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Research Article

Representation of Fuzzy Concept Lattices in the Framework of Classical FCA

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We describe a representation of the fuzzy concept lattices, defined via antitone Galois connections, within the framework of classical Formal Concept Analysis. As it is shown, all needed information is explicitly contained in a given formal fuzzy context and the proposed representation can be obtained without a creation of the corresponding fuzzy concept lattice.

1. Introduction

Formal concept analysis (FCA) is a theory of data analysis for identification of conceptual structures among datasets. The mathematical theory of FCA is based on the notion of concept lattices and it is well developed in the monograph of Ganter and Wille [1]. In this classical approach to FCA the authors provide the crisp case, where an object-attribute model is represented by some binary relation. In practice there are natural examples of object-attribute models for which the relationship between the objects and the attributes is represented by fuzzy relation. Therefore, several attempts to fuzzify FCA have been proposed. There are two kinds of existing approaches to fuzzy FCA based on the structure of the concept lattices. In the first case the concept lattices are fuzzy complete lattices (or complete L-lattices) (cf. [2, 3] or [4]). In the second case the concept lattices are ordinary (crisp) complete lattices. From these approaches we mention a work of Bělohlávek [5–7] based on the logical framework of complete residuated lattices, a work of Georgescu and Popescu to extend this framework to noncommutative logic [8–10], the approaches of Krajči [11], Popescu [12], and Medina et al. [13, 14], or other works on fuzzy concept lattices [15–19]. A nice survey and comparison of some existing approaches to fuzzy concept lattices is available in [20].

Recently, a generalization of Popescu's approach [12] for creating crisp fuzzy concept lattices was introduced (cf. [21–23]) for the so-called one-sided concept lattices. This method is in some manner the most general one and it covers most of the mentioned approaches based on the antitone Galois connections. The main feature of this approach is that it is not fixed to any logical framework and it provides a possibility to create concept lattices also in cases, where particular objects and attributes have assigned different complete lattices representing their truth value structures. Moreover this approach generates all possible antitone Galois connections between the products of complete lattices. According to these facts, in this paper we will use the definition of fuzzy concept lattice as it was proposed in [21].

The main goal of this paper is to describe a representation of the fuzzy concept lattices (represented as crisp complete lattices) in the framework of classical concept lattices. It is a well-known fact that every complete lattice is isomorphic to some concept lattice. Hence, for a given fuzzy formal context, it is possible to create the fuzzy concept lattice first and then find a representation of this fuzzy concept lattice as a classical concept lattice. However, our goal is to describe a representation of the fuzzy concept lattices in the classical FCA framework without any previous creation of the fuzzy concept lattices. We only use the information (knowledge), which is explicitly contained in the given formal fuzzy context.

Since the theory of concept lattices is closely related to antitone Galois connections and closure systems, we give a brief overview of these notions in the preliminary section. Further we describe an approach of creating fuzzy concept lattices as it was defined in [21].

In Section 3 we prove our main theorem, that is, an isomorphism between the fuzzy and the classical concept lattices is described. The proof of this theorem involves the principal ideal representation of complete lattices. At the end of this section we also provide an illustrative example of such representation.

2. Preliminaries

In this section we mention some results concerning the antitone Galois connections and their relationship with the closure systems in complete lattices. Also, we briefly recall the framework of classical FCA as well as the notion of fuzzy concept lattices as it was presented in [21]. In the sequel we will assume that the reader is familiar with the basic notions of lattice theory (cf. [24]).

First we recall the definition of the antitone (contravariant) Galois connections (see [25] or [1]).

Let $(P, \leq)$ and $(Q, \leq)$ be ordered sets and let

$$\varphi : P \longrightarrow Q, \qquad \psi : Q \longrightarrow P \tag{1}$$

be maps between these ordered sets. Such a pair (φ, ψ) of mappings is called an *antitone Galois connection* between the ordered sets if

(a) $p_1 \leq p_2$ implies $\varphi(p_1) \geq \varphi(p_2)$,

(b) $q_1 \leq q_2$ implies $\psi(q_1) \geq \psi(q_2)$,

(c) $p \leq \psi(\varphi(p))$ and $q \leq \varphi(\psi(q))$.

