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Research Article

On the Solution Existence of Variational-Like Inequalities Problems for Weakly Relaxed η-α Monotone Mapping

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We introduce two new concepts of weakly relaxed η-α monotone mappings and weakly relaxed η-α semimonotone mappings. Using the KKM technique, the existence of solutions for variational-like problems with weakly relaxed η-α monotone mapping in reflexive Banach spaces is established. Also, we obtain the existence of solution for variational-like problems with weakly relaxed η-α semimonotone mappings in arbitrary Banach spaces by using the Kakutani-Fan-Glicksberg fixed-point theorem.

1. Introduction

The variational inequality theory provides us with a simple, natural, unified, and elegant framework to study a wide class of linear and nonlinear problems arising in many fields, such as mechanics, engineering sciences, elasticity, optimization, control, programming, economics, transportation, oceanography, and regional. Because of their wide applicability, various extensions and generalizations of the classical variational inequality problem have been proposed and studied in recent years. Variational-like inequalities problems is one of cornerstone in this field. Some special case of generalized variational inequalities and variational-like inequalities have been studied by several authors including Bai et al. [1], Chang et al. [2], dos Santos et al. [3], Xiao and Huang [4], Zhao and Xia [5, 6], and references therein.

It is well known that the monotonicity and generalized monotonicity play an important role of the study in variational inequality theory. In recent years, a substantial number of papers on existence results for solving variational inequality problems and variational-like inequality problems based on different generalization of monotonicity such as pseudomonotonicity, quasimonotonicity, relaxed monotonicity, semimonotonicity, and p-monotonicity (see [7–15]) appeared. In [16], Fang and Huang introduced a new concept of relaxed η-α monotonicity and obtained the existence of solutions for variational-like inequalities with relaxed η-α monotone mappings in reflexive Banach spaces. Recently, Sintunavarat [17] established the existence of solution of mixed equilibrium problem with the weakly relaxed α-monotone bi-function in Banach spaces.

Inspired and motivated by the work of Fang and Huang [16] and Sintunavarat [17], in this paper we introduce the two new concepts of weakly relaxed η-α monotone mappings and weakly relaxed η-α semimonotone mappings as well as two classes of variational-like inequalities with weakly relaxed η-α monotone mappings and weakly relaxed η-α semimonotone mappings. By using the KKM technique, we study some existence of solutions for variational-like inequalities with weakly relaxed η-α monotone mappings in reflexive Banach spaces. We also obtain the solvability of variational-like inequalities with weakly relaxed η-α semimonotone mappings in arbitrary Banach spaces by using the Kakutani-Fan-Glicksberg fixed-point theorem. Our results in this paper extend and improve the results of Fang and Huang [16] and many results in the literature.

2. Preliminaries

In this paper, unless otherwise specified, K is a nonempty closed convex subset of a real reflexive Banach space E with dual space E^*. The following basic knowledge will be useful in our work.

Definition 1 (see [18]). Let $T : K \to E^*$ and $\eta : K \times K \to E$ be two mappings. A mapping T is said to be η-hemicontinuous if, for any fixed $x, y \in K$, the mapping $f : [0, 1] \to \mathbb{R}$ defined by

$$f(t) = \langle T(ty + (1-t)x), \eta(y, x)\rangle \tag{1}$$

is continuous at 0^+.

Definition 2 (see [16]). Let $T : K \to E^*$ and $\eta : K \times K \to E$ be two mappings. A mapping T is said to be η-coercive with respect to a proper function $f : K \to \mathbb{R} \cup \{\infty\}$ if there exists $x_0 \in K$ such that

$$\frac{[\langle Tx - Tx_0, \eta(x, x_0)\rangle + f(x) - f(x_0)]}{\|\eta(x, x_0)\|} \longrightarrow \infty, \tag{2}$$

whenever $\|x\|$ is large enough.

Remark 3. If $f = \delta_K$, where δ_K is the indicator function of K, then Definition 2 coincides with the definition of η-coercivity in the sense of Yang and Chen [15].

Definition 4 (see [16]). A mapping $T : K \to E^*$ is said to be relaxed η-α monotone if there exist a function $\eta : K\times K \to E$ and $\alpha : E \to \mathbb{R}$ with $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in E$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq \alpha(x - y), \quad \forall x, y \in K, \tag{3}$$

where $p > 1$ is a constant.

