

Research Article

Symmetry Reductions, Exact Solutions, and Conservation Laws of a Modified Hunter-Saxton Equation

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We study a modified Hunter-Saxton equation from the Lie group-theoretic point of view. The Lie point symmetry generators of the underlying equation are derived. We utilize the Lie algebra admitted by the equation to obtain the optimal system of one-dimensional subalgebras of the Lie algebra of the equation. These subalgebras are then used to reduce the underlying equation to nonlinear third-order ordinary differential equations. Exact traveling wave group-invariant solutions for the equation are constructed by integrating the reduced ordinary differential equations. Moreover, using the variational method, we construct infinite number of nonlocal conservation laws by the transformation of the dependent variable of the underlying equation.

1. Introduction

The nonlinear partial differential equation

$$(u_t + uu_x)_{xx} = \frac{1}{2}(u_x^2)_x \quad (1)$$

was proposed by Hunter and Saxton [1]. They showed that the weakly nonlinear waves are described by (1), where $u(x, t)$ describes the director field of a nematic liquid crystal, x is a space variable in a reference frame moving with the linearized wave velocity, and t is a slow time variable. Liquid crystals are fluids made up of long rigid molecules. Equation (1) is a high-frequency limit of the Camassa-Holm equation [2]. Hunter and Zheng [3] showed that it is bivariational, bi-Hamiltonian, and a member of the Harry Dym hierarchy of integrable flows. It is well known that (1) under reciprocal transformation reduces to the Liouville equation [2]. Also, an interesting geometric interpretation of the Hunter-Saxton equation [4, 5] is that, for spatially periodic functions, it describes the geodesic flow on the homogeneous space $\text{Vir}(\mathbb{S})/\text{Rot}(S)$ of the Virasoro group $\text{Vir}(S)$ modulo the rotations $\text{Rot}(S)$, with respect to the right-invariant homogeneous \dot{H}^1 metric: $\langle f, g \rangle = \int_{\mathbb{S}} f_x g_x dx$.

In this paper the modified Hunter-Saxton equation [6]

$$u_t - 2u_x u_{xx} - uu_{xxx} - u_{xxt} = 0 \quad (2)$$

is studied from the Lie group analysis standpoint. We first obtain the Lie point symmetry generators and then utilize the Lie algebra admitted by the equation to obtain the optimal system of one-dimensional subalgebras of the Lie algebra of the equation. These subalgebras are later used to reduce the modified Hunter-Saxton equation to nonlinear third-order ordinary differential equations. We then construct exact traveling wave group-invariant solutions for the equation by integrating the reduced ordinary differential equations. Finally, using the variational method, we construct infinite number of nonlocal conservation laws by the transformation of the dependent variable of the underlying equation.

The outline of the paper is as follows. In Section 2, we briefly discuss some main operator identities and their relationship. In Section 3, we present the Lie point symmetry generators of (2). In Section 4, the optimal system of one-dimensional subalgebras of the Lie symmetry algebra of (2) is constructed. Moreover, using the optimal system of subalgebras, symmetry reductions and exact group-invariant

solutions of (2) are obtained. In Section 5, nonlocal conservation laws of (2) are constructed using Noether’s theorem. Finally, in Section 6, concluding remarks are presented.

2. Preliminaries

In this section we present the notations that will be used in the sequel. For details, the reader is referred to [7, 8].

Consider a k th-order system of partial differential equations (PDEs) of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$; namely,

$$E^\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \quad (3)$$

where $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, ..., k th-order partial derivatives; that is, $u_i^\alpha = D_i(u^\alpha)$, $u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$, respectively, with the total differentiation operator with respect to x^i being defined by

$$D_i = \partial_{x^i} + u_i^\alpha \partial_{u^\alpha} + u_{ij}^\alpha \partial_{u_j^\alpha} + \dots, \quad i = 1, \dots, n, \quad (4)$$

and the summation convention is used whenever appropriate.

The conserved vector $T = (T^1, T^2, \dots, T^m)$ of (3), where each $T^i \in \mathcal{A}$, \mathcal{A} is the space of all differential functions, satisfies the equation

$$D_i T^i = 0 \quad (5)$$

along the solutions of (3).

