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Research Article

The Existence and Stability of Solutions for Vector Quasiequilibrium Problems in Topological Order Spaces

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In a topological sup-semilattice, we established a new existence result for vector quasiequilibrium problems. By the analysis of essential stabilities of maximal elements in a topological sup-semilattice, we prove that for solutions of each vector quasi-equilibrium problem, there exists a connected minimal essential set which can resist the perturbation of the vector quasi-equilibrium problem.

1. Introduction

Vector equilibrium problems can unify many nonlinear problems such as vector optimization, vector variational inequality [1], and vector complementarity problems [2]. Recently, not only vector equilibrium problems [3–7] but also vector quasiequilibrium problems [8–14] and the system of vector quasi-equilibrium problems have attracted much attention [15–19].

Topological vector spaces provide the usual mathematical framework in the study of many problems. To avoid the linear feature, sup-semilattices may be good choices. In fact, some results like the existence of KKM points in topological spaces were established in topological sup-semilattices [20], where a two-tuple $(X, \leq)$ is said to be a sup-semilattice, if X is a partially ordered set with the partial ordering $\leq$, in which every pair (x, x') has a least upper bound $x \vee x'$.

The aim of this paper is to study the existence and essential stability of vector quasi-equilibrium problems in topological sup-semilattices. In order to achieve this, firstly, we give a stability result in relation to maximal elements in a topological sup-semilattice. Secondly, a new existence result for vector quasi-equilibrium problems is established, and we show that each vector quasi-equilibrium problem has a connected minimal essential set in its solution set.

2. Preliminaries

Let $(X, \leq)$ be a sup-semilattice. If x and x' are two elements in $(X, \leq)$ and $x \leq x'$, the set $[x, x'] = \{y \in X : x \leq y \leq x'\}$ is called an order interval. Let A, A' be two nonempty finite subsets of X. Then the set $\Delta A = \cup_{x\in A}[x, \sup A]$ is well defined and has the properties: $A \subseteq \Delta A$ and $\Delta A \subseteq \Delta A'$ if $A \subseteq A'$.

Definition 1 (see [20]). A subset $E \subseteq X$ is Δ-convex, if for any nonempty finite subset $A \subseteq E$, we have $\Delta A \subseteq E$. E being a Δ-convex set is equivalent to the following conditions:

(a) if $x, x' \in E$, then its least upper bound $x \vee x' \in E$.

(b) if $x, x' \in E$ and $x \leq x'$, then the order interval $[x, x'] \subset E$.

It is easy to check that the intersection of two Δ-convex sets is Δ-convex as well.

A topological space X is said to be a topological sup-semilattice if X is equipped with a sup-semilattice as its partial ordering denoted by $\leq$, for which $f : X \times X \to X$ with $(x, x') \to x \vee x'$ is a continuous function.

Let Y be a topological vector space and θ the zero element in Y. A subset $C \subset Y$ is called a cone if, for any $y \in C$ and real number $t > 0$, $ty \in C$. A cone C is convex if C is a convex set. If $C \cap -C = \{\theta\}$, it is called a pointed cone.

Definition 2 (see [18]). Let X be a topological sup-semilattice, Y a topological vector space with a cone $C \subset Y$, $\varphi : X \to Y$ a vector-valued function.

(a) $\varphi : X \to Y$ is C_Δ-quasiconcave if, for any nonempty two points subset $A = \{x_1, x_2\} \subset X$ and $y \in Y$, $\varphi(A) \subset y + C \Rightarrow \varphi(\Delta A) \subset y + C$.

(b) $\varphi : X \to Y$ is said to be C_Δ-quasiconcave-like if, for any $x_1, x_2 \in X$, $\varphi(\Delta\{x_1, x_2\}) \in \varphi(x_1) + C$ or $\varphi(\Delta\{x_1, x_2\}) \in \varphi(x_2) + C$.