Remark 5. (1) If $\eta(x, y) = x - y$ for all $x, y \in K$, then (3) reduces to

$$\langle Tx - Ty, x - y\rangle \geq \alpha(x - y), \quad \forall x, y \in K, \tag{4}$$

and T is said to be α-monotone.

(2) If $\eta(x, y) = x - y$ for all $x, y \in K$ and $\alpha(z) = k\|z\|^p$, where $k > 0$ and $p > 1$, then (3) becomes

$$\langle Tx - Ty, x - y\rangle \geq k\|x - y\|^p, \quad \forall x, y \in K, \tag{5}$$

and T is said to be p-monotone (see in [14, 19, 20]).

(3) Every monotone mapping is relaxed η-α monotone with $\eta(x, y) = x - y$ for all $x, y \in K$ and $\alpha \equiv 0$.

Definition 6. Let $F : K \to 2^E$ be a set-valued mapping. Then, F is said to be KKM mapping if, for any finite subset $\{y_1, y_2, \ldots, y_n\}$ of K, we have $\mathrm{co}\{y_1, y_2, \ldots, y_n\} \subset \bigcup_{i=1}^n F(y_i)$, where $\mathrm{co}\{y_1, y_2, \ldots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \ldots, y_n\}$.

Remark 7. Let $F, G : K \to 2^E$. If F is KKM mapping and $F(y) \subset G(y)$ for all $y \in K$, then G is also KKM mapping.

Lemma 8 (see [21]). *Let M be a nonempty subset of a Hausdorff topological vector space X, and let $F : M \to 2^X$ be a KKM mapping. If $F(y)$ is closed in X for all $y \in M$ and compact for some $y \in M$, then $\bigcap_{y\in M} F(y) \neq \emptyset$.*

3. Variational-Like Inequalities Problems with Weakly Relaxed η-α Monotone Mapping

In this section, we introduce the new class of mapping which generalizes several classes. Using KKM technique, we study and prove the existence of solutions for variational-like inequalities problems with mapping in such class in Banach spaces.

Definition 9. A mapping $T : K \to E^*$ is said to be weakly relaxed η-α monotone if there exist a function $\eta : K\times K \to E$ and $\alpha : E \to \mathbb{R}$ with

$$\lim_{t\to 0^+} \alpha(tx) = 0, \tag{6}$$

$$\lim_{t\to 0^+} \frac{d}{dt}\alpha(tx) = 0, \tag{7}$$

for all $t > 0$ and $x \in E$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq \alpha(x - y), \quad \forall x, y \in K. \tag{8}$$

Remark 10. We obtain that the relaxed η-α monotonicity implies weakly relaxed η-α monotonicity. So, class of relaxed η-α monotone mapping is subclass of weakly relaxed η-α monotone mapping class. Also, we get that classes of relaxed α monotone mapping, p-monotone mapping, and monotone mapping are subclass of weakly relaxed η-α monotone mapping class.

Theorem 11. *Let $T : K \to E^*$ be an η-hemicontinuous and weakly relaxed η-α monotone and let $f : K \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. Suppose that the following conditions hold:*

(a) $\eta(x, x) = 0$ *for all* $x \in K$;

(b) *for any fixed* $y, z \in K$*, the mapping* $x \mapsto \langle Tz, \eta(x, y)\rangle$ *is convex.*

Then, the following problems (9) *and* (10) *are equivalent:*

$$\begin{gathered}\text{find } x \in K \\ \text{such that } \langle Tx, \eta(y, x)\rangle + f(y) - f(x) \geq 0, \quad \forall y \in K,\end{gathered} \tag{9}$$

$$\begin{gathered}\text{find } x \in K \\ \text{such that } \langle Ty, \eta(y, x)\rangle + f(y) - f(x) \geq \alpha(y - x), \quad \forall y \in K.\end{gathered} \tag{10}$$

Proof. Suppose that (9) has a solution. So, there exists $x \in K$ such that

$$\langle Tx, \eta(y,x)\rangle + f(y) - f(x) \geq 0, \quad \forall y \in K. \tag{11}$$

