Research Article

A New Application of the Reproducing Kernel Hilbert Space Method to Solve MHD Jeffery-Hamel Flows Problem in Nonparallel Walls

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The present paper emphasizes Jeffery-Hamel flow: fluid flow between two rigid plane walls, where the angle between them is 2α. A new method called the reproducing kernel Hilbert space method (*RKHSM*) is briefly introduced. The validity of the reproducing kernel method is set by comparing our results with HAM, DTM, and HPM and numerical results for different values of H, α, and Re. The results show up that the proposed reproducing kernel method can achieve good results in predicting the solutions of such problems. Comparison between obtained results showed that *RKHSM* is more acceptable and accurate than other methods. This method is very useful and applicable for solving nonlinear problems.

1. Introduction

1.1. Problem Formulation. Consider a system of cylindrical polar coordinates (r, h, z), where the steady two-dimensional flow of an incompressible conducting viscous fluid from a source or sink at channel walls lies in planes and intersects in z-axis. It is assumed that there are no changes with respect to z, that the motion is purely in radial direction and merely depends on r and θ, and that there is no magnetic field along z-axis. Then the governing equations are given as [1].

$$\frac{\rho \partial}{r \partial r} (ru(r, \theta)) = 0, \qquad (1)$$

$$u(r, \theta) \frac{\partial u(r, \theta)}{\partial r} = v \left[\frac{\partial^2 u(r, \theta)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r, \theta)}{\partial r} \right.$$
$$\left. + \frac{1}{r^2} \frac{\partial^2 u(r, \theta)}{\partial \theta^2} - \frac{u(r, \theta)}{r^2} \right] \quad (2)$$
$$- \frac{\sigma B_0^2}{\rho r^2} u(r, \theta) - \frac{1}{\rho} \frac{\partial P}{\partial r},$$

$$\frac{1}{\rho r} \frac{\partial P}{\partial \theta} - \frac{2v}{r^2} \frac{\partial u(r, \theta)}{\partial \theta} = 0, \qquad (3)$$

where B_0 is the electromagnetic induction, σ is the conductivity of the fluid, $u(r, \theta)$ is the velocity along radial direction, P is the fluid pressure, v is the coefficient of kinematic viscosity, and ρ is the fluid density. From (1)

$$f(\theta) = ru(r, \theta), \qquad (4)$$

using dimensionless parameters

$$f(x) = \frac{f(\theta)}{f_{\max}}, \quad x = \frac{\theta}{\alpha}, \qquad (5)$$

where α is the semiangle between the two inclined walls as shown in Figure 1. Substituting (5) into (2) and (3) and eliminating P, we obtain an ordinary differential equation for the normalized function profile $F(x)$ [2]:

$$F'''(x) + 2\alpha \operatorname{Re} F(x) F'(x) + (4 - H) \alpha^2 F'(x) = 0, \qquad (6)$$

with boundary conditions

$$F(0) = 1, \qquad F'(0) = 0, \qquad F(1) = 0. \qquad (7)$$

The Reynolds number is

$$\text{Re} = \frac{f_{\max}\alpha}{\nu} = \frac{U_{\max}r\alpha}{\nu}$$

$$= \begin{pmatrix} \text{divergent channel:} & \alpha > 0, \ f_{\max} > 0 \\ \text{convergent channel:} & \alpha < 0, \ f_{\max} < 0 \end{pmatrix}. \qquad (8)$$

The Hartmann number is

$$H = \sqrt{\frac{\alpha B_0^2}{\rho\nu}}. \qquad (9)$$

Internal flow between two plates is one of the most applicable cases in mechanics, civil and environmental engineering. In simple cases, the one-dimensional flow through tube and parallel plates, which is known as Couette-Poisseuille flow, has exact solution, but in general, like most of fluid mechanics equations, a set of nonlinear equations must be solved which make some problems for analytical solution. Many authors have shown interest in studying two-dimensional incompressible flow between two inclined plates. Jeffery [1] and Hamel et al. [2] were the first persons who discussed this problem, and so, it is known as Jeffery-Hamel problem. The incompressible viscous fluid flow through convergent and divergent channels is one of the most applicable cases in fluid mechanics, electrical, and biomechanical engineering. The MHD Jeffery-Hamel flows in nonparallel walls were investigated analytically for strongly nonlinear ordinary differential equations using homotopy analysis method (HAM). Results for velocity profiles in divergent and convergent channels were proffered for various values of Hartmann and Reynolds numbers in [3]. The mathematical investigations of this problem were underresearched by [3, 4]. Jeffery-Hamel flows are of the Navier-Stokes equations in the particular case of two dimensional flow through a channel with inclined walls [3–13]. One of the most important examples of Jeffery-Hamel problems is this subjected to an applied magnetic field. The equations of magnetohydrodynamics have been solved exactly for the case of two-dimensional steady flow between nonparallel walls of a viscous, incompressible, electrically conducting fluid; this is a straightforward extension of the famous Jeffrey-Hamel problem in ordinary hydrodynamics [9]. It has been indicated that for the Jeffrey-Hamel problem, the equations of magnetohydrodynamics can be curtailed to a set of three ordinary differential equations, two of which are linear and of first order [10]. In addition, these kinds of problems have been well studied in literature [3–13]. Most recent problems such as Jeffery-Hamel flow and other fluid mechanic problems are inherently nonlinear. Except a limited number of these problems, most of them do not have analytical solutions. So, these nonlinear equations should be solved utilizing other methods.

In this paper, the *RKHSM* [14–31] will be used to investigate MHD Jeffery-Hamel flows Problem. In recent years, a lot of attention has been devoted to the study of *RKHSM* to investigate various scientific models. The *RKHSM* which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed.

Recently, a lot of research work has been devoted to the application of *RKHSM* to a wide class of stochastic and deterministic problems involving fractional differential equation, nonlinear oscillator with discontinuity, singular nonlinear two-point periodic boundary value problems, integral equations and nonlinear partial differential equations and so on [14–31]. The method is well suited to physical problems since it makes unnecessary restrictive methods.

The efficiency of the method was used by many authors to investigate several scientific applications. Cui and Lin [15] applied the *RKHSM* to handle the second-order boundary value problems. Wang et al. [24] investigated a class of singular boundary value problems by this method, and the obtained results were good. In [27], the method was used to solve nonlocal boundary value problems. Geng and Cui [18] investigated the approximate solution of the forced Duffing equation with integral boundary conditions by combining the homotopy perturbation method and the *RKHSM*. Recently, the method was appllied the fractional partial differential equations and multipoint boundary value problems [18–22]. For more details about *RKHSM* and the modified forms and its effectiveness, see [14–31] and the references therein. The paper is organized as follows. Section 2 is devoted to several reproducing kernel spaces. Solution representation in $W_2^4[0, 1]$ and a linear operator are introduced in Section 3. Section 4 provides the main results; the exact and approximate solution of system (34) and an iterative method are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Numerical results are given in Section 5. The last Section is the conclusions.