Remark 3. In general cases, C_Δ-quasiconcave, C_Δ-quasiconcave-like, and usual quasiconcave functions are independent of each other. See examples in [18]. Let $\varphi : \mathbb{R} \to \mathbb{R}$, $C = -\mathbb{R}^+$. Then the partial order on $\mathbb{R}$ is "$\leq$" (less than or equal to);, hence, the C_Δ-quasiconcave, C_Δ-quasiconcave-like, and usual quasiconcave property of φ coincide (the usual quasiconcave function φ means that for any x_1, x_2, $y \in \mathbb{R}$, $\varphi(x_1) \leq y$ and $\varphi(x_2) \leq y \Rightarrow \varphi(\lambda x_1 + (1-\lambda)x_2) \leq y$, for all $\lambda \in [0, 1]$).

Lemma 4 (see [18]). *Let X be a topological sup-semilattice, Y a Hausdorff locally convex topological vector space with a closed, convex, and pointed cone $C \subset Y$. If the vector-valued function $\varphi : X \to Y$ is C_Δ-quasiconcave or C_Δ-quasiconcave-like, then the set $A = \{x : \varphi(x) \in \operatorname{int} C\}$ is Δ-convex.*

Now we introduce the vector quasiequilibrium problem (VQEP) that we will consider in this paper.

Let X be a topological sup-semilattice and Y a topological vector space. $C \subset Y$ is a closed, convex, and pointed cone with $\operatorname{int} C \neq \emptyset$. $\varphi : X \times X \to Y$ is a vector-valued function, and G is a multivalued mapping on X. The vector quasi-equilibrium problem ϕ with $\phi = (X, Y, C, \varphi, G)$ is to find $\overline{x} \in X$, such that

$$\overline{x} \in G(\overline{x}) : \varphi(\overline{x}, y) \notin \operatorname{int} C, \quad \forall y \in G(\overline{x}). \tag{1}$$

Let $G(x) = X$, for all $x \in X$; then the VQEP is just a vector equilibrium problem (X, Y, C, φ) (VEP). That is to find $\overline{x} \in X$, such that

$$\varphi(\overline{x}, y) \notin \operatorname{int} C, \quad \forall y \in X. \tag{2}$$

Definition 5 (see [21, 22]). A vector-valued function $\varphi : X \to Y$ is said to be C-continuous on X if, for each $x \in X$ and any open neighborhood $V(\theta)$ of θ in Y, there exists an open neighborhood $O(x)$ of x in X such that

$$\forall x' \in O(x), \quad \varphi(x') \in \varphi(x) + V(\theta) + C. \tag{3}$$

Remark 6. For a function $\varphi : X \to \mathbb{R}$, φ is $\mathbb{R}_+$*-continuous* on X if and only if φ is lower semicontinuous on X.

A maximal element version of the Browder fixed point theorem in a topological sup-semilattice can be found in [18]. We limit it in a metric space as the following lemma.

Lemma 7 (see [18]). *Let (X, d) be a compact sup-semilattice with path connected interval, where d is the metric on X, $S : X \to 2^X$ a multivalued map on X with the conditions: (i) for all $x \in X$, $S(x)$ is Δ-convex; (ii) for all $y \in X$, $S^{-1}(y) = \{x \in X : y \in S(x)\}$ is open in X; (iii) for all $x \in X$, $x \notin S(x)$. Then there exists an $\overline{x} \in X$, such that $S(\overline{x}) = \emptyset$.*

Remark 8. The existence of a metric space with a sup-semilattice can be guaranteed. For instance, let $x^j = (x_1^j, \ldots, x_i^j, \ldots, x_n^j) \in \mathbb{R}^n$, $j = 1, 2$, if $x^1 \leq x^2$ means that $x^2 \in x^1 + \mathbb{R}_+^n$, then $x^1 \vee x^2 = \overline{x}$, where $\overline{x}_i = \max\{x_i^1, x_i^2\}$. Clearly, $(\mathbb{R}^n, \leq)$ with the usual Euclidean metric is a topological sup-semilattice.

Let M denote the collection of S satisfying all the conditions of Lemma 7. For any $S_1, S_2 \in M$, define the metric between S_1 and S_2 as

$$\rho(S_1, S_2) = \sup_{y \in X} h\left(X \setminus S_1^{-1}(y), X \setminus S_2^{-1}(y)\right), \tag{4}$$

where h is the Hausdorff metric induced by d. For each $S \in M$ and each $y \in X$, since $y \notin S(y)$, we have $y \in X \setminus S^{-1}(y)$, that is, $X \setminus S^{-1}(y) \neq \emptyset$. Noting that $X \setminus S^{-1}(y)$ is closed, the metric ρ on M is well defined. Then (M, ρ) is a metric space.

