2}\left\{\frac{\alpha}{b}\left[\left(by+h_1(x)\right)^2+\left(by+h_2(x)\right)^2\right.\right.\\
&\qquad\quad\left.+\cdots+\left(by+h_n(x)\right)^2\right]\\
&\qquad\quad\left.+2H(x)-\frac{\alpha}{b}\sum_{i=1}^{k}h_i^2(x)\right\}\\
&\quad=\frac{1}{2}\frac{\alpha}{b}\left\{\left(by+h_1(x)\right)^2+\left(by+h_2(x)\right)^2\right.\\
&\qquad\qquad\left.+\cdots+\left(by+h_n(x)\right)^2\right\}\\
&\qquad\qquad+\sum_{i=1}^{k}\int_0^x\left(1-\frac{\alpha h_i'(s)}{b}\right)h_i(s)\,ds\\
&\quad\geq\sum_{i=1}^{k}\int_0^x\left(1-\frac{\alpha\rho_i}{b}\right)\frac{h_i(s)}{s}s\,ds\\
&\quad\geq\sum_{i=1}^{k}\int_0^x\left(1-\frac{\alpha\rho_i}{b}\right)\mu_i s\,ds\\
&\quad\geq\frac{1}{2}\sum_{i=1}^{k}\left(1-\frac{\alpha\rho_i}{b}\right)\mu_i x^2=\frac{\delta_4}{2}x^2
\end{aligned}\tag{11}$$

so that

$$V_1\geq\frac{\delta_1\delta_4}{2}x^2,\tag{12}$$

where

$$\delta_4=\sum_{i=1}^{k}\left(1-\frac{\alpha\rho_i}{b}\right)\mu_i.\tag{13}$$

In view of the previous discussion, we can get

$$\begin{aligned}
V(\cdot)\geq{}&\left(\frac{\delta_1\delta_4}{2}\right)x^2+\frac{\delta_3}{4}y^2+\frac{\delta_3}{4}z^2\\
&+\sum_{i=1}^{k}\lambda_i\int_{-r_i}^{0}\int_{t+s}^{t}y^2(\theta)\,d\theta\,ds.
\end{aligned}\tag{14}$$

Using a basic calculation, the time derivative of $V(\cdot)$ along solutions of (6) results in

$$\begin{aligned}
\frac{d}{dt}V(\cdot)={}&c'(t)H(x)+c(t)H'(x)+n\alpha b'(t)G(y)\\
&+n\alpha b(t)G'(y)\\
&+\alpha c'(t)y\sum_{i=1}^{k}h_i(x)+\alpha c(t)y'\sum_{i=1}^{k}h_i\\
&+\alpha c(t)yx'\sum_{i=1}^{k}h_i'(x)\\
&+\frac{1}{2}a'(t)y^2+a(t)yy'+\alpha zz'+y'z+yz'\\
&+\sum_{i=1}^{k}\lambda_i\int_{-r_i}^{0}\left[y^2(t)-y^2(t+s)\right]ds\\
={}&c'(t)H(x)+c(t)y\sum_{i=1}^{k}h_i(x)+n\alpha b'(t)G(y)\\
&+n\alpha b(t)zg(y)\\
&+\alpha c'(t)y\sum_{i=1}^{k}h_i(x)+\alpha c(t)z\sum_{i=1}^{k}h_i(x)\\
&+\alpha c(t)y^2\sum_{i=1}^{k}h_i'(x)\\
&+\frac{1}{2}a'(t)y^2+a(t)yz+\alpha z\\
&\times\left[-a(t)z-nb(t)g(y)-c(t)\sum_{i=1}^{k}h_i(x(t))\right.\\
&\qquad\left.+c(t)\sum_{i=1}^{k}\int_{t-r_i}^{t}h_i'(x(s))y(s)\,ds\right]\\
&+z^2+y\left[-a(t)z-nb(t)g(y)\right.\\
&\qquad-c(t)\sum_{i=1}^{k}h_i(x(t))\\
&\qquad\left.+c(t)\sum_{i=1}^{k}\int_{t-r_i}^{t}h_i'(x(s))y(s)\,ds\right]\\
&+\sum_{i=1}^{k}\lambda_i r_i y^2-\sum_{i=1}^{k}\lambda_i\int_{t-r_i}^{t}y^2(s)\,ds
\end{aligned}$$

$$= c'(t) H(x) + \alpha c'(t) y \sum_{i=1}^{k} h_i(x) + n\alpha b'(t) G(y)$$
$$+ \frac{1}{2} a'(t) y^2$$
$$- \left[nyb(t) g(y) - \alpha c(t) y^2 \sum_{i=1}^{k} h_i'(x) - \sum_{i=1}^{k} \lambda_i r_i y^2 \right]$$
$$- [\alpha a(t) - 1] z^2 + \alpha z c(t) \sum_{i=1}^{k} \int_{t-r_i}^{t} h_i'(x(s)) y(s) ds$$
$$+ yc(t) \sum_{i=1}^{k} \int_{t-r_i}^{t} h_i'(x(s)) y(s) ds$$
$$- \sum_{i=1}^{k} \lambda_i \int_{t-r_i}^{t} y^2(s) ds. \tag{15}$$