The two maps are also called *dually adjoint* to each other. We note that

$$\varphi = \varphi \circ \psi \circ \varphi, \qquad \psi = \psi \circ \varphi \circ \psi \tag{2}$$

and that the conditions (a), (b), and (c) are equivalent to the following one:

(d) $p \leq \psi(q)$ if and only if $\varphi(p) \geq q$.

In what follows we will denote by $\mathbf{Gal}(P, Q)$ the set of all antitone Galois connections between the partially ordered sets P and Q. The class of all complete lattices will be denoted by $\mathbf{CL}$.

The antitone Galois connections between complete lattices are closely related to the notion of closure operator and closure system. Let L be a complete lattice. By a *closure operator* in L we understand a mapping $c : L \rightarrow L$ satisfying

(a) $x \leq c(x)$ for all $x \in L$,

(b) $c(x_1) \leq c(x_2)$ for $x_1 \leq x_2$,

(c) $c(c(x)) = c(x)$ for all $x \in L$ (i.e., c is idempotent).

A subset X of a complete lattice L is called a *closure system* in L if X is closed under arbitrary meets. We note that this condition guarantees that $(X, \leq)$ is a complete lattice, in which the infima are the same as in L, but the suprema in X may not coincide with those from L. It is well known that closure systems and closure operators are in one-to-one correspondence; that is, the closure operator associated with a closure system defines the closure of an element x as the least closed element containing x and the closure system associated with a closure operator c is the family of its fixed points $(\{x : c(x) = x\})$.

The following result (see [25]) relates the relationship between the antitone Galois connections and dually isomorphic closure systems of the complete lattices.

Proposition 1. *Let $L, M \in \mathbf{CL}$ and (φ, ψ) be an antitone Galois connection between L and M. Then the mapping $\varphi \circ \psi : L \rightarrow L$ is a closure operator in L, and similarly, $\psi \circ \varphi : M \rightarrow M$ is a closure operator in M. Moreover, the corresponding closure systems are dually isomorphic.*

Conversely, suppose that X_1 and X_2 are closure systems in L and M, respectively, and $f : X_1 \rightarrow X_2$ is a dual isomorphism between the complete lattices $(X_1, \leq)$ and $(X_2, \leq)$. Then the pair $(\mathsf{C}_{X_1} \circ f, \mathsf{C}_{X_2} \circ f^{-1})$, where $\mathsf{C}_{X_1}, \mathsf{C}_{X_2}$ are closure operators corresponding to X_1 and to X_2, forms an antitone Galois connection between L and M.

We also recall another useful characterization of the antitone Galois connections between complete lattices (see [1]).

Proposition 2. *A map $\varphi : L \rightarrow M$ between complete lattices L and M has a dual adjoint if and only if*

$$\varphi\left(\bigvee_{i \in I} x_i\right) = \bigwedge_{i \in I} \varphi(x_i) \tag{3}$$

holds for any subset $\{x_i : i \in I\}$ of L.

Note that in this case the dual adjoint ψ is uniquely determined by

$$\psi(y) = \bigvee \{x \in L : \varphi(x) \geq y\}. \tag{4}$$

The properties of the antitone Galois connections allow constructing complete lattices (Galois lattices). Formally, let $L, M \in \mathbf{CL}$ and (φ, ψ) be an antitone Galois connection between L and M. Denote by $\mathsf{L}_{\varphi,\psi}$ a subset of $L \times M$ consisting of all pairs (x, y) with $\varphi(x) = y$ and $\psi(y) = x$. Define a partial order on $\mathsf{L}_{\varphi,\psi}$ as follows:

$$(x_1, y_1) \leq (x_2, y_2) \quad \text{iff } x_1 \leq x_2 \quad \text{iff } y_1 \geq y_2. \tag{5}$$

Proposition 3. *Let $L, M \in$ **CL** and (φ, ψ) be an antitone Galois connection between L and M. Then $(\mathsf{L}_{\varphi,\psi}, \leq)$ forms a complete lattice, where*

$$\bigwedge_{i\in I}(x_i, y_i) = \left(\bigwedge_{i\in I} x_i, \varphi\left(\bigwedge_{i\in I} x_i\right)\right),$$
$$\bigvee_{i\in I}(x_i, y_i) = \left(\psi\left(\bigwedge_{i\in I} y_i\right), \bigwedge_{i\in I} y_i\right), \quad (6)$$

for each family $(x_i, y_i)_{i\in I}$ of elements from $\mathsf{L}_{\varphi,\psi}$.