For each $S \in M$, denote by $F(S)$ the set of all maximal elements of S. Then F defines a multivalued mapping from M to X and $F(S) = \cap_{y \in X}(X \setminus S^{-1}(y))$.

Definition 9. For each $S \in M$, a set $e(S)$ is called an essential set of $F(S)$ if it satisfies the following conditions:

(1) $e(S)$ is closed subset of $F(S)$.

(2) For any open set $U \supset e(S)$, there exists an open neighborhood $O(S)$ of $S \in M$ such that $U \cap F(S') \neq \emptyset$, for any $S' \in O(S)$.

A set $m(S)$ is called a minimal essential set of $F(S)$ if it is a minimal element of all essential sets ordered by set inclusion in $F(S)$. A connected component in $F(S)$ is called an essential component, if it includes at least one minimal essential set of $F(S)$.

We recall some notions about multi-valued mappings. Let $G : Y \to 2^P$ be a multi-valued mapping, where Y, P are two topological vector spaces. Then (i) G is said to be upper semicontinuous at $y \in Y$, if for each open set $U \supset G(y)$, there exists an open neighborhood $O(y)$ of y such that $U \supset G(y')$ for any $y' \in O(y)$. (ii) G is lower semi-continuous at $y \in Y$, if for each open set $U \cap G(y) \neq \phi$, there exists an open neighborhood $O(y)$ of y such that $U \cap G(y') \neq \phi$ for any $y' \in O(y)$.

Remark 10. For each $S \in M$, a set $e(S) \subset F(S)$ is essential if F is lower semi-continuous at S. If F is upper semi-continuous at S, then $F(S)$ itself is an essential set. For any two closed sets $A, B \subset F(S)$ with $A \subset B$, if A is essential, then B is also essential. For each $S \in M$ and each $y \in X$, if $S^{-1}(y)$ is open, then $X \setminus S^{-1}(y)$ is closed; hence, $F(S)$ is closed because we have $F(S) = \cap_{y \in X} X \setminus S^{-1}(y)$; consequently, $F(S)$ is compact.

Lemma 11 (see [23]). *Let (X, d) be a metric space, K_1 and K_2 two nonempty compact subsets of X, U_1, and U_2 two nonempty disjoint open subsets of X. If $h(K_1, K_2) < d(U_1, U_2)$, then $h(K_1, (K_1 \setminus U_2) \cup (K_2 \setminus U_1)) \leq h(K_1, K_2)$, where h is the Hausdorff metric defined on X.*

3. The Stability of Maximal Elements on Topological Semilattices

Theorem 12. *$F : M \to 2^X$ is an upper semi-continuous mapping with compact values.*

Proof. For each $S \in M$, by Remark 10, $F(S)$ is compact. Suppose that F is not upper semi-continuous. Then there is a $S \in M$, an open set U with $U \supset F(S)$ and S_n, such that $S_n \to S$ and $F(S_n) \not\subset U$, $n = 1, 2, \ldots$. That is, there exists a point $x_n \in F(S_n)$ such that $x_n \notin U$. Without loss of generality, we may assume that $x_n \to x^*$. Since $S_n \to S$, it holds that $X \setminus S_n^{-1}(y) \to X \setminus S^{-1}(y)$, for all $y \in X$. Since $x_n \in F(S_n)$, we have $x_n \in X \setminus S_n^{-1}(y)$, for all $y \in X$. As n gets close to infinity, we can obtain that $x^* \in X \setminus S^{-1}(y)$, for all $y \in X$, that is, $x^* \in F(S) \subset U$. This results in the fact that $x_n \in U$ while n is large enough, a contradiction with $x_n \notin U$. Therefore, F is definitely upper semi-continuous. □

Theorem 13. *For each $S \in M$, there exists at least a minimal essential set of $F(S)$. If $m(S)$ is a minimal essential set of $F(S)$, then $m(S)$ is connected.*

Proof. For the existence, by Remark 10, each decreasing chain, consisting of essential subsets of $F(S)$, has a minimal element, which is the intersection of the chain. By the Zorn's lemma, the minimal element is just a minimal essential set. For the connectedness, by way of contradiction, suppose that $m(S)$ is not connected. There exist two disjoint closed sets $C_1(S), C_2(S)$ such that $m(S) = C_1(S) \cup C_2(S)$.