Since T is weakly relaxed η-α monotone, we have

$$\langle Ty - Tx, \eta(y,x)\rangle \geq \alpha(y-x), \quad \forall y \in K, \tag{12}$$

and then

$$\begin{aligned}\langle Ty, \eta(y,x)\rangle &+ f(y) - f(x) \\ &\geq \langle Tx, \eta(y,x)\rangle + \alpha(y-x) + f(y) - f(x) \\ &\geq \alpha(y-x), \quad \forall y \in K.\end{aligned} \tag{13}$$

Therefore, $x \in K$ is a solution of problem (10).

Conversely, suppose that $x \in K$ is a solution of problem (10) and y is any point in K with $f(y) < \infty$. For $t \in (0,1)$, we let $y_t := (1-t)x + ty$. From (10), we get that $f(x) < \infty$. It follows from K being convex that $y_t \in K$. From (10), we have

$$\begin{aligned}\langle Ty_t, \eta(y_t,x)\rangle + f(y_t) - f(x) &\geq \alpha(y_t - x) \\ &= \alpha(t(y-x)),\end{aligned} \tag{14}$$

and thus

$$f(y_t) - f(x) \geq \alpha(t(y-x)) - \langle Ty_t, \eta(y_t,x)\rangle. \tag{15}$$

The convexity of f implies that

$$\begin{aligned}f(y_t) - f(x) &= f((1-t)x + ty) - f(x) \\ &\leq t(f(y) - f(x)).\end{aligned} \tag{16}$$

From (15) and (16), we get

$$t(f(y) - f(x)) \geq \alpha(t(y-x)) - \langle Ty_t, \eta(y_t,x)\rangle. \tag{17}$$

By the assumption (b), we have

$$\begin{aligned}\langle Ty_t, \eta(y_t,x)\rangle &= \langle Ty_t, \eta((1-t)x + ty, x)\rangle \\ &\leq (1-t)\langle Ty_t, \eta(x,x)\rangle + t\langle Ty_t, \eta(x,y)\rangle \\ &= t\langle Ty_t, \eta(y,x)\rangle \\ &= t\langle T((1-t)x + ty), \eta(y,x)\rangle.\end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\begin{aligned}t\langle T((1-t)x + ty), \eta(y,x)\rangle &+ t(f(y) - f(x)) \\ &\geq \alpha(t(y-x)),\end{aligned} \tag{19}$$

that is

$$\begin{aligned}\langle T((1-t)x + ty), \eta(y,x)\rangle &+ f(y) - f(x) \\ &\geq \frac{\alpha(t(y-x))}{t},\end{aligned} \tag{20}$$

for all $y \in K$. Taking $t \to 0^+$ in the previous inequality and using η-hemicontinuity of T, we get

$$\langle Tx, \eta(y,x)\rangle + f(y) - f(x) \geq \lim_{t\to 0^+} \frac{\alpha(t(y-x))}{t}. \tag{21}$$

From (6), we get $\lim_{t\to 0^+}(\alpha(t(y-x))/t)$ is indeterminate form. Using L' Hôpital's rule, we obtain that

$$\langle Tx, \eta(y,x)\rangle + f(y) - f(x) \geq \lim_{t\to 0^+} \frac{(d/dt)\alpha(t(y-x))}{1}. \tag{22}$$

By property (7) of weakly relaxed η-α monotone of T, we have

$$\langle Tx, \eta(y,x)\rangle + f(y) - f(x) \geq 0, \tag{23}$$

for all $y \in K$ with $f(y) < \infty$. In case of $f(y) = \infty$, the relation

$$\langle Tx, \eta(y,x)\rangle + f(y) - f(x) \geq 0 \tag{24}$$

is trivial. Therefore, $x \in K$ is also a solution of problem (9). □

Theorem 12. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E, and let E^* be the dual space of E. Suppose that $T : K \to E^*$ is an η-hemicontinuous and weakly relaxed η-α monotone mapping and $f : K \to \mathbb{R} \cup \{\infty\}$ is a proper convex lower semicontinuous function. If the following conditions hold:*

- (a) *$\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in K$,*
- (b) *for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x,y)\rangle$ is convex and lower semicontinuous function,*
- (c) *α is weakly lower semicontinuous; that is, for any net $\{x_\beta\}$, we have that x_β converges to x in $\sigma(E, E^*)$ implies that $\alpha(x) \leq \liminf \alpha(x_\beta)$,*

then the problem (9) *is solvable.*

Proof. Consider the set valued mapping $F : K \to 2^E$ defined by

$$F(y) = \{x \in K : \langle Tx, \eta(y,x)\rangle + f(y) - f(x) \geq 0\}, \tag{25}$$

for all $y \in K$.