Now we briefly recall the basic notions of FCA [1].

Let (B, A, I) be a formal context, that is, $B, A \neq \emptyset$ and $I \subseteq B \times A$. There is a pair of mappings $^{\uparrow} : 2^B \to 2^A$ and $^{\downarrow} : 2^A \to 2^B$, which forms an antitone Galois connection between the power sets 2^B and 2^A

$$X^{\uparrow} = \{y \in A : (x, y) \in I, \forall x \in X\},$$
$$Y^{\downarrow} = \{x \in B : (x, y) \in I, \forall y \in Y\}. \quad (7)$$

The corresponding concept lattice is denoted by $\underline{\mathfrak{B}}(B, A, I)$ (cf. Proposition 3).

Next, we describe the approach proposed in [21]. We start with the definition of a formal fuzzy context.

A 6-tuple $\mathscr{C} = (B, L, A, M, \varphi, \psi)$ is called a *formal fuzzy context* if

(i) $B, A \neq \emptyset$ (B is the set of objects and A is the set of attributes),

(ii) $L : B \to$ **CL**, $M : A \to$ **CL** (recall that **CL** denotes the class of all complete lattices, and thus for $b \in B$, $L(b)$ represents a complete lattice with possible truth values of the object b and similarly for $a \in A$),

(iii) $\varphi = (\varphi_{b,a})_{(b,a)\in B\times A}$, $\psi = (\psi_{b,a})_{(b,a)\in B\times A}$ where for each $b \in B$, $a \in A (\varphi_{b,a}, \psi_{b,a}) \in \mathbf{Gal}(L(b), M(a))$.

Further, define $\uparrow: \prod_{b\in B} L(b) \to \prod_{a\in A} M(a)$ as follows:

$$\uparrow(f)(a) = \bigwedge_{b\in B} \varphi_{b,a}(f(b)), \quad \forall a \in A, \text{ if } f \in \prod_{b\in B} L(b). \quad (8)$$

Similarly we put $\downarrow: \prod_{a\in A} M(a) \to \prod_{b\in B} L(b)$ as follows:

$$\downarrow(g)(b) = \bigwedge_{a\in A} \psi_{b,a}(g(a)), \quad \forall b \in B, \text{ if } g \in \prod_{a\in A} M(a). \quad (9)$$

The following theorem shows the relation between the mappings defined above and the antitone Galois connections between the direct products of the complete lattices. The proof of this theorem can be found in [21].

Proposition 4. *Let $(B, L, A, M, \varphi, \psi)$ be a formal fuzzy context. Then the pair $(\uparrow, \downarrow)$ forms an antitone Galois connection between $\prod_{b\in B} L(b)$ and $\prod_{a\in A} M(a)$.*

Conversely, let (Φ, Ψ) be an antitone Galois connection between $\prod_{b\in B} L(b)$ and $\prod_{a\in A} M(a)$. Then there exists a formal fuzzy context $\mathscr{C} = (B, L, A, M, \varphi, \psi)$, such that $\uparrow = \Phi$ and $\downarrow = \Psi$.

According to this proposition and Proposition 3, the lattice $\mathsf{L}_{\uparrow,\downarrow}$ corresponding to the formal fuzzy context $\mathscr{C} = (B, L, A, M, \varphi, \psi)$ will be denoted by $\mathsf{FCL}(\mathscr{C})$.

Let us note that the second part of Proposition 3 allows a representation of any fuzzy concept lattice (created by antitone Galois connection) as $\mathsf{FCL}(\mathscr{C})$ for a suitable formal fuzzy context $\mathscr{C}$.

As an example we show such representation of the approach based on residuated lattices [5, 7].