2. Preliminaries

2.1. Reproducing Kernel Spaces. In this section, we define some useful reproducing kernel spaces.

Definition 1 (reproducing kernel). Let E be a nonempty abstract set. A function $K : E \times E \to C$ is a reproducing kernel of the Hilbert space H if and only if

$$\forall t \in E, \quad K(\cdot, t) \in H,$$

$$\forall t \in E, \ \forall\varphi \in H, \quad (\varphi(\cdot), K(\cdot, t)) = \varphi(t). \qquad (10)$$

The last condition is called "the reproducing property"; the value of the function φ at the point t is reproduced by the inner product of φ with $K(\cdot, t)$.
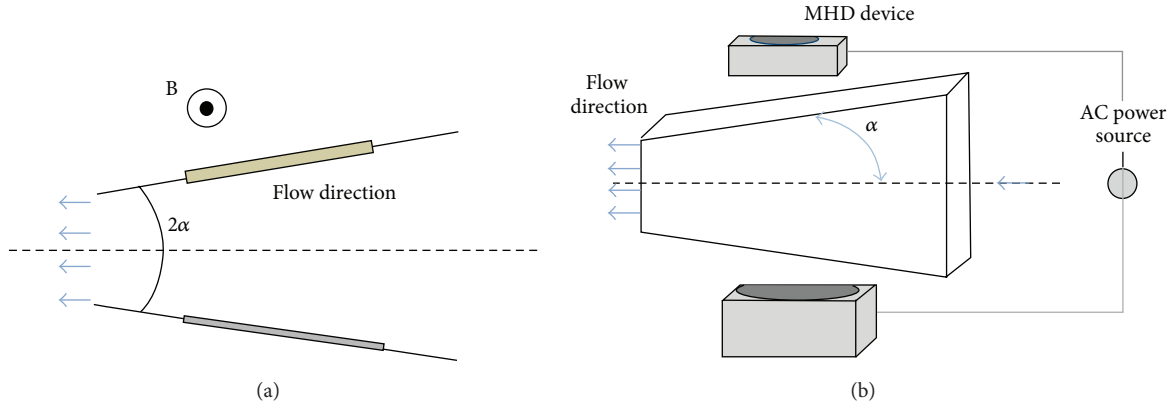
FIGURE 1: Geometry of the MHD Jeffery-Hamel flow in convergent cannel. (a) 2D view and (b) schematic setup of problem.

Definition 2. We define the space W_2^4 by

$$W_2^4[0,1] = \left\{ \begin{array}{l} u \mid u, u', u'', u''', \\ \quad \text{are absolutely continuous in } [0,1] \\ u^{(4)} \in L^2[0,1], \; x \in [0,1], \\ u(0) = 0, \; u(1) = 0, \; u'(0) = 0. \end{array} \right\}. \tag{11}$$

The inner product and the norm in $W_2^4[0,1]$ are defined, respectively, by

$$\langle u, g \rangle_{W_2^4} = \sum_{i=0}^{3} u^{(i)}(0) g^{(i)}(0)$$

$$+ \int_0^1 u^{(4)}(x) g^{(4)}(x)\,dx, \quad u, g \in W_2^4[0,1],$$

$$\|u\|_{W_2^4} = \sqrt{\langle u, u \rangle_{W_2^4}}, \; u \in W_2^4[0,1]. \tag{12}$$

The space $W_2^4[0,1]$ is a reproducing kernel space; that is, for each fixed $y \in [0,1]$ and any $u(x) \in W_2^4[0,1]$, there exists a function $R_y(x)$ such that

$$u(y) = \langle u, R_y \rangle_{W_2^4}. \tag{13}$$

Definition 3. We define the space W_2^2 by

$$W_2^2[0,1] = \left\{ \begin{array}{l} u \mid u, u' \text{ are absolutely continuous in } [0,1] \\ u'' \in L^2[0,1], \; x \in [0,1]. \end{array} \right\}. \tag{14}$$

The inner product and the norm in $W_2^2[0,1]$ are defined, respectively, by

$$\langle u, g \rangle_{W_2^2} = u(0) g(0) + u'(0) g'(0) + \int_0^1 u''(x) g''(x)\,dx,$$

$$(u, g \in W_2^2[0,1]), \tag{15}$$

$$\|u\|_{W_2^2} = \sqrt{\langle u, u \rangle_{W_2^2}}, \quad u \in W_2^2[0,1]. \tag{16}$$

The space $W_2^2[0,1]$ is a reproducing kernel space and its reproducing kernel function T_x is given by

$$T_x(y) = \begin{cases} 1 + xy + \dfrac{y}{2}x^2 - \dfrac{1}{6}x^3, & x \le y, \\ 1 + xy + \dfrac{x}{2}y^2 - \dfrac{1}{6}y^3, & x > y. \end{cases} \tag{17}$$

Theorem 4. *The space $W_2^4[0,1]$ is a complete reproducing kernel space; that is, for each fixed $y \in [0,1]$, there exists $u(x) \in W_2^4[0,1]$, such that*

$$u(y) = \langle u, R_y \rangle_{W_2^4} \tag{18}$$

for any $u(x) \in W_2^4[0,1]$. The reproducing kernel R_y can be denoted by

$$R_y(x) = \begin{cases} \displaystyle\sum_{i=1}^{8} c_i(y) x^{i-1}, & x \le y, \\ \displaystyle\sum_{i=1}^{8} d_i(y) x^{i-1}, & x > y, \end{cases} \tag{19}$$

where

$$c_1(y) = 0,$$

$$c_2(y) = 0,$$

$$c_3(y) = \frac{21}{5680}y^5 + \frac{1}{5680}y^7 - \frac{7}{1136}y^4 - \frac{7}{284}y^3$$
$$+ \frac{2}{71}y^2 - \frac{7}{5680}y^6,$$

$$c_4(y) = \frac{7}{17040}y^5 + \frac{1}{51120}y^7 - \frac{7}{10224}y^4 + \frac{16}{639}y^3$$
$$- \frac{7}{284}y^2 - \frac{7}{51120}y^6,$$

$$c_5(y) = \frac{7}{68160}y^5 + \frac{1}{204480}y^7 - \frac{7}{40896}y^4 + \frac{4}{639}y^3$$
$$- \frac{7}{1136}y^2 - \frac{7}{204480}y^6,$$

$$c_6(y) = \frac{-7}{113600}y^5 - \frac{1}{340800}y^7 + \frac{7}{68160}y^4 + \frac{7}{17040}y^3$$
$$- \frac{1}{2130}y^2 + \frac{7}{340800}y^6,$$

$$c_7(y) = \frac{7}{340800}y^5 + \frac{1}{1022400}y^7 - \frac{7}{204480}y^4 - \frac{7}{51120}y^3$$
$$- \frac{7}{5680}y^2 - \frac{7}{1022400}y^6 + \frac{1}{720}y,$$