It is easy to see that $x \in K$ solves the problem (9); that is,

$$\langle Tx, \eta(y,x)\rangle + f(y) - f(x) \geq 0, \quad \forall y \in K, \tag{26}$$

if and only if $x \in \bigcap_{y\in K} F(y)$. Thus, it is sufficient to prove that $\bigcap_{y\in K} F(y) \neq \emptyset$.

Next, we show that F is a KKM mapping. Assume the contrary, then there exists $\{y_1, y_2, \ldots, y_m\} \subset K$ such that $\mathrm{co}\{y_1, y_2, \ldots, y_m\} \not\subset \bigcup_{i=1}^m F(y_i)$. This implies that there exists $y_0 \in \mathrm{co}\{y_1, y_2, \ldots, y_m\}$ such that $y_0 = \sum_{i=1}^m t_i y_i$, where $t_i \geq 0$, $i = 1, 2, \ldots, m$ and $\sum_{i=1}^m t_i = 1$, but $y_0 \notin \bigcup_{i=1}^m F(y_i)$.

From (25), we have

$$\langle Ty_0, \eta(y_i, y_0)\rangle + f(y_i) - f(y_0) < 0, \quad \text{for } i = 1, 2, \ldots, m. \tag{27}$$

By (b) and (25), we obtain that

$$\begin{aligned} 0 &= \langle Ty_0, \eta(y_0, y_0)\rangle \\ &= \left\langle Ty_0, \eta\left(\sum_{i=1}^{m} t_i y_i, y_0\right)\right\rangle \\ &\le \sum_{i=1}^{m} t_i \langle Ty_0, \eta(y_i, y_0)\rangle \\ &< \sum_{i=1}^{m} t_i [f(y_0) - f(y_i)] \\ &= f(y_0) - \sum_{i=1}^{m} t_i f(y_i) \\ &\le f(y_0) - f(y_0) \\ &= 0, \end{aligned} \tag{28}$$

which is a contradiction. Therefore, F is a KKM mapping.

Now, we define another set-valued mapping $G : K \to 2^E$ by

$$G(y) = \{x \in K : \langle Tx, \eta(y, x)\rangle + f(y) - f(x) \ge \alpha(y - x)\} \tag{29}$$

for all $y \in K$.

Next, we will claim that $F(y) \subset G(y)$ for all $y \in K$. For each $y \in K$, let $x \in F(y)$; then,

$$\langle Tx, \eta(y, x)\rangle + f(y) - f(x) \ge 0. \tag{30}$$

From the weakly relaxed η-α monotonicity of T, we get

$$\begin{aligned} &\langle Ty, \eta(y, x)\rangle + f(y) - f(x) \\ &\quad \ge \alpha(y - x) + [\langle Tx, \eta(y, x)\rangle + f(y) - f(x)] \\ &\quad \ge \alpha(y - x). \end{aligned} \tag{31}$$

This implies that $x \in G(y)$ and hence $F(y) \subset G(y)$ for all $y \in K$. So, G is also a KKM mapping.

By assumption, $x \mapsto \langle Ty, \eta(x, y)\rangle$ and f are two convex lower-semicontinuous functions. Then it is easy to see that they are both weakly lower semicontinuous. From the definition of G and the weakly lower semicontinuity of α, we get that $G(y)$ is weakly closed for all $y \in K$. Since K is closed bounded and convex, it also is weakly compact, and then $G(y)$ is weakly compact in K for each $y \in K$. From Lemma 8 and Theorem 11, we obtain that

$$\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y) \neq \emptyset. \tag{32}$$

So, there exists $x \in K$, such that

$$\langle Tx, \eta(y, x)\rangle + f(y) - f(x) \ge 0, \quad \forall y \in K; \tag{33}$$

that is, problem (9) has a solution. This completes the proof. □

We know that the relaxed η-α monotonicity implies the weakly relaxed η-α monotonicity. Therefore, Theorem 12 can be deduced to the following corollary.