Let $\mathbf{L} = \langle L, \bigwedge, \bigvee, \otimes, \to, 0, 1\rangle$ be a complete residuated lattice and $c \in L$ be an arbitrary element. Denote by ι_c a mapping $\iota_c : L \to L$ with $\iota_c(x) = x \to c$. Since

$$x \leq y \longrightarrow c = \iota_c(y) \quad \text{iff } x \otimes y = y \otimes x \leq c$$
$$\text{iff } y \leq x \longrightarrow c = \iota_c(x) \quad (10)$$

for all $x, y \in L$, we obtain that the pair (ι_c, ι_c) forms an antitone Galois connection between L and L. Hence, if (B, A, I) is $\mathbf{L}$-context, then one can easily obtain formal fuzzy context in our sense by choosing $\varphi_{b,a} = \iota_{I(b,a)}$ and $\psi_{b,a} = \iota_{I(b,a)}$.

3. Representation of Fuzzy Concept Lattices in Classical FCA

In this section we describe the theoretical details regarding the representation of fuzzy concept lattices in the framework of classical concept lattices. We prove a theorem which shows that our representation of a fuzzy concept lattice as a classical concept lattice is correct; that is, we show that both lattices are isomorphic. At the end of this section we provide an illustrative example of such representation.

Let L be any complete lattice. For an element $a \in L$ denote by $\mathrm{id}(a) = \{x \in L : x \leq a\}$ the principal ideal generated by the element a.

Let $\mathscr{C} = (B, L, A, M, \varphi, \psi)$ be a formal fuzzy context. In the sequel we will suppose that $\{L(b) : b \in B\}$ and $\{M(a) : a \in A\}$ form pairwise disjoint family of sets. For each $b \in B$ and each $a \in A$ define a binary relation $I_{b,a} \subseteq L(b) \times M(a)$ by

$$(x, y) \in I_{b,a} \quad \text{iff } x \leq \psi_{b,a}(y) \quad \text{iff } y \leq \varphi_{b,a}(x), \quad (11)$$

and let

$$S = \bigcup_{b\in B} L(b), \qquad T = \bigcup_{a\in A} M(a), \qquad I = \bigcup_{(b,a)\in B\times A} I_{b,a}. \quad (12)$$

Obviously, the triple (S, T, I) forms a classical formal context.

Lemma 5. *Let $b \in B$, $a \in A$ be arbitrary elements and $X \subseteq L(b)$ and $Y \subseteq M(a)$. Then*

$$\forall x \in X, \qquad \forall y \in Y : (x, y) \in I_{b,a},$$
$$\textit{iff} \bigvee X \leq \psi_{b,a}\left(\bigvee Y\right) \quad \textit{iff} \bigvee Y \leq \varphi_{b,a}\left(\bigvee X\right). \quad (13)$$

Proof. Obviously, $(x, y) \in I_{b,a}$ for all $x \in X$, $y \in Y$ if and only if $x \leq \psi_{b,a}(y)$ for all $x \in X$, $y \in Y$. According to Proposition 2 this is equivalent to $\bigvee X \leq \bigwedge_{y\in Y} \psi_{b,a}(y) = \psi_{b,a}(\bigvee Y)$. □

Let $b \in B$ be an object and $a \in A$ be an attribute. For a concept $(X, Y) \in \underline{\mathfrak{B}}(S, T, I)$ we put $X_b = X \cap L(b)$ and $Y_a = Y \cap M(a)$. The next lemma shows that each of the subsets $X_b \subseteq L(b)$ forms a principal ideal in $L(b)$. The same is true for $Y_a \subseteq M(a)$.

Lemma 6. *Let $(X, Y) \in \underline{\mathfrak{B}}(S, T, I)$ be a concept. Then*

$$id\left(\bigvee X_b\right) = X_b, \qquad id\left(\bigvee Y_a\right) = Y_a \tag{14}$$

for all $b \in B$ and for all $a \in A$.

Proof. We prove $\mathrm{id}(\bigvee X_b) = X_b$ for all $b \in B$. Let $b \in B$ be an object. Since any $x \in X_b$ is lower than or equal to $\bigvee X_b$, the inclusion $X_b \subseteq \mathrm{id}(\bigvee X_b)$ is trivial.