$$c_8(y) = \frac{-1}{340800}y^5 - \frac{1}{7156800}y^7 + \frac{7}{204480}y^4 + \frac{1}{51120}y^3$$
$$+ \frac{1}{5680}y^2 + \frac{1}{10224000}y^6 - \frac{1}{5040},$$

$$d_1(y) = \frac{-1}{5040}y^7,$$

$$d_2(y) = \frac{1}{720}y^6,$$

$$d_3(y) = \frac{-1}{2130}y^5 + \frac{1}{5680}y^7 - \frac{7}{1136}y^4 - \frac{7}{284}y^3 + \frac{7}{21}y^2$$
$$- \frac{7}{5680}y^6,$$

$$d_4(y) = \frac{4}{639}y^4 + \frac{1}{17040}y^5 + \frac{7}{40896}y^7 + \frac{16}{639}y^3$$
$$- \frac{7}{284}y^2 - \frac{7}{51120}y^6,$$

$$d_5(y) = \frac{7}{68160}y^5 + \frac{1}{24480}y^7 - \frac{7}{40896}y^4 - \frac{7}{10224}y^3$$
$$- \frac{7}{1136}y^2 - \frac{7}{204480}y^6,$$

$$d_6(y) = \frac{-7}{113600}y^5 - \frac{1}{340800}y^7 + \frac{7}{68160}y^4 + \frac{7}{17040}y^3$$
$$+ \frac{21}{5680}y^2 + \frac{7}{340800}y^6,$$

$$d_7(y) = \frac{7}{34080}y^5 + \frac{1}{1022400}y^7 - \frac{7}{204480}y^4 - \frac{7}{51120}y^3$$
$$- \frac{7}{5680}y^2 - \frac{7}{1022400}y^6,$$

$$d_8(y) = \frac{-1}{340800}y^5 - \frac{1}{7156800}y^7 + \frac{1}{204480}y^4 + \frac{1}{51120}y^3$$
$$+ \frac{1}{5680}y^2 + \frac{7}{1022400}y^6. \tag{20}$$

Proof. By Definition 3, we have

$$\langle u, R_y \rangle_{W_2^4} = \sum_{i=0}^{3} u^{(i)}(0) R_y^{(i)}(0) + \int_0^1 u^{(4)}(x) R_y^{(4)}(x)\, dx,$$
$$\left(u, R_y \in W_2^4[0,1]\right). \tag{21}$$

Through several integrations by parts for (21) we have

$$\langle u(x), R_y(x) \rangle_{W_2^4} = \sum_{i=0}^{3} u^{(i)}(0)\left[R_y^{(i)}(0) - (-1)^{(3-i)} R_y^{(7-i)}(0)\right]$$
$$+ \sum_{i=0}^{3} (-1)^{(3-i)} u^{(i)}(1) R_y^{(7-i)}(1)$$
$$+ \int_0^1 u(x) R_y^{(8)}(x)\, dx. \tag{22}$$

Note that property of the reproducing kernel

$$\langle u, R_y \rangle_{W_2^4} = u(y), \tag{23}$$

R_y, is the solution of the following differential equation:

$$R_y^{(8)}(x) = \delta(x - y), \tag{24}$$

with the boundary conditions

$$R_y^{(3)}(0) - R_y^{(4)}(0) = 0,$$
$$R_y''(0) + R_y^{(5)}(0) = 0,$$
$$R_y^{(4)}(1) = 0, \tag{25}$$
$$R_y^{(5)}(1) = 0,$$
$$R_y^{(6)}(1) = 0,$$

when $x \neq y$,

$$R_y^{(8)}(x) = 0, \tag{26}$$

therefore

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0, 1, 2, 3, 4, 5, 6, \tag{29}$$

$$\partial^7 R_{y^+}(y) - \partial^7 R_{y^-}(y) = 1. \tag{30}$$

$$R_y(x) = \begin{cases} \displaystyle\sum_{i=1}^{8} c_i(y) x^{i-1}, & x \le y, \\ \displaystyle\sum_{i=1}^{8} d_i(y) x^{i-1}, & x > y. \end{cases} \tag{27}$$

Since $R_y(x) \in W_2^4[0,1]$, it follows that

$$R_y(0) = 0, \qquad R_y'(0) = 0, \qquad R_y(1) = 0. \tag{31}$$

Since

From (25)–(31), the unknown coefficients $c_i(y)$ ve $d_i(y)(i = 1, 2, \dots, 8)$ can be obtained.

$$R_y^{(8)}(x) = \delta(x - y), \tag{28}$$

$$R_y(x) = \begin{cases} \dfrac{21}{5680} x^2 y^5 + \dfrac{1}{5680} x^2 y^7 - \dfrac{7}{1136} x^2 y^4 - \dfrac{7}{284} x^2 y^3 + \dfrac{2}{71} x^2 y^2 \\[2mm] -\dfrac{7}{5680} x^2 y^6 + \dfrac{7}{17040} x^3 y^5 + \dfrac{1}{51120} x^3 y^7 - \dfrac{7}{10224} x^3 y^4 + \dfrac{16}{639} x^3 y^3 \\[2mm] -\dfrac{7}{284} x^3 y^2 - \dfrac{7}{51120} x^3 y^6 + \dfrac{7}{68160} x^4 y^5 + \dfrac{1}{204480} x^4 y^7 - \dfrac{7}{40896} x^4 y^4 \\[2mm] +\dfrac{4}{639} x^4 y^3 - \dfrac{7}{1136} x^4 y^2 - \dfrac{7}{204480} x^4 y^6 \dfrac{-7}{113600} x^5 y^5 - \dfrac{1}{340800} x^5 y^7 \\[2mm] +\dfrac{7}{68160} x^5 y^4 + \dfrac{7}{17040} x^5 y^3 - \dfrac{1}{2130} x^5 y^2 + \dfrac{7}{340800} x^5 y^6 + \dfrac{7}{340800} x^6 y^5 \\[2mm] +\dfrac{1}{1022400} x^6 y^7 - \dfrac{7}{204480} x^6 y^4 - \dfrac{7}{51120} x^6 y^3 - \dfrac{7}{5680} x^6 y^2 - \dfrac{7}{1022400} x^6 y^6 \\[2mm] +\dfrac{1}{720} x^6 y - \dfrac{1}{340800} x^7 y^5 - \dfrac{1}{7156800} x^7 y^7 + \dfrac{7}{204480} x^7 y^4 + \dfrac{1}{51120} x^7 y^3 \\[2mm] +\dfrac{1}{5680} x^7 y^2 + \dfrac{1}{10224000} x^7 y^6 - \dfrac{x^7}{5040}, \quad x \le y \\[3mm] \dfrac{21}{5680} y^2 x^5 + \dfrac{1}{5680} y^2 x^7 - \dfrac{7}{1136} y^2 x^4 - \dfrac{7}{284} y^2 x^3 + \dfrac{2}{71} y^2 x^2 - \dfrac{7}{5680} y^2 x^6 \\[2mm] \dfrac{7}{17040} y^3 x^5 + \dfrac{1}{51120} y^3 x^7 - \dfrac{7}{10224} y^3 x^4 + \dfrac{16}{639} y^3 x^3 - \dfrac{7}{284} y^3 x^2 - \dfrac{7}{51120} y^3 x^6 \\[2mm] \dfrac{7}{68160} y^4 x^5 + \dfrac{1}{204480} y^4 x^7 - \dfrac{7}{40896} y^4 x^4 + \dfrac{4}{639} y^4 x^3 - \dfrac{7}{1136} y^4 x^2 \\[2mm] -\dfrac{7}{204480} y^4 x^6 - \dfrac{7}{113600} y^5 x^5 - \dfrac{1}{340800} y^5 x^7 + \dfrac{7}{68160} y^5 x^4 + \dfrac{7}{17040} y^5 x^3 \\[2mm] -\dfrac{1}{2130} y^5 x^2 + \dfrac{7}{340800} y^5 x^6 + \dfrac{7}{340800} y^6 x^5 + \dfrac{1}{1022400} y^6 x^7 - \dfrac{7}{204480} y^6 x^4 \\[2mm] -\dfrac{7}{51120} y^6 x^3 - \dfrac{7}{5680} y^6 x^2 - \dfrac{7}{1022400} y^6 x^6 + \dfrac{1}{720} y^6 x - \dfrac{1}{340800} y^7 x^5 - \dfrac{1}{7156800} x^7 y^7 \\[2mm] +\dfrac{7}{204480} y^7 x^4 + \dfrac{1}{51120} y^7 x^3 + \dfrac{1}{5680} y^7 x^2 + \dfrac{1}{10224000} y^7 x^6 - \dfrac{y^7}{5040}, \quad x > y. \end{cases} \tag{32}$$