Corollary 13. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space E, and let E^* be the dual space of E. Suppose that $T : K \to E^*$ is an η-hemicontinuous and relaxed η-α monotone mapping and $f : K \to \mathbb{R} \cup \{\infty\}$ is a proper convex lower semicontinuous function. If the following conditions hold:*

(a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$,*

(b) *for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y)\rangle$ is convex and lower semicontinuous function,*

(c) *α is weakly lower semicontinuous,*

then the problem (9) *is solvable.*

Remark 14. Since the monotonicity, p-monotonicity, and relaxed α-monotonicity imply relaxed η-α monotonicity, we can be applying Corollary 13 to the other problems for the mapping satisfies these property.

Next, we study and prove that result for the case of K is unbounded set.

Theorem 15. *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space E, and let E^* be the dual space of E. Suppose that $T : K \to E^*$ is an η-hemicontinuous and weakly relaxed η-α monotone mapping and $f : K \to \mathbb{R} \cup \{\infty\}$ is a proper convex lower semicontinuous function. If the following conditions hold:*

(a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$,*

(b) *for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y)\rangle$ is convex and lower semicontinuous function,*

(c) *α is weakly lower semicontinuous,*

(d) *T is weakly η-coercive with respect to f; that is, there exists $x_0 \in K$ such that*

$$\langle Tx, \eta(x_0, x)\rangle + f(x_0) - f(x) < 0, \tag{34}$$

whenever $x \in K$ and $\|x\|$ large enough,

then the problem (9) *is solvable.*

Proof. For $\epsilon > 0$, define $K_\epsilon := \{y \in K : \|y\| \le \epsilon\}$. Consider the following problem:

$$\begin{aligned} &\text{find } x \in K_\epsilon \\ &\text{such that } \langle Tx_\epsilon, \eta(y, x_\epsilon)\rangle + f(y) - f(x_\epsilon) \ge 0, \\ &\qquad \forall y \in K_\epsilon. \end{aligned} \tag{35}$$

Since K_ϵ is bounded, by Theorem 12, we get that the problem (35) has at least one solution $x_\epsilon \in K_\epsilon$.

For x_0 in the weakly η-coercivity condition (d), we fixed $\epsilon > \|x_0\|$. From (35), we can find that $x_\epsilon \in K_\epsilon$ such that

$$\langle Tx_\epsilon, \eta(x_0, x_\epsilon)\rangle + f(x_0) - f(x_\epsilon) \ge 0. \tag{36}$$

Since $x_\epsilon \in K_\epsilon$, we have $\|x_\epsilon\| \leq \epsilon$. If $\|x_\epsilon\| = \epsilon$ for all ϵ, we may choose ϵ large enough so that the weakly η-coercivity condition (d) implies that

$$\langle Tx_\epsilon, \eta(x_0, x_\epsilon)\rangle + f(x_0) - f(x_\epsilon) < 0, \tag{37}$$

which contradicts (36). Therefore, there exists ϵ such that $\|x_\epsilon\| < \epsilon$.

For each $y \in K$, we can choose $0 < t < 1$ such that $ty + (1-t)x_\epsilon \in K_\epsilon$. From (35) and the fact that $ty + (1-t)x_\epsilon \in K_\epsilon$, we have

$$\begin{aligned}\langle Tx_\epsilon, \eta(ty + (1-t)x_\epsilon, x_\epsilon)\rangle + f(ty + (1-t)x_\epsilon) - f(x_\epsilon)\\ \geq 0.\end{aligned} \tag{38}$$

