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Research Article

Hyponormal Toeplitz Operators on the Dirichlet Spaces

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We completely characterize the hyponormality of bounded Toeplitz operators with Sobolev symbols on the Dirichlet space and the harmonic Dirichlet space.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and dA be the normalized Lebesgue area measure on $\mathbb{D}$. $L^\infty(\mathbb{D}, dA)$ and $L^2(\mathbb{D}, dA)$ denote the essential bounded measurable function space and the space of square integral functions on $\mathbb{D}$ with respect to dA, respectively. The Bergman space L_a^2 consists of all analytic functions in $L^2(\mathbb{D}, dA)$. The Sobolev space $W^{1,2}(\mathbb{D})$ is the space of functions $f : \mathbb{D} \to \mathbb{C}$ with the following norm:

$$\|f\| = \left[\left|\int_{\mathbb{D}} f dA\right|^2 + \int_{\mathbb{D}} \left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2 dA\right]^{1/2} < \infty. \quad (1)$$

$W^{1,2}(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f dA \int_{\mathbb{D}} \bar{g} dA + \left\langle \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z}\right\rangle_{L^2(\mathbb{D},dA)} + \left\langle \frac{\partial f}{\partial \bar{z}}, \frac{\partial g}{\partial \bar{z}}\right\rangle_{L^2(\mathbb{D},dA)}. \quad (2)$$

The Dirichlet space $\mathscr{D}$ consists of all analytic functions h in $W^{1,2}(\mathbb{D})$ with $h(0) = 0$. The Sobolev space $W^{1,\infty}(\mathbb{D})$ is defined by

$$W^{1,\infty}(\mathbb{D}) = \left\{u \in W^{1,2}(\mathbb{D}) : u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \in L^\infty(\mathbb{D}, dA)\right\}, \quad (3)$$

with the norm

$$\|u\|_{1,\infty} = \max\left\{\|u\|_\infty, \left\|\frac{\partial u}{\partial z}\right\|_\infty, \left\|\frac{\partial u}{\partial \bar{z}}\right\|_\infty\right\}. \quad (4)$$

Let P be the orthogonal projection of $W^{1,2}(\mathbb{D})$ onto $\mathscr{D}$. P is an integral operator represented by

$$P(f)(z) = \int_{\mathbb{D}} \frac{\partial f}{\partial w} \overline{\left(\frac{\partial K_z}{\partial w}\right)} dA, \quad (5)$$

where $K_z(w) = \sum_{k=1}^\infty (\bar{z}^k w^k / k)$ is the reproducing kernel of $\mathscr{D}$. For $u \in W^{1,\infty}(\mathbb{D})$, the Toeplitz operator T_u with symbol u is defined by

$$T_u h = P(uh) \quad h \in \mathscr{D}. \quad (6)$$

T_u is a bounded operator for $u \in W^{1,\infty}(\mathbb{D})$ on $\mathscr{D}$.

Yu gave a decomposition of the Sobolev space $W^{1,2}(\mathbb{D})$ in [1]. Let $\mathscr{P}_0$ be the set of all the following polynomials:

$$\sum_{j\geq -l} \sum_{l\geq 0} a_{l+j,l} z^{l+j} \bar{z}^l, \quad (7)$$

where j and l run over a finite subset of $\mathbb{Z}$ and $\sum_{l\geq 0} a_{l+j,l} = 0$. Let $\mathscr{A}_0$ denote the closure of $\mathscr{P}_0$ in $W^{1,2}(\mathbb{D})$, and let $\mathscr{A}$ denote $\mathscr{A}_0 + \mathbb{C}$. Since the set of all polynomials in z and $\bar{z}$ is dense in $W^{1,2}(\mathbb{D})$, there is the following decomposition:

$$W^{1,2}(\mathbb{D}) = \mathscr{A} \oplus \mathscr{D} \oplus \overline{\mathscr{D}}. \quad (8)$$

Since $W^{1,\infty}(\mathbb{D}) \subseteq W^{1,2}(\mathbb{D})$ and by the above decomposition, it follows that, if $u \in W^{1,\infty}(\mathbb{D})$, then $u = u_0 + f + \overline{g}$, where $u_0 \in \mathcal{A}$, $f, g \in H(\mathbb{D})$ (the space of the analytic functions on $\mathbb{D}$) with $f(0) = g(0) = 0$.