□

3. Solution Representation in $W_2^4[0,1]$

In this section, the solution of (34) is given in the reproducing kernel space $W_2^4[0,1]$.

On defining the linear operator $L : W_2^4[0,1] \rightarrow W_2^2[0,1]$ as

$$(Lu)(x) = u'''(x)$$

$$+ \left[-2\alpha \operatorname{Re}(x^2 - 1) + (4 - H)\alpha^2 \right] u'(x)$$

$$- 4\alpha x \operatorname{Re} u(x).$$

$$\tag{33}$$

Model problem (6) changes the following problem:

$$Lu = f(x, u, u'), \quad x \in [0, 1],$$

$$u(0) = 0, \qquad u'(0) = 0, \qquad u(1) = 0, \tag{34}$$

where

$$f(x, u, u') = -2\alpha \operatorname{Re} u(x) u'(x) - 4\alpha \operatorname{Re}(x^3 - x)$$

$$+ 2(4 - H)\alpha^2 x, \tag{35}$$

$$u(x) = F(x) + x^2 - 1.$$

Theorem 5. *The operator L defined by* (33) *is a bounded linear operator.*

Proof. We only need to prove $\|Lu\|_{W_2^2}^2 \le M\|u\|_{W_2^4}^2$, where $M > 0$ is a positive constant. By (15) and (16), we have

$$\|Lu\|_{W_2^2}^2 = \langle Lu, Lu \rangle_{W_2^2}$$

$$= [(Lu)(0)]^2 + [(Lu)'(0)]^2 \tag{36}$$

$$+ \int_0^1 [(Lu)''(x)]^2 dx.$$

By (18), we have

$$u(x) = \langle u, R_x \rangle_{W_2^4},$$

$$(Lu)(x) = \langle u, (LR_x) \rangle_{W_2^4}, \tag{37}$$

$$(Lu)'(x) = \langle u, (LR_x)' \rangle_{W_2^4},$$

so

$$|(Lu)(x)| \le \|u\|_{W_2^4} \|LR_x\|_{W_2^4} = M_1 \|u\|_{W_2^4},$$

$$(\text{where } M_1 > 0 \text{ is a positive constant}),$$

$$\left|(Lu)'(x)\right| \le \|u\|_{W_2^4} \left\|(LR_x)'\right\|_{W_2^4} = M_2 \|u\|_{W_2^4}, \tag{38}$$

$$(\text{where } M_2 > 0 \text{ is a positive constant}),$$

thus

$$(Lu)^2(0) + [(Lu)'(0)]^2 \le (M_1^2 + M_2^2)\|u\|_{W_2^4}^2. \tag{39}$$

Since

$$(Lu)'' = \langle u, (LR_x)'' \rangle_{W_2^4}, \tag{40}$$

then

$$\left|(Lu)''\right| \le \|u\|_{W_2^4} \left\|(LR_x)''\right\|_{W_2^4} = M_3 \|u\|_{W_2^4}, \tag{41}$$

$$(\text{where } M_3 > 0 \text{ is a positive constant}),$$

so, we have

$$\left[(Lu)''\right]^2 \le M_3^2 \|u\|_{W_2^4}^2,$$

$$\int_0^1 [(Lu)''(x)]^2 dx \le M_3^2 \|u\|_{W_2^4}^2, \tag{42}$$
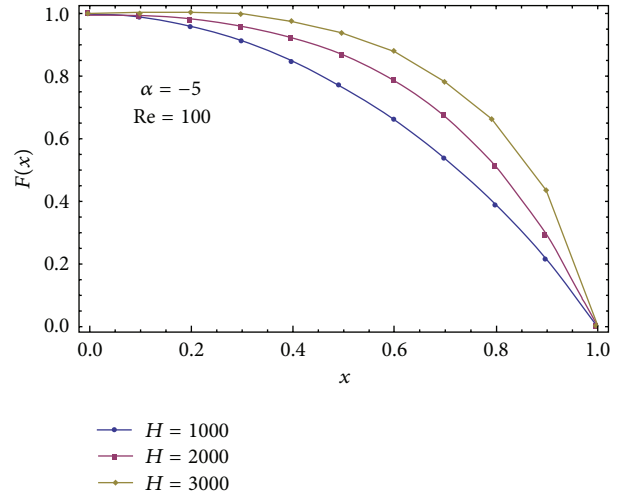
$H = 1000$
$H = 2000$
$H = 3000$

FIGURE 2: A comparison between increasing Hartmann numbers for the velocity profile Re = 100.
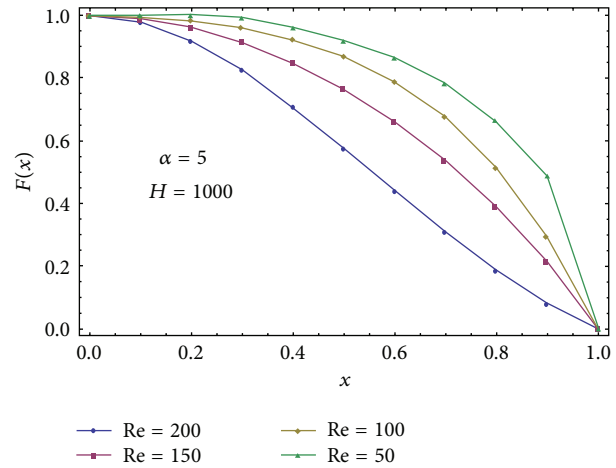