The Lie point symmetry generator in the form of infinite formal sum is given by

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \dots i_s}^\alpha \partial_{u_{i_1 i_2 \dots i_s}^\alpha}, \quad (6)$$

where ξ and η are functions of x and u and are independent of derivatives of u , and the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \dots i_s}^\alpha &= D_{i_1} \dots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \dots i_s}^\alpha, \quad s > 1, \end{aligned} \quad (7)$$

in which W^α is the Lie characteristic function defined by $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

Let $L = L(x, u, u_{(1)}, \dots, u_{(l)}) \in \mathcal{A}$, $l \leq k$ be a Lagrangian associated with a Noether symmetry operator X . If B^1, \dots, B^n are point-dependent gauge terms, then the Noether symmetry operator X is determined by

$$XL + LD_i \xi^i = D_i B^i, \quad i = 1, \dots, n, \quad (8)$$

and the conserved vectors, T^i , corresponding to each X

$$\begin{aligned} T^i &= B^i - L \xi^i - W^\alpha \frac{\partial L}{\partial u_i^\alpha} \\ &\quad - \left[D_j(W^\alpha) - W^\alpha D_j \right] \frac{\partial L}{\partial u_{ij}^\alpha}, \quad i = 1, \dots, n \end{aligned} \quad (9)$$

are obtained via Noether’s theorem (see [9]).

3. Lie Point Symmetries of (2)

In this section, we discuss in brief the Lie symmetry group method to obtain Lie point symmetry generators admitted by (2). For detailed account of this method see [10–14] and the references therein.

A vector field

$$X = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u \quad (10)$$

is a generator of point symmetry of (2), if

$$X^{[3]}(u_t - 2u_x u_{xx} - uu_{xxx} - u_{xxt})|_{(2)} = 0, \quad (11)$$

where the operator $X^{[3]}$ is the third prolongation of the operator X defined by

$$X^{[3]} = X + \zeta_t \partial_{u_t} + \zeta_x \partial_{u_x} + \zeta_{xxx} \partial_{u_{xxx}} + \zeta_{xxt} \partial_{u_{xxt}}, \quad (12)$$

with the coefficients ζ_t , ζ_x , ζ_{xxx} , and ζ_{xxt} being given by

$$\begin{aligned} \zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\ \zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\ \zeta_{xx} &= D_x(\zeta_x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi), \\ \zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi), \\ \zeta_{xxt} &= D_t(\zeta_{xx}) - u_{xxt} D_t(\tau) - u_{xxx} D_t(\xi). \end{aligned} \quad (13)$$

Here, D_t and D_x are the total derivative operators defined by

$$\begin{aligned} D_t &= \partial_t + u_t \partial_u + \dots, \\ D_x &= \partial_x + u_x \partial_u + \dots. \end{aligned} \quad (14)$$

The coefficient functions τ , ξ , and η are independent of the derivatives of u ; thus equating the coefficients of like derivatives of u in the determining equation (11) yields the following over determined system of linear PDEs:

$$\begin{aligned} \tau &= \tau(t), \quad \xi = \xi(t, x), \quad \eta_{uu} = 0, \\ \xi_{xx} - 2\eta_{xu} &= 0, \quad 2\xi_x - \eta_{xxu} = 0, \\ \tau_t - 3\xi_x + \eta_u + \eta_{xxu} &= 0, \\ \eta_t - \eta_{txx} - u\eta_{xxx} &= 0, \\ \xi_t - u\tau_t + 3u\xi_x - u\eta_{xxu} - \eta &= 0, \\ 2\xi_{tx} + 3u\xi_{xx} - 2\eta_x - \eta_{tu} - 3u\eta_{xu} &= 0, \\ \xi_{txx} + u\xi_{xxx} - \xi_t - 2\eta_{xx} - 2\eta_{txu} - 3u\eta_{xxu} &= 0. \end{aligned} \quad (15)$$

Solving the determining equations (15) for τ , ξ , and η , we obtain the three-dimensional Lie algebra spanned by the following Lie point symmetry generators admitted by (2):

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_t - u\partial_u. \quad (16)$$

TABLE 1: Commutator table of the Lie algebra of (2).

	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	0
X_3	$-X_1$	0	0

TABLE 2: Adjoint table of the Lie algebra of (2).

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon X_1$
X_2	X_1	X_2	X_3
X_3	$e^\epsilon X_1$	X_2	X_3

TABLE 3: Subalgebras, group invariants, and group-invariant solutions of (2).

N	X	γ	Group-invariant solution
1	X_2	t	$u = h(\gamma)$
2	$X_1 + \epsilon X_2$	$x - \epsilon t$	$u = h(\gamma)$
3	$X_3 + aX_2$	$x - a \ln t$	$u = (1/t)h(\gamma)$

Here $\epsilon = 0, \pm 1$, and a is an arbitrary constant.

