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Research Article

A Generalized Nonlinear Sum-Difference Inequality of Product Form

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We establish a generalized nonlinear discrete inequality of product form, which includes both nonconstant terms outside the sums and composite functions of nonlinear function and unknown function without assumption of monotonicity. Upper bound estimations of unknown functions are given by technique of change of variable, amplification method, difference and summation, inverse function, and the dialectical relationship between constants and variables. Using our result we can solve both the discrete inequality in Pachpatte (1995). Our result can be used as tools in the study of difference equations of product form.

1. Introduction

Being an important tool in the study of existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of differential equations and integral equations, various generalizations of Gronwall inequalities [1, 2] and their applications have attracted great interests of many mathematicians (such as [3–6]). Some recent works can be found, for example, in [7–10] and some references therein. Along with the development of the theory of integral inequalities and the theory of difference equations, more attention is paid to some discrete versions of Gronwall-Bellman type inequalities (such as [3, 4, 11–13]). Some recent works can be found, for example, in [14–24] and some references therein.

Pachpatte [4] obtained the explicit bound to the unknown function of the following sum-difference inequality:

$$u^2(n) \le \left(c_1^2 + 2\sum_{s=0}^{n-1} f(s)\,u(s)\right)\left(c_2^2 + 2\sum_{s=0}^{n-1} g(s)\,u(s)\right). \quad (1)$$

Pachpatte [3] obtained the estimation of the unknown function of the following inequality:

$$u(n) \le \left(c_1 + \sum_{s=0}^{n} f(s)\,u(s)\right)\left(c_2 + \sum_{s=0}^{n} g(s)\,u(s)\right). \quad (2)$$

Then, the estimation can be used to study the boundedness, asymptotic behavior, and slow growth of the solutions of the sum-difference equation:

$$\begin{aligned} x(n) = k &\left(p_1(n) + \sum_{s=0}^{n-1} f_1(n-s)\,x(s)\right) \\ &\times \left(p_2(n) + \sum_{s=0}^{n-1} f_2(n-s)\,x(s)\right). \end{aligned} \quad (3)$$

However, the bound given on such inequalities in [3, 4] is not directly applicable in the study of certain sum-difference equations. It is desirable to establish new inequalities of the above type, which can be used more effectively in the study of certain classes of sum-difference equations of product form.

In this paper, we establish a new integral inequality of product form

$$\begin{aligned} u(n) \le &\left(p_1(n) + \sum_{s=0}^{n-1} f_1(n,s)\,\varphi_1(u(s))\right) \\ &\times \left(p_2(n) + \sum_{s=0}^{n-1} f_2(n,s)\,\varphi_2(u(s))\right), \quad \forall n \in \mathbb{N}_0, \end{aligned} \quad (4)$$

where p_i, f_i, φ_i $(i = 1, 2)$ may not be monotone. For φ_1, φ_2, we employ a technique of monotonization to construct two functions; the second possesses stronger monotonicity than the first. We can demonstrate that inequalities (1) and (2), considered in [3, 4], respectively, can also be solved with our result. Finally, we expound that we can give estimation of solutions of a class of sum-difference equations of product form.

2. Main Result and Proof

In this section, we proceed to solve the discrete inequality (4) and present explicit bounds on the embedded unknown function. Let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{N} := \{1, 2, \ldots\}$, and $\mathbb{N}_a^b := \{a, a+1, a+2, \ldots, a+n = b\}$ ($a \in \mathbb{N}_0$, $n, b \in \mathbb{N}$). For function $z(n)$, its difference is defined by $\Delta z = z(n+1) - z(n)$. Obviously, the linear difference equation $\Delta z(n) = b(n)$ with the initial condition $z(n_0) = 0$ has the solution $z(n) = \sum_{s=n_0}^{n-1} b(s)$. For convenience, in the sequel we complementarily define that $\sum_{s=n_0}^{n_0-1} b(s) = 0$.

First of all, we monotonize some given functions p_i, f_i, φ_i in the sum; let

$$q_i(n) = \max_{t \in \mathbb{N}_0^n} p_i(t), \quad g_i(n,s) = \max_{t \in \mathbb{N}_0^n} f_i(t,s), \quad i = 1, 2, \tag{5}$$

where q_i and g_i are all nondecreasing in n ($i = 1, 2$) and satisfy

$$q_i(n) \geq p_i(n) \geq 0, \qquad g_i(n,s) \geq f_i(n,s) \geq 0. \tag{6}$$

Let

$$w_1(u) = \max_{s \in [0,u]} \varphi_i(s), \qquad w_2(u) := \max_{s \in [0,u]} \left\{ \frac{\varphi_2(s)}{w_1(s)} \right\} w_1(u), \tag{7}$$