Re = 200   Re = 100
Re = 150   Re = 50

FIGURE 3: A comparison between the increasing values of Re for the velocity profile $H = 1000$.

that is

$$\|Lu\|_{W_2^2}^2 = [(Lu)(0)]^2 + [(Lu)'(0)]^2 + \int_0^1 [(Lu)''(x)]^2 dx$$

$$\le (M_1^2 + M_2^2 + M_3^2)\|u\|_{W_2^4}^2 = M\|u\|_{W_2^4}^2, \tag{43}$$

where $M = (M_1^2 + M_2^2 + M_3^2) > 0$ is a positive constant. □

4. The Structure of the Solution and the Main Results

In (33) it is clear that $L : W_2^4[0,1] \to W_2^2[0,1]$ is a bounded linear operator. Put $\varphi_i = T_{x_i}$ and $\psi_i = L^* \varphi_i$, where L^* is conjugate operator of L. The orthonormal system
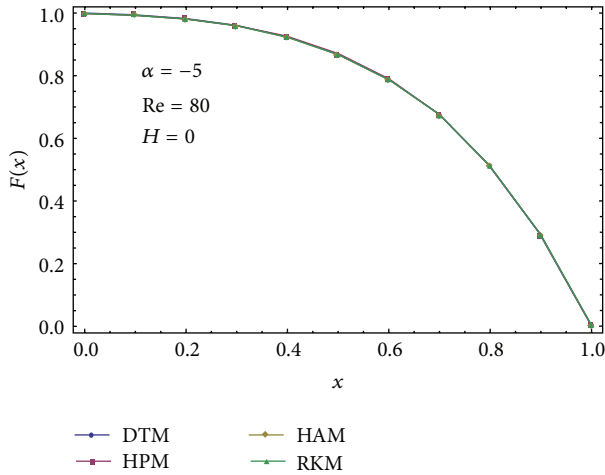
FIGURE 4: A comparison between the DTM, HPM, *RKHSM*, and HAM solutions for the velocity profile Re = 80 and $H = 0$.
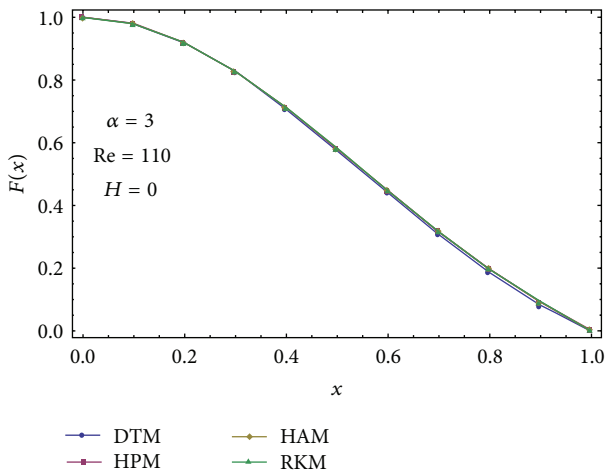

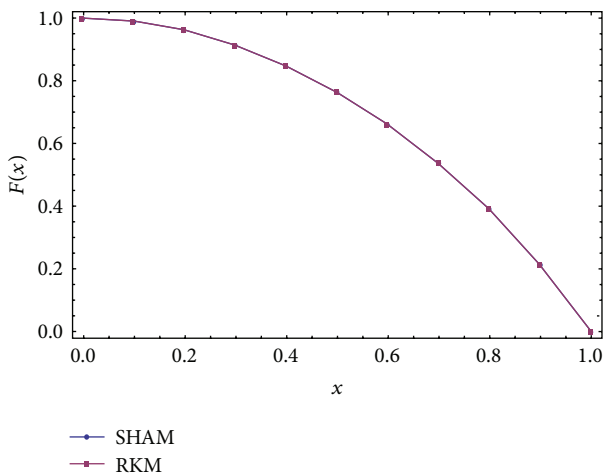FIGURE 5: A comparison between the DTM, HPM, *RKHSM*, and HAM solutions for the velocity profile Re = 110 and $H = 0$.

FIGURE 6: A comparison between the *RKHSM* and SHAM solutions for the velocity profile Re = 50, $\alpha = 5$, and $H = 0$.

$\{\overline{\Psi}_i\}_{i=1}^{\infty}$ of $W_2^4[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i\}_{i=1}^{\infty}$ as

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik} \psi_k(x), \quad (\beta_{ii} > 0, \ i = 1, 2, \ldots). \quad (44)$$

Theorem 6. *For* (33), *if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0, 1]$ *then* $\{\psi_i\}_{i=1}^{\infty}$ *is the complete system of* $W_2^4[0,1]$ *and* $\psi_i(x) = L_y R_x(y)|_{y=x_i}$.

Proof. We have

$$\psi_i(x) = (L^* \varphi_i)(x) = \langle (L^* \varphi_i)(y), R_x(y) \rangle$$
$$= \langle (\varphi_i)(y), L y R_x(y) \rangle = L_y R_x(y)|_{y=x_i}. \quad (45)$$

The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly, $\psi_i(x) \in W_2^4[0,1]$. For each fixed $u(x) \in W_2^4[0,1]$, let $\langle u(x), \psi_i(x) \rangle = 0$, $(i = 1, 2, \ldots)$, which means that

$$\langle u, (L^* \varphi_i) \rangle = \langle Lu, \varphi_i \rangle = \langle Lu, T_{x_i} \rangle = (Lu)(x_i) = 0. \quad (46)$$

Note that, $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of Theorem 6 is complete. □

Theorem 7. *If* $u(x)$ *is the exact solution of* (34), *then*

$$u = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k, u'_k) \overline{\Psi}_i, \quad (47)$$

where $\{x_i\}_{i=1}^{\infty}$ *is a dense set in* $[0, 1]$.