4. Symmetry Reductions and Group-Invariant Solutions of (2)

Here we utilize the Lie point symmetry generators (16) of (2) found in Section 3 to obtain symmetry reduction and construct exact group-invariant solutions for (2).

We first present the optimal system of one-dimensional subalgebras of the Lie algebra admitted by (2). The one-dimensional subalgebras are then used to reduce (2) to ordinary differential equations (ODEs). Exact group-invariant solutions for the underlying equation (2) are constructed by integrating the reduced ODEs.

The results on the classification of the Lie point symmetries of (2) are summarized in the Tables 1, 2, and 3. The commutator table of the Lie point symmetries of (2) and the adjoint representations of the symmetry group of (2) on its Lie algebra are given in Tables 1 and 2, respectively. Tables 1 and 2 are then used to construct the optimal system of one-dimensional subalgebras for (2) which are given in Table 3 (for more details of the approach see [10]).

Case 1. In this case, the group-invariant solution corresponding to the symmetry generator X_2 leads to the trivial solution $u(x, t) = C$, where C is a constant.

Case 2. The group-invariant solution arising from the subalgebra $X_1 + \epsilon X_2$ reduces (2) to the following nonlinear third-order ODE:

$$(h - \epsilon)h''' + 2h'h'' + \epsilon h' = 0, \quad \epsilon \neq 0, \tag{17}$$

where “prime” denotes differentiation with respect to γ . Integrating (17) twice with respect to γ , we obtain

$$h'^2 = \frac{\epsilon h^2 + K_1 h + K_2}{\epsilon - h}, \tag{18}$$

where K_1 and K_2 are arbitrary constants of integration. Separating the variables in (18), integrating, and reverting back to our original variables, one obtains the solution of (2) in quadrature

$$\int \sqrt{\frac{\epsilon - u}{\epsilon u^2 + K_1 u + K_2}} du = K_3 \pm (x - \epsilon t), \tag{19}$$

where K_3 is an arbitrary constant of integration. One can obtain two particular solutions for (17) given by

$$h(\gamma) = \frac{1}{B} \left(\delta \pm \frac{B\gamma}{2} \right)^2 - \frac{A}{B}, \tag{20}$$

$$h(\gamma) = -\frac{\epsilon}{4} (\delta \pm \gamma)^2,$$

where $A = C_1 - \epsilon^2$, $B = -\epsilon$, and C_1, δ are arbitrary constants. Hence, we obtain the two special group-invariant solutions for (2) given by

$$u(x, t) = -\frac{1}{\epsilon} \left[\left(\delta \pm \frac{\epsilon(\epsilon t - x)}{2} \right)^2 - (C_1 - \epsilon^2) \right], \tag{21}$$

$$u(x, t) = -\frac{\epsilon}{4} [\delta \pm (x - \epsilon t)]^2.$$

Likewise, one can obtain the following three group-invariant steady state solutions of (2) for the case when $\epsilon = 0$:

$$u(x, t) = \sqrt{C_1} (C_3 \pm x),$$

$$u(x, t) = \left\{ \frac{9C_2}{4} (C_3 \pm x)^2 \right\}^{1/3},$$

$$\sqrt{C_1 u (C_1 u + C_2)} - C_2 \log \left(\sqrt{C_1 (C_1 u + C_2)} + C_1 \sqrt{u} \right) = C_1^{3/2} (C_3 \pm x). \tag{22}$$

Here, C_1, C_2 , and C_3 are arbitrary constants of integration.

Case 3. Substitution of the group-invariant solution corresponding to the subalgebra $X_3 + aX_2$ into (2) results in the reduced nonlinear third-order ODE

$$(h - a)H''' - h'' + 2h'h'' + ah' + h = 0, \tag{23}$$

where “prime” denotes differentiation with respect to γ .

5. Conservation Laws of (2)

In this section, we obtain nonlocal conservation laws of (2) using the variational approach.