where $w_i(u)$ is nondecreasing in u ($i = 1, 2$) and $w_2(u)/w_1(u)$ is also nondecreasing in u and satisfies

$$w_i(u) \geq \varphi_i(u), \quad i = 1, 2, \tag{8}$$

$$W_1(x) = \int_{x_0}^{x} \frac{ds}{w_1(s)}, \quad x_0 > 0, \tag{9}$$

$$W_2(x) := \int_{x_0}^{x} \frac{w_1\left(W_1^{-1}(s)\right) ds}{w_2\left(W_1^{-1}(s)\right)}, \quad x_0 > 0, \tag{10}$$

$$W_3(x) := \int_{x_0}^{x} \frac{ds}{w_1\left(W_1^{-1}\left(W_2^{-1}(s)\right)\right)}, \quad x_0 > 0, \tag{11}$$

where W_1^{-1}, W_2^{-1} denote the inverse function of W_1, W_2, respectively.

Theorem 1. *Let f, g be nonnegative and given functions on $\mathbb{N}_0$. Suppose that u is a nonnegative and unknown function. Then, the discrete inequality* (4) *gives*

$$u(n) \leq W_1^{-1}\left(W_2^{-1}\left(W_3^{-1}\left(A(n) + B(n)\right)\right)\right), \quad \forall n \in \mathbb{N}_0^b, \tag{12}$$

where W_1, W_2, W_3 are defined by (9), (10), *and* (11), *respectively, W_1^{-1}, W_2^{-1}, W_3^{-1} denote the inverse functions of W_1, W_2, W_3, respectively,*

$$\begin{aligned} A(n) &= W_3\left(W_2\left(W_1\left(q_1(n)\, q_2(n)\right) + \sum_{t=0}^{n-1} q_2(n)\, g_1(n,t)\right)\right. \\ &\qquad \left. + \sum_{s=0}^{n-1} q_1(n)\, g_2(n,t)\right), \\ B(n) &= \sum_{t=0}^{n-1}\left(g_2(n,t) \sum_{s=0}^{t} g_1(n,s) + g_1(n,t) \sum_{s=0}^{t-1} g_2(n,s)\right), \end{aligned} \tag{13}$$

and b is the largest natural number such that

$$\begin{aligned} b = \operatorname{Max}\Big\{ n \in \mathbb{N}_0 : & \ A(n) + B(n) \in \operatorname{Dom}\left(W_3^{-1}\right), \\ & W_3^{-1}\left(A(n) + B(n)\right) \in \operatorname{Dom}\left(W_2^{-1}\right), \\ & W_2^{-1}\left(W_3^{-1}\left(A(n) + B(n)\right)\right) \in \operatorname{Dom}\left(W_1^{-1}\right) \Big\}. \end{aligned} \tag{14}$$

Proof. Using (5), (6), (7), and (8), we observe that

$$\begin{aligned} u(n) &\leq \left(p_1(n) + \sum_{s=0}^{n-1} f_1(n,s)\, \varphi_1(u(s))\right) \\ &\quad \times \left(p_2(n) + \sum_{s=0}^{n-1} f_2(n,s)\, \varphi_2(u(s))\right) \\ &\leq \left(q_1(n) + \sum_{s=0}^{n-1} g_1(n,s)\, w_1(u(s))\right) \\ &\quad \times \left(q_2(n) + \sum_{s=0}^{n-1} g_2(n,s)\, w_2(u(s))\right) \\ &\leq \left(q_1(T) + \sum_{s=0}^{n-1} g_1(T,s)\, w_1(u(s))\right) \\ &\quad \times \left(q_2(T) + \sum_{s=0}^{n-1} g_2(T,s)\, w_2(u(s))\right), \quad \forall n \in \mathbb{N}_0^T, \end{aligned} \tag{15}$$

where $T \in \mathbb{N}_0^J$ is chosen arbitrarily. Let $v(n)$ denote the function on the right-hand side of (15), namely,

$$\begin{aligned} v(n) &= \left(q_1(T) + \sum_{s=0}^{n-1} g_1(T,s)\, w_1(u(s))\right) \\ &\quad \times \left(q_2(T) + \sum_{s=0}^{n-1} g_2(T,s)\, w_2(u(s))\right), \quad \forall n \in \mathbb{N}_0^T, \end{aligned} \tag{16}$$

which is a nonnegative and nondecreasing function on $\mathbb{N}_0^T$ with $v(0) = q_1(T) q_2(T)$. Then (4) is equivalent to