Proof. From (44) and uniqueness of solution of (34) we have

$$u = \sum_{i=1}^{\infty} \langle u, \overline{\psi}_i \rangle_{W_2^4} \overline{\psi}_i$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u, L^* T_{x_k} \rangle_{W_2^4} \overline{\psi}_i$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle Lu, T_{x_k} \rangle_{W_2^2} \overline{\psi}_i \quad (48)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle f(x, u, u'), T_{x_k} \rangle_{W_2^2} \overline{\psi}_i$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k, u'_k) \overline{\psi}_i.$$

Now the approximate solution $u_n(x)$ can be obtained by truncating the n-term of the exact solution $u(x)$:

$$u_n = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(x_k, u_k, u'_k) \overline{\psi}_i. \quad (49)$$

□

TABLE 1: The comparison between the numerical results and DTM, HPM, HAM, and *RKHSM* solutions for Re = 110, $\alpha = 3$, and $H = 0$.

x	DTM [5]	HPM [5]	HAM [5]	*RKHSM*	Numerical [5]
0.0	1.000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9789771156	0.9791761778	0.9792357062	0.9792357171	0.9792357085
0.2	0.9182598446	0.9190424983	0.9192658842	0.91926585	0.9192658898
0.3	0.8243664466	0.8260939720	0.8265336102	0.82653635	0.8265336182
0.4	0.7065763476	0.7096036928	0.7102211838	0.7102315393	0.7102211890
0.5	0.5751498602	0.5798357741	0.5804994700	0.5804817201	0.5804994634
0.6	0.4397114086	0.4463900333	0.4469350941	0.4468796913	0.4469350697
0.7	0.3081560927	0.3170877938	0.3174084545	0.3174013727	0.3174084270
0.8	0.1862239095	0.1975366451	0.1976410661	0.1976321	0.1976410889
0.9	0.0784362201	0.09124214542	0.09123022879	0.0912030082	0.0912304211
1.0	0.0000000015	0.0000000007	-0.00000047	$8.052549207 \times 10^{-8}$	0.0

TABLE 2: The numerical results for Re = 50, $H = 1000$.

x	HAM [3]	*RKHSM* ($\alpha = 5$)	Error	HAM [3]	*RKHSM* ($\alpha = -5$)	Error
0	1.000000000	1.0000000000	0.0	1.000000000	1.00000000	0.0
0.05	0.997605126	0.997605447	3.203×10^{-7}	0.999197467	0.99919702	4.432×10^{-7}
0.10	0.990427215	0.990432890	0.56744×10^{-6}	0.99675704	0.9967562	8.409×10^{-7}
0.15	0.978485626	0.9784839628	0.16638×10^{-6}	0.992578975	0.992578	9.754×10^{-7}
0.20	0.961810074	0.96179	0.20074×10^{-5}	0.98649281	0.98649340	5.900×10^{-7}
0.25	0.940436864	0.9403939	0.42964×10^{-5}	0.978250927	0.9782510	7.24×10^{-8}
0.30	0.91440365	0.9145	0.96349×10^{-5}	0.967519314	0.9675443	0.24985×10^{-5}
0.35	0.883742856	0.8833	0.44285×10^{-4}	0.953865319	0.95382	0.45319×10^{-5}
0.40	0.848473706	0.8484738539	1.473×10^{-7}	0.936742176	0.936821	0.78823×10^{-5}
0.45	0.808592961	0.808592834	1.279×10^{-7}	0.915470063	0.915531	0.60936×10^{-5}
0.50	0.764064241	0.7640637445	4.967×10^{-7}	0.889213540	0.889241	0.27459×10^{-5}
0.55	0.714805913	0.7148062	2.867×10^{-7}	0.856955292	0.8565	0.45529×10^{-4}
0.60	0.660677266	0.660670	0.72666×10^{-6}	0.817466464	0.817199	0.26746×10^{-4}
0.65	0.601462467	0.6014683135	0.58461×10^{-6}	0.769274094	0.770	0.7259×10^{-4}
0.70	0.536852087	0.53685274	6.525×10^{-7}	0.710627559	0.710014	0.61355×10^{-4}
0.75	0.466421078	0.4664202	8.783×10^{-7}	0.639465773	0.63946970	0.39331×10^{-6}
0.80	0.389601905	0.389602099	1.934×10^{-7}	0.553390063	0.55336107	0.28992×10^{-5}
0.85	0.305651801	0.305645	0.68011×10^{-6}	0.449648596	0.44963621	0.12386×10^{-5}
0.90	0.213611172	0.2136120	8.277×10^{-7}	0.325142373	0.32516167	0.19298×10^{-5}
0.95	0.112250324	0.112249347	9.775×10^{-7}	0.176465831	0.17656197	0.9614×10^{-5}
1.00	0.000000000	8.3437×10^{-8}	8.3437×10^{-8}	0.000000000	3.614×10^{-7}	3.614×10^{-7}

TABLE 3: The comparison between the numerical results and DTM, HPM, HAM, and *RKHSM* solutions for Re = 80, $\alpha = -5$, and $H = 0$.

x	DTM [5]	HPM [5]	HAM [5]	*RKHSM*	Numerical
0	1.000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.10	0.9959603887	0.9960671874	0.9959606242	0.99595999	0.9959606278
0.20	0.9832745481	0.9836959424	0.9832755258	0.983275	0.9832755381
0.30	0.9601775551	0.9610758773	0.9601798911	0.96017	0.96017991139
0.40	0.9235170706	0.9249245156	0.9235215737	0.923519	0.9235215894
0.50	0.8684511349	0.8701997697	0.8684588997	0.86845826	0.86845887772
0.60	0.7880785402	0.7898325937	0.7880910186	0.78809	0.78809092032
0.70	0.6731248448	0.6745334968	0.6731437690	0.67314	0.6731436346
0.80	0.5119644061	0.5128373095	0.5119909939	0.5119873503	0.5119910891
0.90	0.2915280122	0.2918936991	0.2915580178	0.2915582665	0.29155874261
1.00	0.0000000000	0.0000000001	-0.000001149	2.851385×10^{-9}	0.0

Lemma 8. *If $u \in W_2^4[0,1]$, then there exists $M_1 > 0$, such that*

$$\|u\|_{C^2[0,1]} \leq M_1 \|u\|_{W_2^4}, \tag{50}$$

where $\|u\|_{C^2[0,1]} = \max_{x \in [0,1]} |u(x)| + \max_{x \in [0,1]} |u'(x)| + \max_{x \in [0,1]} |u''(x)|$.

Lemma 9. *If* $\|u_n - u\|_{W_2^4} \to 0$, $x_n \to x$, $(n \to \infty)$ *and* $f(x, u, u')$ *is continuous for* $x \in 0, 1]$, *then*

$$\begin{aligned} f\left(x_n, u_{n-1}(x_n), u'_{n-1}(x_n)\right) \\ \longrightarrow f\left(x, u(x), u'(x)\right) \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{51}$$

Proof. Since $\|u_n - u\|_{W_2^4} \to 0$ $(n \to \infty)$, by Lemma 8, we know that u_n is convergent uniformly to $u(x)$, therefore, the proof is complete. □

Remark 10. (i) If (34) is linear, that is, $f(x, u) = f(x)$, then the analytical solution of (34) can be obtained directly by (47).

(ii) If (34) is nonlinear; that is, f depends on u and u' then the solution of (34) can be obtained by the following iterative method.