For the space $\mathcal{A}_0$, there is the following proposition.

Proposition 1 (see [1]). *Let $\phi \in W^{1,\infty}(\mathbb{D})$. Then $\phi\mathcal{A}_0 \subset \mathcal{A}_0$.*

A bounded linear operator A on a Hilbert space is called hyponormal if $A^*A - AA^*$ is a positive operator. There is an extensive literature on hyponormal Toeplitz operators on $H^2(\mathbb{T})$ (the Hardy space on $\mathbb{T}$) [2–4]. The corresponding problems for the Toeplitz operators on the Bergman space have been characterized in [5–9]. In the case of the Dirichlet space and the harmonic Dirichlet space, Lu and Yu proved that there are no nonconstant hyponormal Toeplitz operators with certain symbols [10]. In this paper, we completely characterize the Toeplitz operators T_u with $u \in W^{1,\infty}(\mathbb{D})$ on Dirichlet space $\mathcal{D}$ and harmonic Dirichlet space $\mathcal{D}_h$.

2. Case on the Dirichlet Space

In this section, the hyponormality of T_u with $u \in W^{1,\infty}(\mathbb{D})$ on $\mathcal{D}$ will be discussed.

Theorem 2. *Let $u = u_0 + c + f + \overline{g} \in W^{1,\infty}(\mathbb{D})$ with $u_0 + c \in \mathcal{A}$, $f, g \in H(\mathbb{D})$, and $f(0) = g(0) = 0$. Then T_u is hyponormal on $\mathcal{D}$ if and only if $u \in \mathcal{A}$.*

Proof. By Proposition 1, we only need to prove the necessity with $u(z) = f(z) + \overline{g(z)} = \sum_{k=1}^{\infty} f_k z^k + \sum_{k=1}^{\infty} \overline{g}_k \overline{z}^k$.

Let $h(z) = \sum_{k=1}^{\infty} h_k z^k \in \mathcal{D}$. Simple calculations imply that

$$\begin{aligned} T_{f+\overline{g}}h(z) &= T_f h(z) + T_{\overline{g}}h(z) \\ &= f(z)h(z) + P(\overline{g}h)(z) \\ &= f(z)h(z) + \langle \overline{g}h, K_z \rangle \\ &= \sum_{m=2}^{\infty}\left(\sum_{k=1}^{m-1} f_k h_{m-k}\right)z^m + \sum_{l=2}^{\infty} h_l \left(\sum_{k=1}^{l-1} \overline{g}_k z^{l-k}\right) \\ &= \sum_{m=2}^{\infty}\left(\sum_{k=1}^{m-1} f_k h_{m-k}\right)z^m + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \overline{g}_k h_{k+m} z^m. \end{aligned} \tag{9}$$

Furthermore,

$$\begin{aligned} \frac{\partial\left(T_{f+\overline{g}}h\right)}{\partial z} &= \sum_{m=1}^{\infty}(m+1)\left(\sum_{k=1}^{m} f_k h_{m-k+1}\right)z^m \\ &\quad + \sum_{m=0}^{\infty}(m+1)\left(\sum_{k=1}^{\infty} \overline{g}_k h_{k+m+1}\right)z^m \\ &= \sum_{k=1}^{\infty} \overline{g}_k h_{k+1} + \sum_{m=1}^{\infty}(m+1) \\ &\quad \times \left(\sum_{k=1}^{m} f_k h_{m-k+1} + \sum_{k=1}^{\infty} \overline{g}_k h_{k+m+1}\right) z^m. \end{aligned} \tag{10}$$