By the above inequality and the convexity of f and mapping in (b), we get

$$\begin{aligned} t\langle Tx_\epsilon, \eta(y, x_\epsilon)\rangle &+ tf(y) - tf(x_\epsilon)\\ &= t\langle Tx_\epsilon, \eta(y, x_\epsilon)\rangle + (1-t)\langle Tx_\epsilon, \eta(x_\epsilon, x_\epsilon)\rangle\\ &\quad + tf(y) + (1-t)f(x_\epsilon) - f(x_\epsilon)\\ &\geq \langle Tx_\epsilon, \eta(ty + (1-t)x_\epsilon, x_\epsilon)\rangle + f(ty + (1-t)x_\epsilon)\\ &\quad - f(x_\epsilon)\\ &\geq 0,\end{aligned} \tag{39}$$

for all $y \in K$. This implies that

$$\langle Tx_\epsilon, \eta(y, x_\epsilon)\rangle + f(y) - f(x_\epsilon) \geq 0, \tag{40}$$

for all $y \in K$. Therefore, $x_\epsilon \in K$ is a solution of the problem (9). This completes the proof. □

It is easy to see that the relaxed η-α monotonicity implies the weakly relaxed η-α monotonicity. So, Theorem 15 can be deduced to the following corollary.

Corollary 16. *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space E, and let E^* be the dual space of E. Suppose that $T : K \to E^*$ is an η-hemicontinuous and relaxed η-α monotone mapping and $f : K \to \mathbb{R} \cup \{\infty\}$ is a proper convex lower semicontinuous function. If the following conditions hold:*

(a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$,*

(b) *for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y)\rangle$ is convex and lower semicontinuous function,*

(c) *α is weakly lower semicontinuous,*

(d) *T is weakly η-coercive with respect to f,*

then the problem (9) *is solvable.*

Since the η-coercivity with respect to f implies that the weakly η-coercivity with respect to f, we can utilize Corollary 16 to the result of Fang and Huang [16].

Corollary 17 (Theorem 2.3 [16]). *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space E, and let E^* be the dual space of E. Suppose that $T : K \to E^*$ is an η-hemicontinuous and relaxed η-α monotone mapping and $f : K \to \mathbb{R}\cup\{\infty\}$ is a proper convex lower semicontinuous function. If the following conditions hold:*

(a) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$,*

(b) *for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y)\rangle$ is convex and lower semicontinuous function,*

(c) *α is weakly lower semicontinuous,*

(d) *T is η-coercive with respect to f,*

then the problem (9) *is solvable.*

Remark 18. Theorems 11, 12, and 15 are improvement of the results of Fang and Huang [16] from the corresponding results of variational-like inequality problems for relaxed η-α monotone mapping to weakly relaxed η-α monotone mapping. Also, these results are extension of the known results of Hartman and Stampacchia [22] and corresponding results in [14, 19, 23].

4. Variational-Like Inequalities Problems with Weakly Relaxed η-α Semimonotone Mapping

Through this section, let E be an arbitrary Banach space with its dual space E^*, let E^{**} denote the dual space of E^*, and let K be a nonempty closed convex subset of E^{**}.

Definition 19. A mapping $A : K \times K \to E^{**}$ is said to be weakly relaxed η-α semimonotone if the following conditions hold:

(a) for each fixed $u \in K$, $A(u, \cdot)$ is weakly relaxed η-α monotone; that is, there exist mappings $\eta : K \times K \to E^*$ and $\alpha : E^{**} \to \mathbb{R}$ such that

$$\begin{gathered}\lim_{t\to 0^+} \alpha(tx) = 0,\\ \lim_{t\to 0^+} \frac{d}{dt}\alpha(tx) = 0,\end{gathered} \tag{41}$$

for all $t > 0$ and $x \in E^{**}$ and

$$\begin{aligned}\langle A(u, v) - A(u, w), \eta(v, w)\rangle \geq \alpha(v - w)\\ \forall v, w \in K;\end{aligned} \tag{42}$$

(b) for each fixed $v \in K$, $A(\cdot, v)$ is completely continuous; that is, for any net $\{u_\beta\}$ with u_β converges to u_0 in $\sigma(E^{**}, E^*)$, we have that $\{A(u_\beta, v)\}$ converges to $A(u_0, v)$ in the norm topology of E^*.

Remark 20. We obtain that relaxed η-α semimonotonicity due to Fang and Huang [16] implies weakly relaxed η-α semimonotonicity. Therefore, the class of relaxed η-α semimonotone mappings is subclass of the class of weakly relaxed η-α semimonotone mappings.