We construct an iterative sequence $u_n(x)$, putting

$$\begin{aligned} \text{any fixed } u_0(x) \in W_2^4[0,1], \\ u_n(x) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x), \end{aligned} \tag{52}$$

where

$$A_1 = \beta_{11} f\left(x_1, u_0(x_1), u'_0(x_1)\right),$$

$$A_2 = \sum_{k=1}^{2} \beta_{2k} f\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k)\right),$$

$$\vdots \tag{53}$$

$$A_n = \sum_{k=1}^{n} \beta_{nk} f\left(x_k, u_{k-1}(x_k), u'_{k-1}(x_k)\right).$$

Next we will prove that u_n given by the iterative formula (52) converges to the exact solution (47).

Theorem 11. *Suppose that the following conditions are satisfied:* (i) $\|u_n\|_{W_2^4}$ *is bounded;* (ii) $\{x_i\}_{i=1}^{\infty}$ *is a dense in* $[0,1]$; (iii) $f(x, u, u') \in W_2^2[0,1]$ *for any* $u \in W_2^4[0,1]$. *Then* u_n *in iterative formula (52) converges to the exact solution of (47) in* $W_2^4[0,1]$ *and*

$$u = \sum_{i=1}^{\infty} A_i \overline{\psi}_i, \tag{54}$$

where A_i *is given by (53).*

Proof. (i) First, we will prove the convergence of $u_n(x)$. By (52), we have

$$u_{n+1}(x) = u_n(x) + A_{n+1} \overline{\psi}_{n+1}(x). \tag{55}$$

From the orthogonality of $\{\overline{\Psi}_i(x)\}_{i=1}^{\infty}$, it follows that

$$\begin{aligned} \|u_{n+1}\|_{W_2^4}^2 &= \|u_n\|_{W_2^4}^2 + (A_{n+1})^2 \\ &= \|u_{n-1}\|_{W_2^4}^2 + (A_n)^2 + (A_{n+1})^2 \\ &= \cdots = \sum_{i=1}^{n+1} (A_i)^2. \end{aligned} \tag{56}$$

From boundedness of $\|u_n\|_{W_2^4}$, we have

$$\sum_{i=1}^{\infty} (A_i)^2 < \infty, \tag{57}$$

that is,

$$\{A_i\} \in l^2 \quad (i = 1, 2, \ldots). \tag{58}$$

Let $m > n$, in view of $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned} &\|u_m - u_n\|_{W_2^4}^2 \\ &= \|u_m - u_{m-1} + u_{m-1} - u_{m-2} + \cdots + u_{n+1} - u_n\|_{W_2^4}^2 \\ &\leq \|u_m - u_{m-1}\|_{W_2^4}^2 + \cdots + \|u_{n+1} - u_n\|_{W_2^4}^2 \\ &= \sum_{i=n+1}^{m} (A_i)^2 \longrightarrow 0 \quad (m, n \longrightarrow \infty). \end{aligned} \tag{59}$$

Considering the completeness of $W_2^4[0,1]$, there exists $u(x) \in W_2^4[0,1]$, such that

$$u_n(x) \xrightarrow{\|\cdot\|_{W_2^4}} u(x), \quad \text{as } n \longrightarrow \infty. \tag{60}$$

(ii) Second, we will prove that $u(x)$ is the solution of (34).

By Lemma 8 and Theorem 11 (i), we know that u_n converges uniformly to u. It follows that, on taking limits in (52),

$$u = \sum_{i=1}^{\infty} A_i \overline{\psi}_i. \tag{61}$$

Since

$$\begin{aligned} (Lu)(x_j) &= \sum_{i=1}^{\infty} A_i \left\langle L\overline{\Psi}_i(x), \varphi_j(x) \right\rangle_{W_2^2} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \overline{\Psi}_i(x), L^* \varphi_j(x) \right\rangle_{W_2^4} \\ &= \sum_{i=1}^{\infty} A_i \left\langle \overline{\Psi}_i(x), \overline{\Psi}_j(x) \right\rangle_{W_2^4}, \end{aligned} \tag{62}$$

TABLE 4: The errors of DTM, HPM, HAM, and *RKHSM* for $F(x)$ results when Re = 110, $\alpha = 3$, and $H = 0$.

x	DTM [5]	HPM [5]	HAM [5]	*RKHSM*
0.0	0.0	0.0	0.0	0.0
0.1	0.0002	0.000059	0.0000000023	8.6×10^{-9}
0.2	0.0010	0.00022	0.0000000056	3.98×10^{-8}
0.3	0.0021	0.00043	0.000000008	0.0000027318
0.4	0.0036	0.00061	0.0000000052	0.0000103503
0.5	0.0053	0.00066	0.0000000066	0.0000177433
0.6	0.0072	0.00054	0.000000024	0.0000553784
0.7	0.0092	0.00032	0.000000027	0.0000070543
0.8	0.0114	0.000104	0.000000022	0.0000089889
0.9	0.0127	0.000011	0.00000019	0.0000274129
1.0	0.0000	0.000000	0.0000004	$8.052549207 \times 10^{-8}$

TABLE 5: The errors of DTM, HPM, HAM, and *RKHSM* for $F(x)$ results when for Re = 80, $\alpha = -5$, and $H = 0$.

x	DTM [5]	HPM [5]	HAM [5]	*RKHSM*
0.0	0.0	0.0	0.0	0.0
0.1	0.00000023	0.000106	0.000000003	6.378×10^{-7}
0.2	0.00000099	0.00042	0.000000012	5.381×10^{-7}
0.3	0.0000023	0.00089	0.00000002	0.9114×10^{-6}
0.4	0.0000045	0.0014	0.000000015	2.5894×10^{-6}
0.5	0.0000077	0.0017	0.000000021	6.177×10^{-7}
0.6	0.000012	0.0017	0.000000098	9.203×10^{-7}
0.7	0.000018	0.0013	0.00000013	3.6346×10^{-6}
0.8	0.000026	0.0008	0.000000095	7.388×10^{-6}
0.9	0.000030	0.00033	0.00000072	4.761×10^{-7}
1.0	0.0000	0.0000000001	0.0000011	2.8513856×10^{-9}

it follows that

$$\sum_{j=1}^{n} \beta_{nj} (Lu)(x_j) = \sum_{i=1}^{\infty} A_i \left\langle \overline{\Psi}_i(x), \sum_{j=1}^{n} \beta_{nj} \overline{\Psi}_j(x) \right\rangle_{W_2^4}$$

$$= \sum_{i=1}^{\infty} A_i \left\langle \overline{\Psi}_i(x), \overline{\Psi}_n(x) \right\rangle_{W_2^4} = A_n. \tag{63}$$