Therefore

$$\begin{aligned} \left\|T_{f+\overline{g}}h\right\|^2 &= \left|\sum_{k=1}^{\infty} \overline{g}_k h_{k+1}\right|^2 \\ &\quad + \sum_{m=1}^{\infty}\left|\sum_{k=1}^{m} f_k h_{m-k+1} + \sum_{k=1}^{\infty} \overline{g}_k h_{k+m+1}\right|^2. \end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned} T^*_{f+\overline{g}}h(z) &= \left\langle T^*_{f+\overline{g}}h, K_z\right\rangle = \left\langle h, T_{f+\overline{g}}K_z\right\rangle \\ &= \sum_{m=2}^{\infty}\frac{1}{m}\left(\sum_{l=1}^{m-1} l g_{m-l} h_l\right) z^m \\ &\quad + \sum_{m=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{m+k}{m}\overline{f}_k h_{k+m}\right) z^m, \\ \frac{\partial\left(T^*_{f+\overline{g}}h\right)}{\partial z} &= \sum_{k=1}^{\infty}(k+1)\overline{f}_k h_{k+1} \\ &\quad + \sum_{m=1}^{\infty}\left[\sum_{k=1}^{\infty}(m+k+1)\overline{f}_k h_{m+k+1} \right. \\ &\quad \left. + \sum_{l=1}^{m} l g_{m-l+1} h_l\right] z^m, \\ \left\|T^*_{f+\overline{g}}h\right\|^2 &= \left|\sum_{k=1}^{\infty}(k+1)\overline{f}_k h_{k+1}\right|^2 \\ &\quad + \sum_{m=1}^{\infty}\frac{1}{m+1}\left|\sum_{k=1}^{\infty}(m+k+1)\overline{f}_k h_{m+k+1} \right. \\ &\quad \left. + \sum_{l=1}^{m} l g_{m-l+1} h_l\right|^2. \end{aligned} \tag{12}$$

Denote $e_i(z) = (1/i)z^i$ for $i \ge 1$. Since T_u is hyponormal, we have

$$\left\|T_u e_i\right\|^2 - \left\|T_u^* e_i\right\|^2 \ge 0 \quad \text{for } i \ge 1. \tag{13}$$

For $i \ge 2$, $\|T_u e_i\|^2 - \|T_u^* e_i\|^2 \ge 0$ implies that

$$\begin{aligned} \frac{1}{i^2}\left(\sum_{k=1}^{i-1}|g_k|^2 + \sum_{l=1}^{\infty}|f_l|^2\right) &\ge \sum_{k=1}^{i-1}\frac{1}{i-k}|f_k|^2 + \sum_{l=1}^{\infty}\frac{1}{i+l}|g_l|^2 \\ &\ge \frac{1}{i}|f_1|^2. \end{aligned} \tag{14}$$

Hence

$$\left(\sum_{i=2}^{N}\frac{1}{i^2}\right)\left(\sum_{l=1}^{\infty}|g_l|^2+\sum_{l=1}^{\infty}|f_l|^2\right)\geq\left(\sum_{i=2}^{N}\frac{1}{i}\right)|f_1|^2 \quad \text{for } N\geq 2. \tag{15}$$

Letting $N\to\infty$, since $\sum_{i=2}^{N}(1/i^2)$ and $(\sum_{l=1}^{\infty}|g_l|^2+\sum_{l=1}^{\infty}|f_l|^2)$ are convergent and $\sum_{i=2}^{N}(1/i)$ is disconvergent, we get $f_1=0$. Similarly, by choosing i, we get $f_l=0$ for $l\geq 1$. Note that $\|T_u e_1\|^2-\|T_u^* e_1\|^2\geq 0$ implies that $\sum_{l=1}^{\infty}|f_l|^2\geq\sum_{l=1}^{\infty}(1/(l+1))|g_l|^2$. Thus $g_l=0$ for $l\geq 1$ and the proof is finished. □

The following corollary generalizes Theorems 1 and 2 in [10]. Denote

$$\begin{aligned}\Omega=\{u: u=f+\overline{g}, f,g\in H^{\infty}(\mathbb{D}),\\ |f|^2 dA \text{ is a } \mathscr{D}\text{-Carleson measure}\},\end{aligned} \tag{16}$$

where $H^{\infty}(\mathbb{D})$ is the space of the bounded analytic functions on $\mathbb{D}$.