Let $A : K \times K \to E^{**}$ and $\eta : K \times K \to E^{*}$ be two mappings and $f : K \to \mathbb{R} \cup \{\infty\}$ is a proper convex lower-semicontinuous function. We consider the following problem:

$$\text{find } u \in K$$
$$\text{such that } \langle A(u,u), \eta(v,u) \rangle + f(v) - f(u) \geq 0, \quad \forall v \in K. \tag{43}$$

Theorem 21. *Let E be a real Banach space and let $K \subset E^{**}$ be a nonempty bounded closed convex set. Suppose that $A : K \times K \to E^{**}$ is a weakly relaxed η-α semimonotone mapping and let $f : K \to \mathbb{R} \cup \{\infty\}$ be a proper convex lower semicontinuous function. If the following conditions hold:*

(a) *$\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in K$,*

(b) *for any fixed $y, z, w \in K$, the mapping $x \mapsto \langle A(z, w), \eta(x,y) \rangle$ is convex and lower semicontinuous function,*

(c) *for each $x \in K$, $A(x,\cdot) : K \to E^{**}$ is finite-dimensional continuous; that is, for any finite-dimensional subspace $F \subset E^{**}$, $A(x,\cdot) : K \cap F \to E^{**}$ is continuous,*

(d) *α is convex lower semicontinuous,*

then the problem (43) *is solvable.*

Proof. Let $F \subset E^{**}$ be a finite-dimensional subspace with $K_F := F \cap K \neq \emptyset$. For each $w \in K$, consider the following problem:

$$\text{find } u_0 \in K_F$$
$$\text{such that } \langle A(w,u_0), \eta(v,u_0) \rangle + f(v) - f(u_0) \geq 0, \quad \forall v \in K_F. \tag{44}$$

It follows from K_F being bounded closed and convex and $A(w,\cdot)$ being continuous on K_F and weakly relaxed η-α monotone for each fixed $w \in K$ that from Theorem 12, we obtain that problem (44) has a solution $u_0 \in K_F$.

Next, define a set-valued mapping $T : K_F \to 2^{K_F}$ as follows:

$$Tw = \{u \in K_F : \langle A(w,u), \eta(v,u) \rangle + f(v) - f(u) \geq 0, \forall v \in K_F\}, \quad \forall w \in K_F. \tag{45}$$

By Theorem 11, we get that, for each $w \in K_F$,

$$Tw = \{u \in K_F : \langle A(w,u), \eta(v,u) \rangle + f(v) - f(u) \geq \alpha(v-u), \forall v \in K_F\}. \tag{46}$$

It is known that every convex lower-semicontinuous function in Banach spaces is weakly lower semicontinuous. Therefore, condition (b) and the proper convex lower semicontinuity of f and α implies that T has nonempty bounded closed and convex values. By the complete continuity of $A(\cdot,u)$, we have that T is upper semicontinuous. Using the Kakutani-Fan-Glicksberg fixed-point theorem, we obtain that T has a fixed-point $w_0 \in K_F$ and thus

$$\langle A(w_0,w_0), \eta(v,w_0) \rangle + f(v) - f(w_0) \geq 0, \quad \forall v \in K_F. \tag{47}$$

Now, define

$$\mathcal{F} := \{F \subset E^{**} : F \text{ is finite dimensional with } F \cap K \neq \emptyset\},$$
$$W_F := \{u \in K : \langle A(u,v), \eta(v,u) \rangle + f(v) - f(u) \geq \alpha(v-u), \forall v \in K_F\}, \quad \forall F \in \mathcal{F}. \tag{48}$$

From (47), using Theorem 11, we obtain that W_F is nonempty and bounded. Here, we denote $\overline{W_F}$ by the $\sigma(E^{**}, E^{*})$-closure of W_F in E^{**} and thus $\overline{W_F}$ is $\sigma(E^{**}, E^{*})$-compact in E^{**}.