$$u(n) \leq v(n), \quad \forall n \in \mathbb{N}_0^T. \tag{17}$$

Using the difference formula

$$\Delta x(n)\, y(n) = x(n+1)\,\Delta y(n) + y(n)\,\Delta x(n) \tag{18}$$

and the monotonicity of w_i and v, from (16) and (17), we observe that

$$\begin{aligned}
\Delta v(n) &= \left(q_1(T) + \sum_{s=0}^{n-1} g_1(T,s)\, w_1(u(s))\right) \\
&\quad \times \Delta\left(q_2(T) + \sum_{s=0}^{n-1} g_2(T,s)\, w_2(u(s))\right) \\
&\quad + \Delta\left(q_1(T) + \sum_{s=0}^{n-1} g_1(T,s)\, w_1(u(s))\right) \\
&\quad \times \left(q_2(T) + \sum_{s=0}^{n-1} g_2(T,s)\, w_2(u(s))\right) \\
&= \left(q_1(T) + \sum_{s=0}^{n} g_1(T,s)\, w_1(u(s))\right) \\
&\quad \times g_2(T,n)\, w_2(u(n)) \\
&\quad + g_1(T,n)\, w_1(u(n)) \\
&\quad \times \left(q_2(T) + \sum_{s=0}^{n-1} g_2(T,s)\, w_2(u(s))\right) \\
&\le \left(q_1(T) + \sum_{s=0}^{n} g_1(T,s)\, w_1(v(s))\right) \\
&\quad \times g_2(T,n)\, w_2(v(n)) \\
&\quad + g_1(T,n)\, w_1(v(n)) \\
&\quad \times \left(q_2(T) + \sum_{s=0}^{n-1} g_2(T,s)\, w_2(v(s))\right) \\
&= w_1(v(n)) \Bigg(q_2(T)\, g_1(T,n) \\
&\qquad + q_1(T)\, g_2(T,n) \frac{w_2(v(n))}{w_1(v(n))} \\
&\qquad + g_2(T,n) \frac{w_2(v(n))}{w_1(v(n))} \\
&\qquad \times \sum_{s=0}^{n} g_1(T,s)\, w_1(v(s)) \\
&\qquad + g_1(T,n) \sum_{s=0}^{n-1} g_2(T,s)\, w_2(v(s)) \Bigg) \\
&\le w_1(v(n)) \Bigg(q_2(T)\, g_1(T,n) \\
&\qquad + q_1(T)\, g_2(T,n) \frac{w_2(v(n))}{w_1(v(n))} \\
&\qquad + \left(g_2(T,n) \sum_{s=0}^{n} g_1(T,s) \right. \\
&\qquad \left. + g_1(T,n) \sum_{s=0}^{n-1} g_2(T,s) \right) \\
&\qquad \times w_2(v(n)) \Bigg), \\
&\qquad \forall n \in \mathbb{N}_0^T,
\end{aligned} \tag{19}$$

for all $n \in \mathbb{N}_0^T$. From (19), we have

$$\begin{aligned}
\frac{\Delta v(n)}{w_1(v(n))} &\le q_2(T)\, g_1(T,n) + q_1(T)\, g_2(T,n) \frac{w_2(v(n))}{w_1(v(n))} \\
&\quad + \left(g_2(T,n) \sum_{s=0}^{n} g_1(T,s) + g_1(T,n) \right. \\
&\quad \left. \times \sum_{s=0}^{n-1} g_2(T,s) \right) w_2(v(n)), \quad \forall n \in \mathbb{N}_0^T.
\end{aligned} \tag{20}$$

On the other hand, by the mean value theorem for integrals, for arbitrarily given integers $n, n+1 \in \mathbb{N}_0^T$, there exists ξ in the open interval $(v(n), v(n+1))$ such that

$$\begin{aligned}
&W_1(v(n+1)) - W_1(v(n)) \\
&\quad = \int_{v(n)}^{v(n+1)} \frac{ds}{w_1(s)} = \frac{\Delta v(n)}{w_1(\xi)} \le \frac{\Delta v(n)}{w_1(v(n))}, \quad \forall n \in \mathbb{N}_0^T,
\end{aligned} \tag{21}$$

where W_1 is defined by (9). From (20) and (21), we have

$$\begin{aligned}
&W_1(v(n+1)) - W_1(v(n)) \\
&\quad \le q_2(T)\, g_1(T,n) + q_1(T)\, g_2(T,n) \frac{w_2(v(n))}{w_1(v(n))} \\
&\qquad + \left(g_2(T,n) \sum_{s=0}^{n} g_1(T,s) \right. \\
&\qquad \left. + g_1(T,n) \sum_{s=0}^{n-1} g_2(T,s) \right) w_2(v(n)),
\end{aligned} \tag{22}$$