Now we prove the opposite inclusion. Let $y \in Y$ be an arbitrary element. There is an attribute $a \in A$ such that $y \in M(a)$ and $(x, y) \in I_{b,a}$ for all $x \in X_b$, which is equivalent to $y \leq \varphi_{b,a}(x)$ for all $x \in X_b$ by (11). Let $x_1 \in \mathrm{id}(\bigvee X_b)$. From the basic properties of antitone Galois connections and due to Proposition 2 we obtain

$$\varphi_{b,a}(x_1) \geq \varphi_{b,a}\left(\bigvee X_b\right) = \bigwedge_{x \in X_b} \varphi_{b,a}(x) \geq y, \tag{15}$$

which yields $(x_1, y) \in I_{b,a}$ and consequently $(x_1, y) \in I$ for all $x_1 \leq \bigvee X_b$. Since this holds for all $y \in Y$, we obtain $\mathrm{id}(\bigvee X_b) \subseteq Y^{\downarrow} = X$. □

Lemma 7. *Let $(X, Y) \in \underline{\mathfrak{B}}(S, T, I)$ be a concept. Then*

$$\begin{aligned} \bigvee Y_a &= \bigwedge_{b \in B} \varphi_{b,a}\left(\bigvee X_b\right), \quad \forall a \in A, \\ \bigvee X_b &= \bigwedge_{a \in A} \psi_{b,a}\left(\bigvee Y_a\right), \quad \forall b \in B. \end{aligned} \tag{16}$$

Proof. We prove $\bigvee Y_a = \bigwedge_{b \in B} \varphi_{b,a}(\bigvee X_b)$ for all $a \in A$. Let $a \in A$ be an arbitrary element. Since $Y_a \subseteq Y$, we obtain $(x, y) \in I_{b,a}$ for all $b \in B$, $x \in X_b$ and for all $y \in Y_a$. Due to Lemma 5 this is equivalent to $\bigvee Y_a \leq \varphi_{b,a}(\bigvee X_b)$ for all $b \in B$ which yields $\bigvee Y_a \leq \bigwedge_{b \in B} \varphi_{b,a}(\bigvee X_b)$.

Conversely, denote $y = \bigwedge_{b \in B} \varphi_{b,a}(\bigvee X_b)$. Then $y \leq \varphi_{b,a}(\bigvee X_b)$ for all $b \in B$, and hence according to Lemma 5 $(x, y) \in I_{b,a}$ for all $b \in B$, $x \in X_b$. The pair (X, Y) forms a concept; thus $y \in Y_a$, and we obtain $y = \bigwedge_{b \in B} \varphi_{b,a}(\bigvee X_b) \leq \bigvee Y_a$. □

Now we can provide our main theorem.

Theorem 8. *Let $\mathscr{C} = (B, L, A, M, \varphi, \psi)$ be a formal fuzzy context. Then $\mathsf{FCL}(\mathscr{C}) \cong \underline{\mathfrak{B}}(S, T, I)$ and this isomorphism is given by the following correspondence:*

$$(f, g) \longmapsto \left(\bigcup_{b \in B} id(f(b)), \bigcup_{a \in A} id(g(a))\right). \tag{17}$$

Proof. Let $H : \mathsf{FCL} \to 2^S \times 2^T$ be the mapping defined in the theorem.

First, we prove that the range of H is a subset of $\underline{\mathfrak{B}}(S, T, I)$, that is,

$$H(f, g) = \left(\bigcup_{b \in B} \mathrm{id}(f(b)), \bigcup_{a \in A} \mathrm{id}(g(a))\right) \in \underline{\mathfrak{B}}(S, T, I) \tag{18}$$

for all $f \in \prod_{b \in B} L(b)$, $g \in \prod_{a \in A} M(a)$ satisfying $\uparrow(f) = g$ and $\downarrow(g) = f$.