It is known that, for any $F_i \in \mathcal{F}$, $i \in \mathbb{N}$, we have $W_{\cap_i F_i} \subset \cap W_{F_i}$. Therefore, $\{\overline{W_F} : F \in \mathcal{F}\}$ has the finite intersection property; that is,

$$\bigcap_{F \in \mathcal{F}} \overline{W_F} \neq \emptyset. \tag{49}$$

Next, we show that, for $u \in \bigcap_{F \in \mathcal{F}} \overline{W_F}$,

$$\langle A(u,u), \eta(v,u) \rangle + f(v) - f(u) \geq 0, \quad \forall v \in K. \tag{50}$$

Indeed, for $v \in K$, let $F \in \mathcal{F}$ be such that $v \in K_F$ and $u \in K_F$. Then, there exists a net $\{u_\beta\}$ in W_F such that u_β converges to u in $\sigma(E^{**}, E^{*})$. From the definition of W_F, we have

$$\langle A(u_\beta, v), \eta(v, u_\beta) \rangle + f(v) - f(u_\beta) \geq \alpha(v - u_\beta). \tag{51}$$

By the complete continuity of $A(\cdot, v)$ and the proper convex lower semicontinuity of f and α, we get

$$\langle A(u,v), \eta(v,u) \rangle + f(v) - f(u) \geq \alpha(v-u), \quad \forall v \in K. \tag{52}$$

Again, using Theorem 11, we conclude that

$$\langle A(u,v), \eta(v,u) \rangle + f(v) - f(u) \geq 0, \quad \forall v \in K. \tag{53}$$

This implies that u is a solution of the problem (43). This completes the proof. ☐

Theorem 22. *Let E be a real Banach space and let $K \subset E^{**}$ be a nonempty unbounded closed convex set. Suppose that $A : K \times K \to E^{**}$ is a weakly relaxed η-α semimonotone mapping and let $f : K \to \mathbb{R} \cup \{\infty\}$ be a proper convex lower semicontinuous function. If the following conditions hold:*

(a) *$\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in K$,*

(b) *for any fixed $y, z, w \in K$, the mapping $x \mapsto \langle A(z, w), \eta(x,y) \rangle$ is convex and lower semicontinuous function,*

(c) *for each* $x \in K$, $A(x,\cdot) : K \to E^*$ *is finite-dimensional continuous,*

(d) α *is convex lower semicontinuous,*

(e) *there exists a point* $u_0 \in K$ *such that* $\liminf_{\|u\|\to\infty} [\langle A(u,u), \eta(u,u_0)\rangle + f(u) - f(u_0)] > 0$,

then the problem (43) *is solvable.*

Proof. For $\epsilon > 0$, we denote by B_ϵ the closed ball with radius ϵ and center at 0 in E^{**}. By Theorem 21, the problem

$$\langle A(u,u), \eta(v,u)\rangle + f(v) - f(u) \geq 0, \quad \forall v \in B_\epsilon \tag{54}$$

has a solution $u_\epsilon \in B_\epsilon$.

Let ϵ be large enough such that $u_0 \in B_\epsilon$. Therefore,

$$\langle A(u_\epsilon,u_\epsilon), \eta(u_0,u_\epsilon)\rangle + f(u_0) - f(u_\epsilon) \geq 0. \tag{55}$$

By condition (e), we get that $\{u_\epsilon\}$ is bounded. So, we may suppose that u_ϵ converges to u in $\sigma(E^{**}, E^*)$ as $\epsilon \to \infty$. It follows from Theorem 11 that

$$\langle A(u_\epsilon,v), \eta(v,u_\epsilon)\rangle + f(v) - f(u_\epsilon) \geq \alpha(v - u_\epsilon), \quad \forall v \in K. \tag{56}$$

Letting $\epsilon \to \infty$, we have

$$\langle A(u,v), \eta(v,u)\rangle + f(v) - f(u) \geq \alpha(v-u), \quad \forall v \in K. \tag{57}$$

Again from Theorem 11, we get

$$\langle A(u,u), \eta(v,u)\rangle + f(v) - f(u) \geq 0, \quad \forall v \in K. \tag{58}$$

This show that $u \in K$ is a solution of the problem (43). This completes the proof. □

Remark 23. Theorems 21 and 22 extend and improve Theorems 3.1 and 3.2 of Fang and Huang [16] and Theorems 2.1 to 2.6 of Chen [7].

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