Equation (2) does not have a Lagrangian as it is an evolution equation. By making a substitution $u = v_x$, (2) becomes

$$v_{xt} - 2v_{xx}v_{xxx} - v_x v_{xxxx} - v_{xxx}t = 0. \tag{24}$$

Equation (24) has a usual Lagrangian

$$L = -\frac{1}{2} \left(v_t v_x + v_x v_{xx}^2 + v_{xt} v_{xx} \right). \quad (25)$$

If $X = \tau \partial_t + \xi \partial_x + \eta \partial_v$, is the Noether symmetry operator, then from the Noether operator determining equation (8) we obtain

$$\begin{aligned} & -\frac{1}{2} v_x \zeta_t - \frac{1}{2} (v_t + v_{xx}^2) \zeta_x - \left(v_x v_{xx} + \frac{1}{2} v_{xt} \right) \zeta_{xx} \\ & - \frac{1}{2} v_{xx} \zeta_{xt} + (\tau_t + \tau_v v_t) L + (\xi_x + \xi_v v_x) L \\ & = B_t^1 + B_v^1 v_t + B_x^2 + B_v^2 v_x. \end{aligned} \quad (26)$$

Expansion of (26) and then equating the coefficients of the various monomials in the first- and second-order partial derivatives of v to zero yield the following determining equations:

$$\begin{aligned} \tau_x = \tau_v = 0, \quad \xi_t = \xi_v = 0, \\ \eta_v = 0, \quad 4\xi_x - \tau_t - 3\eta_v = 0, \\ \eta_v - \xi_x, \quad \xi_t - \eta_x = 0, \\ B_v^1 = -\frac{1}{2}\eta_x, \quad B_v^2 = -\frac{1}{2}\eta_t, \\ B_t^1 + B_x^2 = 0. \end{aligned} \quad (27)$$

The solution of (27) is $\tau = c_1$, $\xi = c_2$, $\eta = a(t)$, $B^1 = c(t, x)$, and $B^2 = -1/2a'(t)v + d(t, x)$, where c_1, c_2 are arbitrary constants, $a(t)$ is an arbitrary function, and $c_t + d_x = 0$. We set $c = d = 0$. Thus, we obtain the following Noether symmetry operators associated with the Lagrangian (25) for (24):

$$\begin{aligned} X_1 = \partial_t, \quad B^1 = 0, \quad B^2 = 0, \\ X_2 = \partial_x, \quad B^1 = 0, \quad B^2 = 0, \\ X_3 = a(t) \partial_v, \quad B^1 = 0, \quad B^2 = -\frac{1}{2} a'(t) v. \end{aligned} \quad (28)$$

Hence, by invoking (9), we obtain the following conserved vectors corresponding to the Noether symmetry operators (28):

$$(i) X_1 = \partial_t, B^1 = 0, B^2 = 0,$$

$$\begin{aligned} T^1 &= \frac{1}{2} v_x v_{xx}^2 + \frac{1}{2} v_t v_{xxx}, \\ T^2 &= -\frac{1}{2} v_t^2 + \frac{1}{2} v_t v_{xx}^2 + v_t v_{xxt} + v_t v_x v_{xxx} \\ &\quad - \frac{1}{2} v_{tt} v_{xx} - v_x v_{tx} v_{xx} - \frac{1}{2} v_{tx}^2. \end{aligned} \quad (29)$$

Hence, from (5), we have

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= -v_t(v_{xt} - 2v_{xx}v_{xxx} - v_x v_{xxxx} - v_{xxt}) \\ &\quad + \frac{1}{2} v_t v_{xxt} - \frac{1}{2} v_{xx} v_{xtt} \\ &= \frac{1}{2} v_t v_{xxt} - \frac{1}{2} v_{xx} v_{xtt} \\ &= D_t\left(\frac{1}{2} v_t v_{xxt}\right) - D_x\left(\frac{1}{2} v_{xx} v_{tt}\right). \end{aligned} \quad (30)$$

Taking these terms across and including them into the conserved flows, we get

$$D_t\left(T^1 - \frac{1}{2} v_t v_{xxt}\right) + D_x\left(T^2 + \frac{1}{2} v_{xx} v_{tt}\right) = 0. \quad (31)$$

If we let

$$\begin{aligned} \tilde{T}^1 &= T^1 - \frac{1}{2} v_t v_{xxt} = \frac{1}{2} v_x v_{xx}^2, \\ \tilde{T}^2 &= T^2 + \frac{1}{2} v_{xx} v_{tt} \\ &= -\frac{1}{2} v_t^2 + \frac{1}{2} v_t v_{xx}^2 + v_t v_{xxt} + v_t v_x v_{xxx} \\ &\quad - v_x v_{tx} v_{xx} - \frac{1}{2} v_{tx}^2, \end{aligned} \quad (32)$$

one can readily verify that the new components of the conserved vector \tilde{T} satisfy the equation $D_i \tilde{T}^i|_{(24)} = 0$.