The following chain of equivalent assertions shows $(\bigcup_{b \in B} \mathrm{id}(f(b)))^{\uparrow} = \bigcup_{a \in A} \mathrm{id}(g(a))$:

$$\begin{aligned} & y \in \left(\bigcup_{b \in B} \mathrm{id}(f(b))\right)^{\uparrow} \\ & \quad \text{iff } \forall x \in \bigcup_{b \in B} \mathrm{id}(f(b)) : (x, y) \in I \\ & \quad \text{iff } \exists a \in A, \forall b \in B, \forall x \in L(b) : x \leq f(b) \\ & \quad \Longrightarrow (x, y) \in I_{b,a} \\ & \quad \text{iff } \exists a \in A, \forall b \in B, \forall x \in L(b) : x \leq f(b) \\ & \quad \Longrightarrow y \leq \varphi_{b,a}(x) \\ & \quad \text{iff } \exists a \in A, \forall b \in B : y \leq \varphi_{b,a}(f(b)) \\ & \quad \text{iff } \exists a \in A : y \leq \bigwedge_{b \in B} \varphi_{b,a}(f(b)) = \uparrow(f)(a) = g(a) \\ & \quad \text{iff } y \in \bigcup_{a \in A} \mathrm{id}(g(a)). \end{aligned} \tag{19}$$

In the same way one can prove $(\bigcup_{a \in A} \mathrm{id}(g(a)))^{\downarrow} = \bigcup_{b \in B} \mathrm{id}(f(b))$.

Since in any lattice L we have $u \leq v$ if and only if $\mathrm{id}(u) \subseteq \mathrm{id}(v)$, we obtain that the mapping H satisfies

$$(f_1, g_1) \leq (f_2, g_2) \quad \text{iff } H(f_1, g_1) \leq H(f_2, g_2) \tag{20}$$

for all $(f_1, g_1), (f_2, g_2) \in \mathsf{FCL}(\mathscr{C})$. Let us remark that this condition ensures that the mapping H is injective.

Finally, we show that the mapping H is surjective. Let $(X, Y) \in \underline{\mathfrak{B}}(S, T, I)$ be a concept. Define $f \in \prod_{b \in B} L(b)$ and $g \in \prod_{a \in A} M(a)$ by

$$\begin{aligned} f(b) &= \bigvee (X \cap L(b)) = \bigvee X_b, \\ g(a) &= \bigvee (Y \cap M(a)) = \bigvee Y_a. \end{aligned} \tag{21}$$

According to Lemma 6 $\mathrm{id}(f(b)) = X_b$ for all $b \in B$ and $\mathrm{id}(g(a)) = Y_a$ for all $a \in A$, thus

$$(X, Y) = \left(\bigcup_{b \in B} X_b, \bigcup_{a \in A} Y_a\right) = \left(\bigcup_{b \in B} \mathrm{id}(f(b)), \bigcup_{a \in A} \mathrm{id}(g(a))\right). \tag{22}$$

Moreover, Lemma 7 shows that $\uparrow(f)(a) = g(a)$ for all $a \in A$ and $\downarrow(g)(b) = f(b)$ for all $b \in B$, which completes the proof. □

Table 1: Example of formal fuzzy context.