Corollary 3. *Let $u\in\Omega$. Then T_u is hyponormal on $\mathscr{D}$ if and only if u is a constant function.*

3. Case on the Harmonic Dirichlet Space

In this section, we will characterize the hyponormality of T_u with $u\in W^{1,\infty}(\mathbb{D})$ on $\mathscr{D}_h$.

The harmonic Dirichlet space $\mathscr{D}_h$ consists of all harmonic functions in $W^{1,2}(\mathbb{D})$. It is a closed subspace of $W^{1,2}(\mathbb{D})$, and hence it is a Hilbert space with the following reproducing kernel:

$$R_z(w)=\overline{K_z(w)}+K_z(w)+1=\ln\frac{1}{1-z\overline{w}}+\ln\frac{1}{1-\overline{z}w}+1. \tag{17}$$

Let Q be the orthogonal projection of $W^{1,2}(\mathbb{D})$ onto $\mathscr{D}_h$. Q is an integral operator represented by

$$\begin{aligned}Q(f)(z)&=\langle f,R_z\rangle\\&=\int_{\mathbb{D}}\frac{\partial f}{\partial w}\overline{\left(\frac{\partial K_z}{\partial w}\right)}dA(w)\\&\quad+\int_{\mathbb{D}}\frac{\partial f}{\partial\overline{w}}\overline{\left(\frac{\partial\overline{K_z}}{\partial\overline{w}}\right)}dA(w)+\int_{\mathbb{D}}f\,dA.\end{aligned} \tag{18}$$

For $u\in W^{1,\infty}(\mathbb{D})$, the Toeplitz operator $\widetilde{T_u}$ with symbol u is defined by

$$\widetilde{T_u}h=Q(uh)\qquad h\in\mathscr{D}_h. \tag{19}$$

$\widetilde{T_u}$ is a bounded operator for $u\in W^{1,\infty}(\mathbb{D})$ on $\mathscr{D}_h$ (see [11]).

Theorem 4. *Let $u=u_0+c+f+\overline{g}\in W^{1,\infty}(\mathbb{D})$ with $u_0+c\in\mathscr{A}$, $f,g\in H(\mathbb{D})$, and $f(0)=g(0)=0$. Then $\widetilde{T_u}$ is hyponormal on $\mathscr{D}_h$ if and only if $u\in\mathscr{A}$.*

Proof. By Proposition 1, we only need to prove the necessity with $u(z)=f(z)+\overline{g(z)}=\sum_{k=1}^{\infty}f_k z^k+\sum_{k=1}^{\infty}\overline{g}_k\overline{z}^k$.

Let $h(z)=a_0+\sum_{k=1}^{\infty}a_k z^k+\sum_{k=1}^{\infty}b_k\overline{z}^k=\sum_{k=0}^{\infty}a_k z^k+\sum_{k=1}^{\infty}b_k\overline{z}^k\in\mathscr{D}_h$. Since $\widetilde{T_u}$ is hyponormal on $\mathscr{D}_h$, we have $\|\widetilde{T}_{f+\overline{g}}h\|^2-\|\widetilde{T}^*_{f+\overline{g}}h\|^2\geq 0$. Note that

$$\begin{aligned}\widetilde{T}_{f+\overline{g}}h(z)&=Q[(f+\overline{g})h](z)=\langle(f+\overline{g})h,R_z\rangle\\&=\int_{\mathbb{D}}\frac{\partial[(f+\overline{g})h]}{\partial w}\frac{z}{1-z\overline{w}}dA\\&\quad+\int_{\mathbb{D}}\frac{\partial[(f+\overline{g})h]}{\partial\overline{w}}\frac{\overline{z}}{1-\overline{z}w}dA+\int_{\mathbb{D}}(f+\overline{g})h\,dA\\&=\sum_{k=1}^{\infty}\frac{1}{k+1}(b_k f_k+a_k\overline{g}_k)\\&\quad+\left[a_0 f_1+\sum_{k=1}^{\infty}(b_k f_{k+1}+a_{k+1}\overline{g}_k)\right]z\\&\quad+\sum_{m=2}^{\infty}\left[\sum_{k=1}^{m-1}a_{m-k}f_k+a_0 f_m+\sum_{k=1}^{\infty}(b_k f_{k+m}+a_{k+m}\overline{g}_k)\right]z^m\\&\quad+\left[a_0\overline{g}_1+\sum_{k=1}^{\infty}(a_k\overline{g}_{k+1}+b_{k+1}f_k)\right]\overline{z}\\&\quad+\sum_{m=2}^{\infty}\left[\sum_{k=1}^{m-1}b_{m-k}\overline{g}_k+a_0\overline{g}_m+\sum_{k=1}^{\infty}(a_k\overline{g}_{k+m}+b_{k+m}f_k)\right]\overline{z}^m.\end{aligned} \tag{20}$$