(ii) Consider $X_2 = \partial_x$, $B^1 = 0$, $B^2 = 0$,

$$\begin{aligned} T^1 &= -\frac{1}{2} v_x^2 + \frac{1}{2} v_x v_{xxx} - \frac{1}{2} v_{xx}^2, \\ T^2 &= v_x v_{xxt} + v_x^2 v_{xxx} - \frac{1}{2} v_{xt} v_{xx}. \end{aligned} \quad (33)$$

Invoking (5), we obtain

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= -v_x(v_{xt} - 2v_{xx}v_{xxx} - v_x v_{xxxx} - v_{xxt}) \\ &\quad + \frac{1}{2} v_x v_{xxt} - \frac{1}{2} v_{xx} v_{xxt} \\ &= \frac{1}{2} v_x v_{xxt} - \frac{1}{2} v_{xx} v_{xxt} \\ &= D_x\left(\frac{1}{2} v_x v_{xxt}\right) - D_t\left(\frac{1}{2} v_{xx}^2\right). \end{aligned} \quad (34)$$

Taking the terms across and adding them into the conserved flows yield

$$D_t\left(T^1 + \frac{1}{2} v_{xx}^2\right) + D_x\left(T^2 - \frac{1}{2} v_x v_{xxt}\right) = 0. \quad (35)$$

Considering

$$\begin{aligned} \tilde{T}^1 &= T^1 + \frac{1}{2}v_{xx}^2 = \frac{1}{2}v_x v_{xxx} - \frac{1}{2}v_x^2, \\ \tilde{T}^2 &= T^2 - \frac{1}{2}v_x v_{xxt} \\ &= \frac{1}{2}v_x v_{xxt} + v_x^2 v_{xxx} - \frac{1}{2}v_{xt} v_{xx}, \end{aligned} \tag{36}$$

we obtain the modified new components of the conserved vector \tilde{T} so that the equation, $D_i \tilde{T}^i|_{(24)} = 0$, is satisfied.

(iii) Consider $X_3 = a(t)\partial_v$, $B^1 = 0$, $B^2 = -(1/2)a(t)v$,

$$\begin{aligned} T^1 &= \frac{1}{2}a(t)v_x - \frac{1}{2}a(t)v_{xxx}, \\ T^2 &= -\frac{1}{2}a'(t)v + \frac{1}{2}a(t)v_t - \frac{1}{2}a(t)v_{xx}^2 \\ &\quad - a(t)v_{xxt} - a(t)v_x v_{xxx}. \end{aligned} \tag{37}$$

In this case, we have infinite number of conservation laws. Using (5), we have

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= a(t)(v_{xt} - 2v_{xx}v_{xxx} - v_x v_{xxxx} - v_{xxt}) \\ &\quad - \frac{1}{2}(a'(t)v_{xxx} + a(t)v_{xxt}) \\ &= -\frac{1}{2}(a'(t)v_{xxx} + a(t)v_{xxt}) \\ &= D_t\left(-\frac{1}{2}a(t)v_{xxx}\right). \end{aligned} \tag{38}$$

Taking the term across and adding it into the conserved flows, we obtain

$$D_t\left(T^1 + \frac{1}{2}a(t)v_{xxx}\right) + D_x(T^2) = 0. \tag{39}$$

If we choose

$$\begin{aligned} \tilde{T}^1 &= T^1 + \frac{1}{2}a(t)v_{xxx} = \frac{1}{2}a(t)v_x, \\ \tilde{T}^2 &= T^2, \end{aligned} \tag{40}$$

we obtain $D_i \tilde{T}^i|_{(24)} = 0$.

6. Concluding Remarks

In this paper, we studied the modified Hunter-Saxton equation (2) using the Lie symmetry group of infinitesimal transformations of the equation. We found that the underlying equation admits a three-dimensional Lie algebra spanned by the vector fields of translations in time and space and the scaling of time and the dependent variable. We obtained the optimal system of one-dimensional subalgebras of the Lie algebra of the equation. These subalgebras were then used to reduce the underlying equation to nonlinear third-order ordinary

differential equations. Exact group-invariant solutions called traveling wave solutions were constructed by integrating the reduced ODEs. Furthermore, we constructed infinite number of nonlocal conservation laws for the underlying equation by the transformation of the dependent variable of the equation and making use of Noether's theorem.

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