Since $C_i(S)$ is not essential, there is an open set W_i with $W_i \supset C_i(S)$ such that for any $\varepsilon > 0$, there exists a $S_i \in M$ with $\rho(S, S_i) < \varepsilon$ and $F(S_i) \cap W_i = \emptyset$, $i = 1, 2$. Clearly, $C_i(S)$ is compact, then there is an open set U_i with $C_i(S) \subset U_i \subset W_i$, $i = 1, 2$, such that $U_1 \cap U_2 = \emptyset$. For $U_1 \cup U_2 \supset m(S)$, because $m(S)$ is essential, there is a number $\delta < 2d(U_1, U_2)$, such that $F(T) \cap (U_1 \cup U_2) \neq \emptyset$ for each T satisfying $\rho(T, S) < \delta$. Therefore, we can select a $S_i \in M$ such that $\rho(S, S_i) < \delta/4$ and $F(S_i) \cap U_i = \emptyset$, $i = 1, 2$. Then $\rho(S_1, S_2) < \rho(S_1, S) + \rho(S, S_2) < \delta/2 < d(U_1, U_2)$.

Define a multi-valued mapping $S' : M \to 2^X$ as

$$S' = \begin{cases} S_1(x), & x \in U_1, \\ S_2(x), & x \in U_2, \\ S_1(x) \cap S_2(x), & x \in X \setminus (U_1 \cup U_2). \end{cases} \quad (5)$$

We show that $S' \in M$.

(a) For each $x \in X$, since $x \notin S_1(x)$ and $x \notin S_2(x)$, we have $x \notin S'(x)$;

(b) for each $x \in X$, because $S_1(x)$ and $S_2(x)$ are Δ-convex sets, it follows that $S'(x)$ is Δ-convex;

(c) for each $y \in X$, we have

$$S'^{-1}(y) = \left(S_1^{-1}(y) \cap S_2^{-1}(y)\right) \cup \left(S_2^{-1}(y) \cap U_2\right) \cup \left(S_1^{-1}(y) \cap U_1\right). \quad (6)$$

Noting that $S_1^{-1}(y)$, $S_2^{-1}(y)$, U_1, and U_2 are open sets, it follows that $S'^{-1}(y)$ is open.

Through a direct calculation, $X \setminus S'^{-1}(y)$ can be written as

$$\begin{aligned} &\left(\left(X \setminus S_1^{-1}(y)\right) \cap \left(X \setminus S_2^{-1}(y)\right)\right) \\ &\cup \left(\left(X \setminus S_1^{-1}(y)\right) \cap (X \setminus U_2)\right) \\ &\cup \left(\left(X \setminus S_2^{-1}(y)\right) \cap (X \setminus U_1)\right). \end{aligned} \quad (7)$$

Take any $x \in X \setminus S'^{-1}(y)$. Note that if $x \in U_1$, then $x \in X \setminus U_2$; if $x \in U_2$, then $x \in X \setminus U_1$; if $x \in X \setminus (U_1 \cup U_2)$, then $x \in (X \setminus U_1) \cap (X \setminus U_2)$. Consequently, we can obtain that if $x \in (X \setminus S_1^{-1}(y)) \cap (X \setminus S_2^{-1}(y))$, then

$$\begin{aligned} x \in &\left(\left(X \setminus S_1^{-1}(y)\right) \cap (X \setminus U_2)\right) \\ &\cup \left(\left(X \setminus S_2^{-1}(y)\right) \cap (X \setminus U_1)\right). \end{aligned} \quad (8)$$

Therefore, we have

$$\begin{aligned} X \setminus S'^{-1}(y) &= \left(\left(X \setminus S_1^{-1}(y)\right) \cap (X \setminus U_2)\right) \\ &\cup \left(\left(X \setminus S_2^{-1}(y)\right) \cap (X \setminus U_1)\right). \end{aligned} \quad (9)$$