Using $h_i'(x) \le \rho_i$, $c(t) \le L$, and the estimate $2|ef| \le e^2 + f^2$, we have

$$\alpha z c(t) \sum_{i=1}^{k} \int_{t-r_i}^{t} h_i'(x(s)) y(s) ds$$
$$\le \alpha c(t) |z| \sum_{i=1}^{k} \int_{t-r_i}^{t} c|y(s)| ds$$
$$\le \frac{1}{2} \alpha c(t) z^2 \sum_{i=1}^{k} r_i \rho_i + \frac{1}{2} \alpha c(t) \sum_{i=1}^{k} \int_{t-r_i}^{t} \rho_i y^2(s) ds$$
$$\le \frac{1}{2} L\alpha c z^2 \sum_{i=1}^{k} r_i + \frac{1}{2} L\alpha c \sum_{i=1}^{k} \int_{t-r_i}^{t} y^2(s) ds,$$
$$yc(t) \sum_{i=1}^{k} \int_{t-r_i}^{t} h_i'(x(s)) y(s) ds \tag{16}$$
$$\le c(t) |y| \sum_{i=1}^{k} \int_{t-r_i}^{t} \rho_i |y(s)| ds$$
$$\le \frac{1}{2} c(t) y^2 \sum_{i=1}^{k} r_i \rho_i + \frac{1}{2} c(t) \sum_{i=1}^{k} \int_{t-r_i}^{t} \rho_i y^2(s) ds$$
$$\le \frac{1}{2} L c y^2 \sum_{i=1}^{k} r_i + \frac{1}{2} L c \sum_{i=1}^{k} \int_{t-r_i}^{t} y^2(s) ds,$$

where

$$c = \max \rho_i. \tag{17}$$

Noting the previous discussion, it follows that

$$\frac{d}{dt} V(\cdot) \le c'(t) H(x) + n\alpha b'(t) G(y) + \alpha c'(t) y \sum_{i=1}^{k} h_i(x)$$
$$+ \frac{1}{2} a'(t) y^2$$
$$- \left[nyb(t) g(y) - \alpha c(t) y^2 \sum_{i=1}^{k} h_i'(x) \right.$$
$$\left. - \sum_{i=1}^{k} \lambda_i r_i y^2 \right]$$
$$- \frac{1}{2} \left[2(\alpha a(t) - 1) - L\alpha c \sum_{i=1}^{k} r_i \right] z^2 + \frac{1}{2} L c y^2 \sum_{i=1}^{k} r_i$$
$$+ \sum_{i=1}^{k} \left[\frac{1}{2} L c (\alpha + 1) - \lambda_i \right] \int_{t-r_i}^{t} y^2(s) ds. \tag{18}$$

If $y = 0$, then

$$nyb(t) g(y) - \alpha c(t) y^2 \sum_{i=1}^{k} h_i'(x) - \sum_{i=1}^{k} \lambda_i r_i y^2 = 0. \tag{19}$$

If $y \ne 0$, then it follows that

$$nyb(t) g(y) - \alpha c(t) y^2 \sum_{i=1}^{k} h_i'(x) - \sum_{i=1}^{k} \lambda_i r_i y^2$$
$$= \left[nb(t) \frac{g(y)}{y} - \alpha c(t) \sum_{i=1}^{k} h_i'(x) - \sum_{i=1}^{k} \lambda_i r_i \right] y^2$$
$$\ge \left[nb(t) b - \alpha c(t) \sum_{i=1}^{k} \rho_i - \sum_{i=1}^{k} \lambda_i r_i \right] y^2 \tag{20}$$
$$= c(t) \left[\frac{b(t)}{c(t)} nb - \alpha \sum_{i=1}^{k} \rho_i \right] y^2 - \sum_{i=1}^{k} \lambda_i r_i y^2$$
$$\ge \delta_1 \left[nb - \alpha \sum_{i=1}^{k} \rho_i - \delta_1^{-1} \sum_{i=1}^{k} \lambda_i r_i \right] y^2.$$