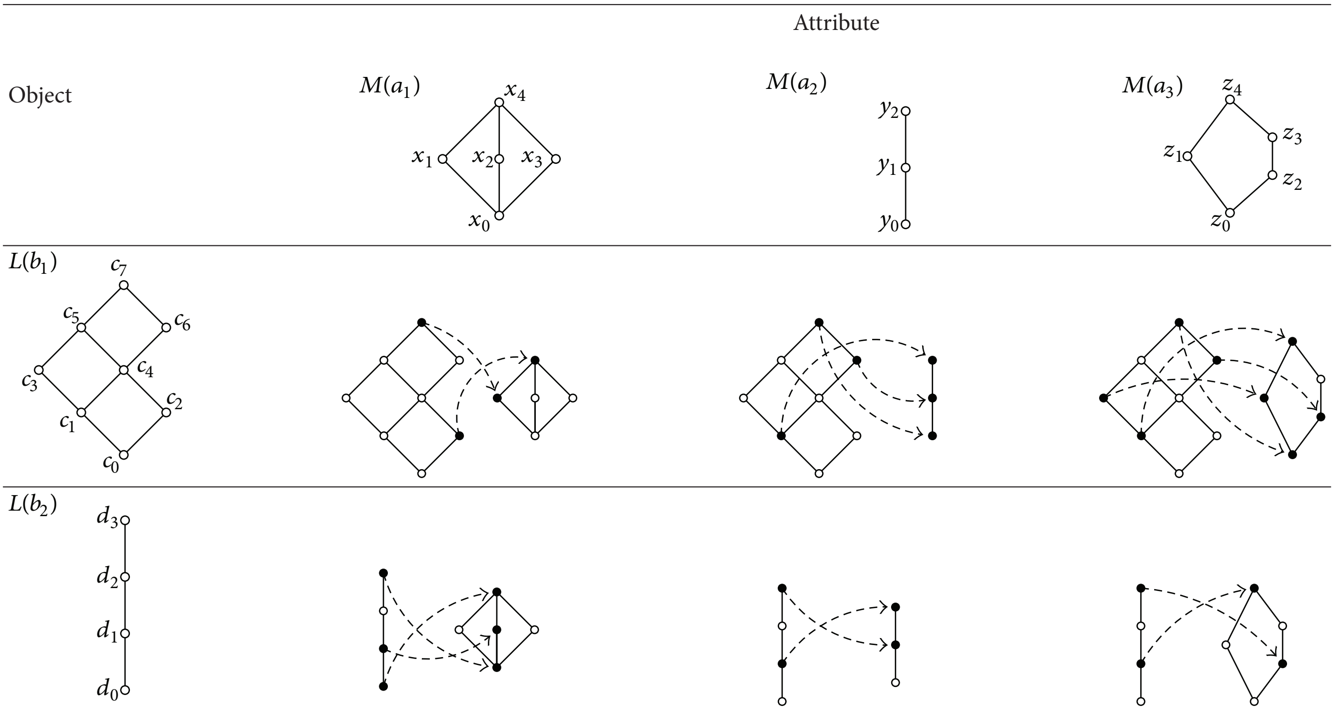

Table 2: Formal context corresponding to the formal fuzzy context from Table 1.

	x_0	x_1	x_2	x_3	x_4	y_0	y_1	y_2	z_0	z_1	z_2	z_3	z_4
c_0	1	1	1	1	1	1	1	1	1	1	1	1	1
c_1	1	1	0	0	0	1	1	1	1	1	1	1	1
c_2	1	1	1	1	1	1	1	0	1	0	1	0	0
c_3	1	1	0	0	0	1	0	0	1	1	0	0	0
c_4	1	1	0	0	0	1	1	0	1	0	1	0	0
c_5	1	1	0	0	0	1	0	0	1	0	0	0	0
c_6	1	1	0	0	0	1	1	0	1	0	1	0	0
c_7	1	1	0	0	0	1	0	0	1	0	0	0	0
d_0	1	1	1	1	1	1	1	1	1	1	1	1	1
d_1	1	0	1	0	0	1	1	1	1	1	1	1	1
d_2	1	0	0	0	0	1	1	0	1	0	1	0	0
d_3	1	0	0	0	0	1	1	0	1	0	1	0	0

In practice, one can use the result of this theorem as follows. Let $\mathscr{C} = (B, L, A, M, \varphi, \psi)$ be a given formal fuzzy context. There is given the formal context (S, T, I) as we described. The lattice $\mathsf{FCL}(\mathscr{C})$ can be fully reconstructed from the concept lattice $\underline{\mathfrak{B}}(S, T, I)$ using the inverse mapping H^{-1}. In this case

$$H^{-1}(X, Y) = (f, g), \qquad f(b) = \bigvee (X \cap L(b)), \qquad g(a) = \bigvee (Y \cap M(a)), \tag{23}$$

for all $b \in B$ and for all $a \in A$.

At the end of this section, we provide an illustrative example.

Example 9. We will consider the following formal fuzzy context $\mathscr{C} = (B, L, A, M, \varphi, \psi)$, where $B = \{b_1, b_2\}$, $A = \{a_1, a_2, a_3\}$ and the mappings $L : B \to \mathsf{CL}$, $M : A \to \mathsf{CL}$ and the system of antitone Galois connections $(\varphi_{b,a}, \psi_{b,a})_{(b,a)\in B\times A}$ are introduced in Table 1. Note that for more legibility we only indicate the corresponding dual isomorphism of the closure systems, since it uniquely determines the corresponding antitone Galois connection (cf. Proposition 1).