Since $\rho(S_1, S_2) < d(U_1, U_2)$, by Lemma 11, we have

$$\begin{aligned} h&\left(X \setminus S'^{-1}(y), X \setminus S_1^{-1}(y)\right) \\ &\le h\left(X \setminus S_2^{-1}(y), X \setminus S_1^{-1}(y)\right) \\ &\le \rho(S_2, S) + \rho(S, S_1) < \frac{\delta}{2}. \end{aligned} \quad (10)$$

That is, $\rho(S', S_1) < \delta/2$. This results in the fact that

$$\rho(S', S) < \rho(S', S_1) + \rho(S_1, S) < \delta. \quad (11)$$

Consequently, we have $F(S') \cap (U_1 \cup U_2) \neq \emptyset$. If there is a point $x \in U_i \cap F(S')$, then $x \in U_i$ and $x \in X \setminus S_i^{-1}(y)$, for all $y \in X$, that is, $x \in U_i$ and $x \in F(S_i)$ which contradicts with $F(S_i) \cap U_i = \emptyset$, $i = 1, 2$. Therefore, $m(S)$ is connected. □

4. The Existence and Stability of Solutions for VQEP

This section gives an existence result in relation to VQEP in topological sup-semilattices and induces the existence of minimal essentially stable sets for each VQEP in the set of its solutions.

Theorem 14. *Let $\phi = (X, Y, C, \varphi, G)$ be a VQEP, where X is a compact topological sup-semilattice with path connected intervals, Y is a Hausdorff locally convex topological vector space, and $G : X \to 2^X$ is a multi-valued mapping with nonempty and Δ-convex values. If the VQEP satisfies that*

(i) *for all* $x \in X$, $\varphi(x,x) \notin \operatorname{int} C$*;*

(ii) *for all* $y \in X$, $x \to \varphi(x,y)$ *is* C*-continuous;*

(iii) *for all* $x \in X$, $y \to \varphi(x,y)$ *is* C_Δ*-quasiconcave-like or* C_Δ*-quasiconcave;*

(iv) *for all* $y \in X$, $\{x \in X : y \in G(x)\}$ *is open in* X*,*

(v) $\{x \in X : x \in G(x)\}$ *is closed in* X*,*

then the VQEP has a solution.

Proof. Denote $\{x \in X : x \in G(x)\}$ by K. Let $B : X \to 2^X$ such that $B(x) = \{y \in X : \varphi(x,y) \in \operatorname{int} C\}$, for all $x \in X$. Define

$$S(x) = \begin{cases} B(x) \cap G(x), & x \in K, \\ G(x), & x \in X \setminus K. \end{cases} \quad (12)$$

Then for each $x \in X$, if $x \in K$, we have $S(x) = B(x) \cap G(x)$, by the condition (i), $x \notin B(x)$, hence, $x \notin S(x)$; if $x \in X \setminus K$, from the definition of K, we have $x \notin G(x) = S(x)$.

Since $y \to \varphi(x,y)$ is C_Δ-quasiconcave-like or C_Δ-quasiconcave, by Lemma 4, we have that $B(x)$ is Δ-convex. Then $B(x) \cap G(x)$ is a Δ-convex set, noting that G has Δ-convex values, we have that $S(x)$ is also Δ-convex.

For each $y \in X$, we can check that

$$S^{-1}(y) = \left(B^{-1}(y) \cap G^{-1}(y)\right) \cup \left(G^{-1}(y) \cap (X \setminus K)\right). \quad (13)$$

Take a point $x \in B^{-1}(y) = \{x \in X : \varphi(x,y) \in \operatorname{int} C\}$, since $\operatorname{int} C$ is open, there is an open set $V(\theta)$ such that $V(\theta) + \varphi(x,y) \subset \operatorname{int} C$, then, by the condition (ii), there exists an open neighborhood $O(x)$ in X such that for all $x' \in O(x)$,

$$\varphi(x', y) \in \varphi(x,y) + V(\theta) + C \subset \operatorname{int} C + C \subset \operatorname{int} C. \quad (14)$$

That is, $O(x) \subset B^{-1}(y)$; hence, $B^{-1}(y)$ is open. Noting that $X \setminus K$ and $G^{-1}(y)$ are open sets in X. We can obtain that $S^{-1}(y)$ is also open in X.