If $n = 1$, then

$$(Lu)(x_1) = f\left(x_1, u_0(x_1), u_0'(x_1)\right). \tag{64}$$

If $n = 2$, then

$$\beta_{21} (Lu)(x_1) + \beta_{22} (Lu)(x_2)$$

$$= \beta_{21} f\left(x_1, u_0(x_1), u_0'(x_1)\right) \tag{65}$$

$$+ \beta_{22} f\left(x_2, u_1(x_2), u_1'(x_2)\right).$$

From (64) and (65), it is clear that

$$(Lu)(x_2) = f\left(x_2, u_1(x_2), u_1'(x_2)\right). \tag{66}$$

Furthermore, it is easy to see by induction that

$$(Lu)(x_j) = f\left(x_j, u_{j-1}(x_j), u_{j-1}'(x_j)\right). \tag{67}$$

Notice that $\{x_i\}_{i=1}^{\infty}$ is dense on interval $[0, 1]$, for any $y \in [0, 1]$, there exists subsequence $\{x_{n_j}\}$, such that $x_{n_j} \to y$, as $j \to \infty$. Hence, by the convergence of $u_n(x)$ and Lemma 9, we have

$$(Lu)(y) = f\left(y, u(y), u'(y)\right), \tag{68}$$

that is, $u(x)$ is the solution of (34) and

$$u = \sum_{i=1}^{\infty} A_i \overline{\psi}_i, \tag{69}$$

where A_i is given by (53). □

Corollary 12. *Assume that the conditions of Theorem 11 hold; then u_n in (52) satisfies $\|u_n - u\|_{C^2[0,1]} \to 0$, $n \to \infty$, where u is the solution of* (34).

Theorem 13. *Assume that u is the solution of* (34) *and r_n is the error between the approximate solution u_n and the exact solution u. Then the error sequence r_n is monotone decreasing in the sense of $\| \cdot \|_{W_2^4}$ and $\|r_n(x)\|_{W_2^4} \to 0$.*

Proof. From (47) and (49), it follows that

$$\|r_n\|_{W_2^4} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f\left(x_k, u_k, u_k'\right) \overline{\psi}_i(x) \right\|_{W_2^4}$$

$$= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} f\left(x_k, u_k, u_k'\right) \right)^2. \tag{70}$$

Equation (70) shows that the error r_n is decreasing in the sense of $\| \cdot \|_{W_2^4}$. □

5. Numerical Results

All computations are performed by Maple 15. Results obtained by the method are compared with the homotopy analysis method [3], three analytical methods [5], homotopy perturbation method [6], and a new spectral-homotopy analysis method [8]. The *RKHSM* does not require discretization of the variables, that is, time and space; it is not effected by computation round off errors and one is not faced with necessity of large computer memory and time. The accuracy of the *RKHSM* for the MHD Jeffery-Hamel flows problem is controllable and absolute errors are small with present choice of x (see Tables 1–5). The numerical results that we obtained justify the advantage of this methodology.

5.1. Result and Discussion. In this study the purpose is to apply the *RKHSM* to obtain an approximate solution of the Jeffery-Hamel problem. The obtained results of *RKHSM* solution and numerical ones are shown in the tables and figures. In Table 2 a comparison of the HAM and *RKHSM* is shown. Tables 1 and 3 show the comparison between the numerical results and DTM, HPM, HAM, and *RKHSM* solutions. Tables 4 and 5 indicate the errors of DTM, HPM, HAM, and *RKHSM* for $F(x)$ results. Our results further show
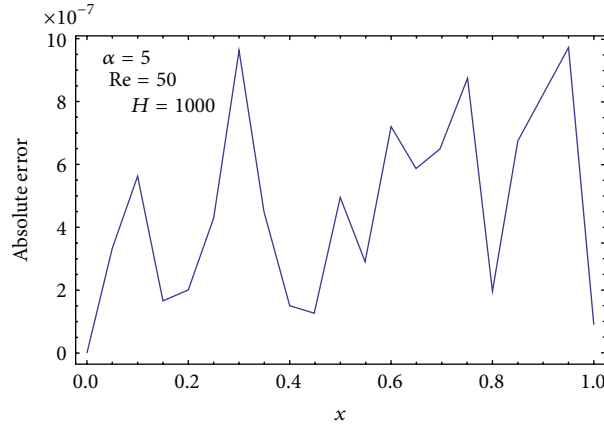
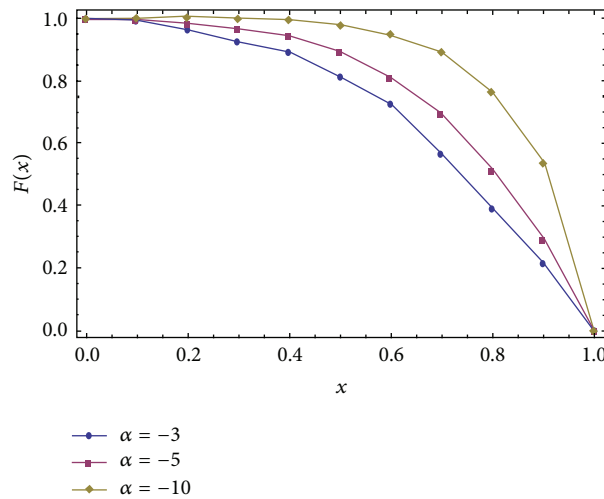FIGURE 7: Absolute error for Re = 50 and H = 1000.

- $\alpha = -3$
- $\alpha = -5$
- $\alpha = -10$

FIGURE 8: A comparison between different values of α for velocity in convergent channel for Re = 50 and H = 1000.

that the fluid velocity increases with increasing Hartman numbers. Numerical simulations show that for fixed Hartmann numbers, the fluid velocity increases with Reynolds numbers in the case of convergent channels but decreases with Re in the case of divergent channels. Figure 2 indicates that increasing the Hartmann number leads to higher velocity which has a great effect on the performance of the system. In Figure 3 we give a comparison between the *RKHSM* and the HAM solutions for several Re numbers at H = 1000. In Figure 4 we can see a comparison between the DTM, HPM, *RKHSM* and HAM solutions for the velocity profile Re = 80 and H = 0. There is a comparison between the DTM, HPM, *RKHSM*, and HAM solutions for the velocity profile Re = 110 and H = 0 in Figure 5. In Figure 6 we compare *RKHSM* and SHAM solutions. We can see absolute error for Re = 50 and H = 1000 in Figure 7. The comparison of numerical results and *RKHSM* solution for velocity in convergent channel for Re = 50 and H = 1000 is given with Figure 8. The solutions show that the results of the present method are in excellent agreement with those of the numerical ones. Moreover, *RKHSM* has been used to investigate the effects of the parameters of the problem.

6. Conclusion

In this paper, we introduce an algorithm for solving the MHD Jeffery-Hamel flows problem with boundary conditions by using the *RKHSM*. The approximate solution obtained by the present method is uniformly convergent. Clearly, the series solution methodology can be applied to much more complicated nonlinear differential equations and boundary value problems. However, if the problem becomes nonlinear, then the *RKHSM* does not require discretization or perturbation and it does not make closure approximation. Results show that the present method is an accurate and reliable analytical method for MHD Jeffery-Hamel flows problem with boundary conditions.

Conflict of Interests

The authors declare that they do not have any competing or conflict of interests.

Acknowledgment

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