for all $n \in \mathbb{N}_0^T$. By setting $n = t$ in (22) and substituting $t = 0, 1, 2, \ldots, n-1$ successively, we obtain

$$\begin{aligned} W_1(v(n)) \le{}& W_1(v(0)) + \sum_{t=0}^{n-1} q_2(T) g_1(T,t) + \sum_{t=0}^{n-1} q_1(T) g_2(T,t) \frac{w_2(v(t))}{w_1(v(t))} \\ &+ \sum_{t=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right) w_2(v(t)) \\ \le{}& W_1(v(0)) + \sum_{t=0}^{T-1} q_2(T) g_1(T,t) + \sum_{t=0}^{n-1} q_1(T) g_2(T,t) \frac{w_2(v(t))}{w_1(v(t))} \\ &+ \sum_{t=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right) w_2(v(t)), \quad \forall n \in \mathbb{N}_0^T. \end{aligned} \tag{23}$$

Let $x(n)$ denote the function on the right-hand side of (23); namely,

$$\begin{aligned} x(n) ={}& W_1(v(0)) + \sum_{t=0}^{T-1} q_2(T) g_1(T,t) + \sum_{t=0}^{n-1} q_1(T) g_2(T,t) \frac{w_2(v(t))}{w_1(v(t))} \\ &+ \sum_{t=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right) w_2(v(t)), \quad \forall n \in \mathbb{N}_0^T. \end{aligned} \tag{24}$$

Then $x(0) = W_1(v(0)) + \sum_{t=0}^{T-1} q_2(T) g_1(T,t)$, x is a nonnegative and nondecreasing function on $\mathbb{N}_0^T$, and (23) is equivalent to

$$v(n) \le W_1^{-1}(x(n)), \quad \forall n \in \mathbb{N}_0^T. \tag{25}$$

From (24), we obtain

$$\begin{aligned} \Delta x(n) ={}& q_1(T) g_2(T,n) \frac{w_2(v(n))}{w_1(v(n))} + \left(g_2(T,n)\sum_{s=0}^{n} g_1(T,s) + g_1(T,n)\sum_{s=0}^{n-1} g_2(T,s)\right) w_2(v(n)) \\ \le{}& q_1(T) g_2(T,n) \frac{w_2\left(W_1^{-1}(x(n))\right)}{w_1\left(W_1^{-1}(x(n))\right)} \\ &+ \left(g_2(T,n)\sum_{s=0}^{n} g_1(T,s) + g_1(T,n)\sum_{s=0}^{n-1} g_2(T,s)\right) w_2\left(W_1^{-1}(x(n))\right), \quad \forall n \in \mathbb{N}_0^T. \end{aligned} \tag{26}$$

From (26), we have

$$\begin{aligned} \frac{w_1\left(W_1^{-1}(x(n))\right)\Delta x(n)}{w_2\left(W_1^{-1}(x(n))\right)} \le{}& q_1(T) g_2(T,n) \\ &+ \left(g_2(T,n)\sum_{s=0}^{n} g_1(T,s) + g_1(T,n)\sum_{s=0}^{n-1} g_2(T,s)\right) w_1\left(W_1^{-1}(x(n))\right), \quad \forall n \in \mathbb{N}_0^T. \end{aligned} \tag{27}$$

Once again, performing the same procedure as in (21), (22), and (23), (27) gives

$$\begin{aligned} W_2(x(n)) \le{}& W_2(x(0)) + \sum_{s=0}^{n-1} q_1(T) g_2(T,t) \\ &+ \sum_{s=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right) \\ &\times w_1\left(W_1^{-1}(x(t))\right) \end{aligned}$$

$$\leq W_2(x(0)) + \sum_{s=0}^{T-1} q_1(T) g_2(T,t) + \sum_{s=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right) \times w_1\left(W_1^{-1}(x(t))\right), \quad \forall n \in \mathbb{N}_0^T, \tag{28}$$

where W_2 is defined by (10). Let $z(n)$ denote the function on the right-hand side of (28); namely,

$$z(n) = W_2(x(0)) + \sum_{s=0}^{T-1} q_1(T) g_2(T,t) + \sum_{s=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right) w_1\left(W_1^{-1}(x(t))\right), \quad \forall n \in \mathbb{N}_0^T. \tag{29}$$