Thus, there is an $\overline{x} \in X$ such that $S(\overline{x}) = \emptyset$ by Lemma 7. If $\overline{x} \in X \setminus K$, then $G(\overline{x}) = \emptyset$, a contradiction to the fact that G has nonempty values. Therefore, $\overline{x} \in K$ and $B(\overline{x}) \cap G(\overline{x}) = \emptyset$, that is, $\overline{x} \in G(\overline{x})$, $\varphi(\overline{x}, y) \notin \operatorname{int} C$, for all $y \in X$. □

By Theorem 14 and its proof, we can also obtain the existence result for VQEP as the following.

Corollary 15. *Let* $\phi = (X, Y, C, \varphi, G)$ *be a VQEP, where* X *is a compact topological sup-semilattice with path connected intervals,* Y *is a Hausdorff topological vector space, and* $G : X \to 2^X$ *is a multi-valued mapping with nonempty and* Δ*-convex values. If the VQEP satisfies that*

(i) *for all* $x \in X$, $\varphi(x,x) \notin \operatorname{int} C$*;*

(ii) *for all* $y \in X$, $\{x \in X : \varphi(x,y) \in \operatorname{int} C\}$ *is open in* X*;*

(iii) *for all* $x \in X$, $\{y \in X : \varphi(x,y) \in \operatorname{int} C\}$ *is* Δ*-convex;*

(iv) *for all* $y \in X$, $\{x \in X : y \in G(x)\}$ *is open in* X*,*

(v) $\{x \in X : x \in G(x)\}$ *is closed in* X*,*

then the VQEP has a solution.

By Theorem 14, for the special case of VQEP without the feasible mapping G, we can obtain the existence result concerning VEP as the following.

Corollary 16. *Let* (X, Y, C, φ) *be a vector equilibrium problem, where* X *is a compact topological sup-semilattice with path connected intervals,* Y *is a Hausdorff locally convex topological vector space. If the VEP satisfies the following conditions:*

(i) *for all* $x \in X$, $y \to \varphi(x,y)$ *is* C_Δ*-quasiconcave or* C_Δ*-quasiconcave-like;*

(ii) *for all* $y \in X$, $x \to \varphi(x,y)$ *is* C*-continuous;*

(iii) *for all* $x \in X$, $\varphi(x,x) \notin \operatorname{int} C$*,*

then this VEP has a solution.

Example 17. Let $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, $C = -\mathbb{R}_+$. The $(X, \leq)$ is a sup-semilattice, in which $x^1 \leq x^2$ means that $x^2 \in x^1 + \mathbb{R}^2_+$, for all $x^1, x^2 \in X$.

(a) For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, the function φ is defined as

$$\varphi(x,y) = (1 - y_1)(1 - y_2) - (1 - x_1)(1 - x_2). \quad (15)$$

It can be easily checked that for each $x \in X$, $\varphi(x, \cdot)$ is C_Δ quasiconcave and C_Δ quasiconcave-like but not a usual quasiconcave function.

Denote by D the set $(1 \times [0,1]) \cup ([0,1] \times 1)$. For each $x = (x_1, x_2) \in X$, the multi-valued mapping G satisfies that

$$G(x) = \begin{cases} (x_1, 1] \times [0,1] \cup [0,1] \times (x_2, 1], & x \in X \setminus D, \\ (1,1), & x \in D. \end{cases} \quad (16)$$

Note that G is not a usual convex but a Δ-convex multi-valued mapping. For each $y = (y_1, y_2) \in X$, if $y \in X \setminus (1,1)$, then $G^{-1}(y) = [0, y_1) \times [0, y_2)$; if $y = (1,1)$, then $G^{-1}(y) = X$. Thus, $G^{-1}(y)$ is open in X for each $y \in X$. Then φ and G satisfy all the conditions in Theorem 14. We can find that $\overline{x} = (1,1)$ is the unique solution for the VQEP, $(X, \mathbb{R}, C, \varphi, G)$.