In this case, our representation gives the formal context (S, T, I) (see Table 2). From this context we obtain the following concept lattice $\underline{\mathfrak{B}}(S, T, I)$ (see Figure 1).

Finally, we obtain the fuzzy concept lattice $\mathsf{FCL}(\mathscr{C})$ (Figure 2) consisting of pairs (f, g), such that $f(b_i) = \bigvee(X \cap$

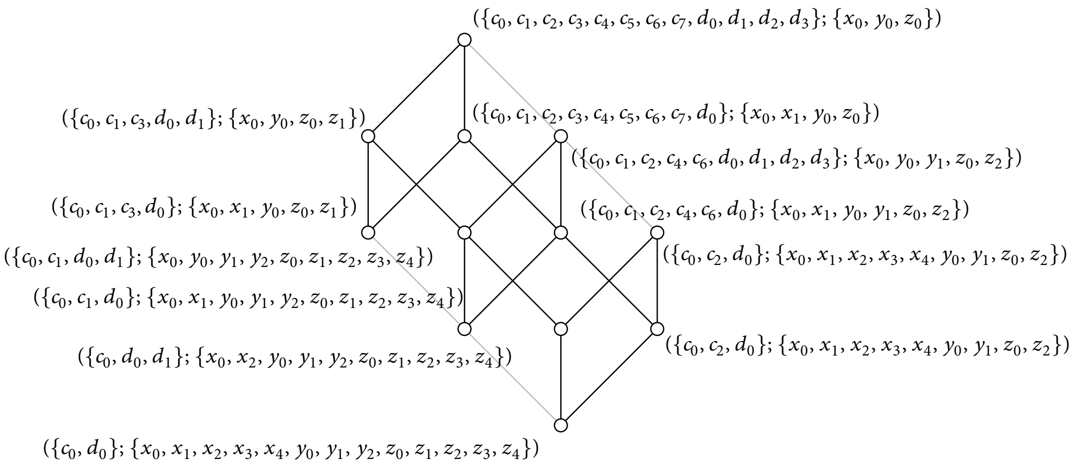

Figure 1: Concept lattice $\underline{\mathfrak{B}}(S, T, I)$ corresponding to Table 2.

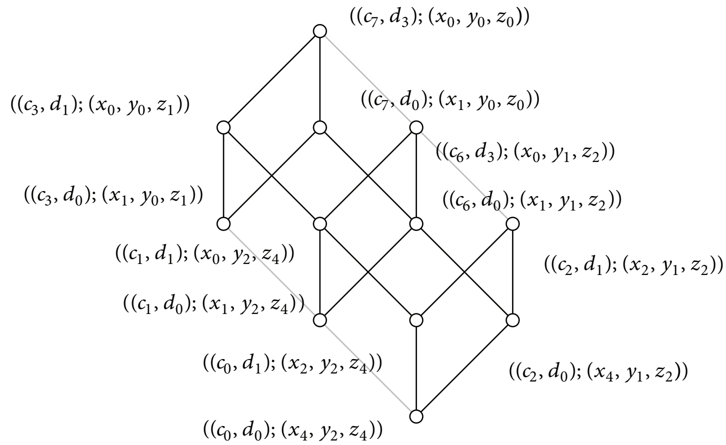

Figure 2: Fuzzy concept lattice $\mathsf{FCL}(\mathscr{C})$.

$L(b_i))$ for $i \in \{1, 2\}$ and $g(a_i) = \bigvee(Y \cap M(a_i))$ for $i = \{1, 2, 3\}$, where $(X, Y) \in \underline{\mathfrak{B}}(S, T, I)$.

4. Conclusion

In this paper we described a representation of the fuzzy concept lattices in the framework of the classical FCA. This representation transforms a fuzzy formal context into a binary formal context. As it was shown, this transformation maintains all the information given by the lattice structure of a concept lattice, since the corresponding concept lattices are isomorphic. Consequently, the well developed theory of classical FCA can be used for studying the fuzzy concept lattices or an arbitrary algorithm for classical concept lattices can be used for the creation of the fuzzy concept lattices.

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