Then $z(0) = W_2(x(0)) + \sum_{s=0}^{T-1} q_1(T) g_2(T,t)$, z is a nonnegative and nondecreasing function on $\mathbb{N}_0^T$, and (28) is equivalent to

$$x(n) \leq W_2^{-1}(z(n)), \quad \forall n \in \mathbb{N}_0^T. \tag{30}$$

From (29) and (30), we obtain

$$\begin{aligned}\Delta z(n) &= \left(g_2(T,n)\sum_{s=0}^{n} g_1(T,s) + g_1(T,n)\sum_{s=0}^{n-1} g_2(T,s)\right) w_1\left(W_1^{-1}(x(n))\right) \\ &\leq \left(g_2(T,n)\sum_{s=0}^{n} g_1(T,s) + g_1(T,n)\sum_{s=0}^{n-1} g_2(T,s)\right) w_1\left(W_1^{-1}\left(W_2^{-1}(z(n))\right)\right),\end{aligned} \tag{31}$$

for all $n \in \mathbb{N}_0^T$. From (31), we have

$$\frac{\Delta z(n)}{w_1\left(W_1^{-1}\left(W_2^{-1}(z(n))\right)\right)} \leq g_2(T,n)\sum_{s=0}^{n} g_1(T,s) + g_1(T,n)\sum_{s=0}^{n-1} g_2(T,s), \quad \forall n \in \mathbb{N}_0^T. \tag{32}$$

Once again, performing the same procedure as in (21), (22), and (23), (32) gives

$$W_3(z(n)) \leq W_3(z(0)) + \sum_{t=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right), \quad \forall n \in \mathbb{N}_0^T. \tag{33}$$

Using (17), (25), and (30), from (33) we have

$$\begin{aligned} u(n) &\leq v(n) \leq W_1^{-1}(x(n)) \leq W_1^{-1}\left(W_2^{-1}(z(n))\right) \\ &\leq W_1^{-1}\left(W_2^{-1}\left(W_3^{-1}\left(W_3(z(0)) + \sum_{t=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right)\right)\right)\right) \\ &\leq W_1^{-1}\left(W_2^{-1}\left(W_3^{-1}\left(W_3\left(W_2(x(0)) + \sum_{s=0}^{T-1} q_1(T) g_2(T,t)\right) + \sum_{t=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right)\right)\right)\right) \\ &\leq W_1^{-1}\left(W_2^{-1}\left(W_3^{-1}\left(W_3\left(W_2\left(W_1(q_1(T) q_2(T)) + \sum_{t=0}^{T-1} q_2(T) g_1(T,t)\right) + \sum_{s=0}^{T-1} q_1(T) g_2(T,t)\right) + \sum_{t=0}^{n-1}\left(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\right)\right)\right)\right), \\ &\qquad \forall n \in \mathbb{N}_0^T. \end{aligned} \tag{34}$$

As $n = T$, (34) yields

$$u(T) \leq W_1^{-1}\Bigg(W_2^{-1}\Bigg(W_3^{-1}\Bigg(W_3\Bigg(W_2\Bigg(W_1\left(q_1(T)q_2(T)\right) + \sum_{t=0}^{T-1} q_2(T)g_1(T,t)\Bigg) + \sum_{s=0}^{T-1} q_1(T)g_2(T,t)\Bigg) + \sum_{t=0}^{T-1}\Bigg(g_2(T,t)\sum_{s=0}^{t} g_1(T,s) + g_1(T,t)\sum_{s=0}^{t-1} g_2(T,s)\Bigg)\Bigg)\Bigg)\Bigg). \tag{35}$$

Since $T \in \mathbb{N}$, and $T \leq b$ is chosen arbitrarily in (35), the estimation (12) is derived. This completes the proof of Theorem 1. □

3. Application

We consider a sum-difference equation of product form

$$x(n) = \left(a(n) + \sum_{s=0}^{n-1} f(n,s)\varphi_1(x(s))\right) \times \left(b(n) + \sum_{s=0}^{n-1} g(n,s)\varphi_2(x(s))\right), \quad \forall n \in \mathbb{N}_0. \tag{36}$$

From (36), we have

$$|x(n)| \leq \left(a(n) + \sum_{s=0}^{n-1} f(n,s)\varphi_1(|x(s)|)\right) \times \left(b(n) + \sum_{s=0}^{n-1} g(n,s)\varphi_2(|x(s)|)\right), \quad \forall n \in \mathbb{N}_0. \tag{37}$$

Let $u(n) = |x(n)|$, $p_1(n) = |a(n)|$, $p_2(n) = |b(n)|$, $f_1(n,s) = |f(n,s)|$, and $f_2(n,s) = |g(n,s)|$ in (37); then (37) is the inequality of the form (4). Applying our result we get the estimation of solution of the sum-difference equations of product form (36).

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