Since $0 \le \delta_1 \le c(t) \le b(t)$, $1 \le b(t)/c(t)$, and $\alpha a(t) \ge \alpha a > 1$, then

$$\begin{aligned}
&\frac{1}{2}a'(t)y^2 \\
&\quad - \left[nyb(t)g(y) - \alpha c(t)y^2\sum_{i=1}^{k}h_i'(x) - \sum_{i=1}^{k}\lambda_i r_i y^2\right] \\
&\le \left[\delta_2 - \delta_1\left(nb - \alpha\sum_{i=1}^{k}\rho_i\right) + \sum_{i=1}^{k}\lambda_i r_i\right]y^2 \\
&= -\left(\delta_5 - \sum_{i=1}^{k}\lambda_i r_i\right)y^2, \\
&\left[2(\alpha a(t) - 1) - L\alpha c\sum_{i=1}^{k}r_i\right]z^2 \ge \left(\delta_6 - L\alpha c\sum_{i=1}^{k}r_i\right)z^2,
\end{aligned} \tag{21}$$

where $\delta_5 = \delta_1(nb - \alpha\sum_{i=1}^{k}\rho_i) - \delta_2 > 0$ and $\delta_6 \equiv 2(\alpha a - 1) > 0$. Thus, we get

$$\begin{aligned}
\frac{d}{dt}V(\cdot) &\le c'(t)H(x) + n\alpha b'(t)G(y) + \alpha c'(t)y\sum_{i=1}^{k}h_i(x) \\
&\quad - \left\{\delta_5 - \sum_{i=1}^{k}\lambda_i r_i\right\}y^2 - \frac{1}{2}\left(\delta_6 - L\alpha c\sum_{i=1}^{k}r_i\right)z^2 \\
&\quad + \frac{1}{2}Lcy^2\sum_{i=1}^{k}r_i \\
&\quad + \sum_{i=1}^{k}\left[\frac{1}{2}Lc(\alpha+1) - \lambda_i\right]\int_{t-r_i}^{t}y^2(s)\,ds.
\end{aligned} \tag{22}$$

Let $\lambda_i = (1/2)Lc(\alpha+1)$. Hence,

$$\begin{aligned}
&\frac{d}{dt}V(\cdot) \\
&\quad \le c'(t)H(x) + n\alpha b'(t)G(y) + \alpha c'(t)y\sum_{i=1}^{k}h_i(x) \\
&\qquad - \left\{\delta_5 - \frac{1}{2}Lc(\alpha+2)\sum_{i=1}^{k}r_i\right\}y^2 \\
&\qquad - \frac{1}{2}\left(\delta_6 - L\alpha c\sum_{i=1}^{k}r_i\right)z^2.
\end{aligned} \tag{23}$$

If $c'(t) = 0$, then $c'(t)H(x) + n\alpha b'(t)G(y) + \alpha c'(t)y\sum_{i=1}^{k}h_i(x) = n\alpha b'(t)G(y) \le 0$ since $b'(t) \le 0$ and $G(y) \ge 0$. For those t's such that $c'(t) < 0$, we have

$$\begin{aligned}
&c'(t)H(x) + n\alpha b'(t)G(y) + \alpha c'(t)y\sum_{i=1}^{k}h_i(x) \\
&\quad = c'(t)\left[H(x) + n\alpha\frac{b'(t)}{c'(t)}G(y) + \alpha y\sum_{i=1}^{k}h_i(x)\right] \\
&\quad \le c'(t)\left[H(x) + n\alpha G(y) + \alpha y\sum_{i=1}^{k}h_i(x)\right] \\
&\quad \le c'(t)\delta_4 H(x) \le 0.
\end{aligned} \tag{24}$$

Thus,

$$\begin{aligned}
\frac{d}{dt}V(\cdot) \le &-\left\{\delta_5 - \frac{1}{2}Lc(\alpha+2)\sum_{i=1}^{k}r_i\right\}y^2 \\
&- \frac{1}{2}\left(\delta_6 - L\alpha c\sum_{i=1}^{k}r_i\right)z^2.
\end{aligned} \tag{25}$$

Therefore, if

$$\sum_{i=1}^{k}r_i < \min\left\{\frac{2\delta_5}{Lc(\alpha+2)}, \frac{\delta_6}{L\alpha c}\right\}, \tag{26}$$

then we have

$$\frac{d}{dt}V(\cdot) \le -\beta\left(y^2 + z^2\right) \quad \text{for some } \beta > 0. \tag{27}$$

The proof for Theorem 1 is complete. □

Let $p(t) \ne 0$.