(b) For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, let $T(x) = B_\varepsilon(x)$, where $B_\varepsilon(x)$ is the ε-neighborhood of x in X,

$$\varphi(x,y) = (1 + x_1 - y_1)(1 + x_2 - y_2), \quad (17)$$

and G is the same as the setting in (a). Then the function φ and the mappings G and T meet all the conditions in Theorem 14. The set of solutions for the VEP $(X, \mathbb{R}, C, \varphi)$ is the overall X which is also the set of solutions for $(X, \mathbb{R}, C, \varphi, T)$. The solution of $(X, \mathbb{R}, C, \varphi, G)$ is just one point $\overline{x} = (1,1)$.

To study the stability of vector quasi-equilibrium problems, let (X, d) be a metric space and define the set M' as

$$M' = \{\phi = (X, Y, C, \varphi, G) : \phi \text{ satisfies all the conditions in Theorem 14}\} \quad (18)$$

For each $\phi \in M'$, by the proof of Theorem 14, we can find that a point $\overline{x} \in X$ is a solution of ϕ if and only if $\overline{x}$ is a maximal element of S defined in the proof. Let $F'(\phi)$ denote all the solutions of ϕ. Then F' is a multi-valued mapping from M' to X. For any two ϕ_1, ϕ_2, define the metric $\rho'(\phi_1, \phi_2)$ between ϕ_1 and ϕ_2 as

$$\rho'(\phi_1, \phi_2) = \rho(S_1, S_2), \tag{19}$$

where S_1 and S_2 are multi-valued mappings corresponding to ϕ_1 and ϕ_2 in the proof of Theorem 14. Then (M', ρ') is a metric space. Instead of M, S and $F(S)$ by M', ϕ and $F'(\phi)$ in Definition 9, we can also define essential sets $e(\phi)$, minimal essential sets $m(\phi)$ of $F'(\phi)$, and essential component in $F'(\phi)$. If an essential set $e(\phi)$ is singleton set $\{x^*\}$, x^* is called an essential solution of ϕ.

From Theorems 12 and 13, we have the following results.

Theorem 18. *$F' : M' \to 2^X$ is an upper semi-continuous mapping with compact values. For each VQEP $\phi \in M'$, there exists at least a connected minimal essential set $m(\phi)$ of $F'(\phi)$.*

Remark 19. For each $\phi \in M'$, $y \in X$, let

$$A_\phi(y) = \{x : y \notin G(x), \text{or } \varphi(x, y) \notin \operatorname{int} C \text{ and } x \in G(x)\}. \tag{20}$$

For any $\phi_1, \phi_2 \in M'$, from the definition of the metric between S_1 and S_2, then

$$\rho'(\phi_1, \phi_2) = \sup_{y \in X} h(A_{\phi_1}(y), A_{\phi_2}(y)), \tag{21}$$

which gives an overall consideration of φ and G. If ϕ_1 and ϕ_2 are two VEP, then

$$\begin{aligned} &\rho'(\phi_1, \phi_2) \\ &\quad = \sup_{y \in X} h(\{x : \varphi_1(x, y) \notin \operatorname{int} C\}, \\ &\qquad\qquad x\{x : \varphi_2(x, y) \notin \operatorname{int} C\}). \end{aligned} \tag{22}$$

For the essential stability of solutions for VQEP, clearly, the class of perturbations induced by the metric ρ' is different from the perturbation of uniform topology in [3, 14] and also different from the perturbation of best response defined in [16]. For example, the existence of essential sets of solutions for VQEP in topological vector spaces is proved in [3], and the uniform metric for two VQEP $\phi_1 = (X, X, C, \varphi_1, G_1)$ and $\phi_2 = (X, X, C, \varphi_2, G_2)$ is defined as

$$\begin{aligned} \rho''(\phi_1, \phi_2) = {} & \sup_{(x,y) \in X \times X} \|\varphi_1(x, y) - \varphi_2(x, y)\| \\ & + \sup_{x \in X} h(G_1(x), G_2(x)), \end{aligned} \tag{23}$$

where X is a compact convex subset of a Banach space. Naturally, the feasible mapping G requires closed values, which is not a requirement in Theorem 14, however, where each inverse image being open is necessary.

By Theorem 18, each connected component including a connected minimal essential set of solutions is essential; that is, the existence of essential components can be induced.

Corollary 20. *Let $\phi \in M'$. There is an essential component in $F'(\phi)$. If $F'(\phi) = \{x^*\}$ is a singleton, then x^* is an essential solution of ϕ.*

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