Theorem 3. *One assumes that all the assumptions of Theorem 1 and the assumption*

$$\int_0^t |p(s)|\,ds < \infty \tag{28}$$

hold. If

$$\begin{aligned}
&\frac{b}{\rho_i} > \alpha > \frac{1}{a}, \\
&\frac{1}{2}a'(t) \le \delta_2 < \delta_1\left(nb - \alpha\sum_{i=1}^{k}\rho_i\right), \\
&\sum_{i=1}^{k}r_i < \min\left\{\frac{2\delta_5}{Lc(\alpha+2)}, \frac{\delta_6}{Lc\alpha}\right\},
\end{aligned} \tag{29}$$

then all solutions of (2) *are bounded.*

Proof. Equation (2) is equivalent to the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -a(t)z - nb(t)g(y) - c(t)\sum_{i=1}^{k} h_i(x(t)) \\ &\quad + c(t)\sum_{i=1}^{k}\int_{t-r_i}^{t} h_i'(x(s))\,y(s)\,ds + p(t). \end{aligned} \tag{30}$$

Along any solution $(x(t), y(t), z(t))$ of (6), we have

$$\dot{V}_{(3)}(\cdot) = \dot{V}_{(2)}(\cdot) + (y + \alpha z)\,p(t). \tag{31}$$

Since $\dot{V}_{(2)}(\cdot) \le 0$, then it follows that

$$\begin{aligned} \dot{V}_{(3)}(\cdot) &\le (y + \alpha z)\,p(t) \le (|y| + \alpha|z|)\,|p(t)| \\ &\le \delta_8(|y| + |z|)\,|p(t)|, \end{aligned} \tag{32}$$

where $\delta_8 \equiv \max\{1, \alpha\}$. Noting that $|m| < 1 + m^2$, we get

$$\begin{aligned} \dot{V}_{(3)}(\cdot) &\le \delta_8\left(2 + y^2 + z^2\right)|p(t)| \le 2\delta_8|p(t)| + \delta_8\|X\|^2|p(t)| \\ &\le 2\delta_8|p(t)| + \left(\frac{\delta_8}{\delta_7}\right)V(\cdot)\,|p(t)| \end{aligned} \tag{33}$$

recalling that $\delta_7\|X\|^2 \le V(\cdot)$.

Let $\eta = \max(2\delta_8, \delta_8/\delta_7)$, then

$$\dot{V}_{(3)}(\cdot) \le \eta|p(t)| + \eta V(\cdot)\,|p(t)| \tag{34}$$

or

$$\dot{V}_{(3)}(\cdot) - \eta V(\cdot)\,|p(t)| \le \eta|p(t)|. \tag{35}$$

Multiplying each side of this estimate by the integrating factor $\exp(-\eta\int_0^t |p(s)|ds)$, we get

$$\begin{aligned} &\dot{V}_{(3)}(\cdot)\exp\left(-\eta\int_0^t |p(s)|\,ds\right) \\ &\quad - \eta V(\cdot)\,|p(t)|\exp\left(-\eta\int_0^t |p(s)|\,ds\right) \\ &\le \eta|p(t)|\exp\left(-\eta\int_0^t |p(s)|\,ds\right). \end{aligned} \tag{36}$$

Integrating each side of this estimate from 0 to t, we obtain

$$\begin{aligned} &V(\cdot)\exp\left(-\eta\int_0^t |p(s)|\,ds\right) - V(0) \\ &\le 1 - \exp\left(-\eta\int_0^t |p(s)|\,ds\right) \end{aligned} \tag{37}$$

or

$$V(\cdot) \le V(0)\exp\left(\eta\int_0^t |p(s)|\,ds\right) + \exp\left(\eta\int_0^t |p(s)|\,ds\right) - 1, \tag{38}$$

where $(0, x(0), y(0), z(0)) = 0$.

Since $\int_0^t |p(s)|ds \le A$ for all t, this implies

$$V(\cdot) \le V(0)\,e^{\eta A} + \left(e^{\eta A} - 1\right) \quad \text{for } t \ge 0. \tag{39}$$

Since the right-hand side of the last estimate is a constant and $V(\cdot) \to \infty$ when $x^2 + y^2 + z^2 \to \infty$, it follows that there exists a positive constant D such that

$$|x(t)| \le D, \quad |y(t)| \le D, \quad |z(t)| \le D \quad \forall t \ge 0. \tag{40}$$

From the system (30) this implies that

$$|x(t)| \le D, \quad |\dot{x}(t)| \le D, \quad |\ddot{x}(t)| \le D \quad \forall t \ge 0. \tag{41}$$

The proof for Theorem 3 is complete. □

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