Thus

$$\begin{aligned}\|\widetilde{T}_{f+\overline{g}}h\|^2&=\left|\int_{\mathbb{D}}\widetilde{T}_{f+\overline{g}}h\,dA\right|^2\\&\quad+\int_{\mathbb{D}}\left|\frac{\partial\widetilde{T}_{f+\overline{g}}h}{\partial z}\right|^2dA+\int_{\mathbb{D}}\left|\frac{\partial\widetilde{T}_{f+\overline{g}}h}{\partial\overline{z}}\right|^2dA\\&=\left|\sum_{k=1}^{\infty}\frac{1}{k+1}(b_k f_k+a_k\overline{g}_k)\right|^2\\&\quad+\left|a_0 f_1+\sum_{k=1}^{\infty}(b_k f_{k+1}+a_{k+1}\overline{g}_k)\right|^2\\&\quad+\sum_{m=2}^{\infty}\left|\sum_{k=1}^{m-1}a_{m-k}f_k+a_0 f_m\right.\\&\qquad\left.+\sum_{k=1}^{\infty}(b_k f_{k+m}+a_{k+m}\overline{g}_k)\right|^2\\&\quad+\left|a_0\overline{g}_1+\sum_{k=1}^{\infty}(a_k\overline{g}_{k+1}+b_{k+1}f_k)\right|^2\end{aligned}$$

$$+\sum_{m=2}^{\infty}\left|\sum_{k=1}^{m-1}b_{m-k}\overline{g}_k+a_0\overline{g}_m+\sum_{k=1}^{\infty}\left(a_k\overline{g}_{k+m}+b_{k+m}f_k\right)\right|^2. \tag{21}$$

Similarly, we have

$$\begin{aligned}\widetilde{T}^*_{f+\overline{g}}h(z)&=\left\langle\widetilde{T}^*_{f+\overline{g}}h,R_z\right\rangle=\left\langle h,\widetilde{T}_{f+\overline{g}}R_z\right\rangle=\left\langle h,(f+\overline{g})R_z\right\rangle\\&=\sum_{k=1}^{\infty}k\left(a_k\overline{f}_k+b_kg_k\right)\\&\quad+\left\{\frac{a_0}{2}g_1+\sum_{k=1}^{\infty}\left[kb_kg_{k+1}+(k+1)a_{k+1}\overline{f_k}\right]\right\}z\\&\quad+\sum_{m=2}^{\infty}\frac{1}{m}\left\{\sum_{k=1}^{m-1}ka_kg_{m-k}+\frac{a_0}{m+1}g_m+\sum_{k=1}^{\infty}\left[(m+k)a_{k+m}\overline{f}_k+kb_kg_{k+m}\right]\right\}z^m\\&\quad+\left\{\frac{a_0}{2}\overline{f}_1+\sum_{k=1}^{\infty}\left[ka_k\overline{f}_{k+1}+(k+1)b_{k+1}g_k\right]\right\}\overline{z}\\&\quad+\sum_{m=2}^{\infty}\frac{1}{m}\left\{\sum_{k=1}^{m-1}kb_k\overline{f}_{m-k}+\frac{a_0}{m+1}\overline{f}_m+\sum_{k=1}^{\infty}\left[(m+k)b_{k+m}g_k+ka_k\overline{f}_{k+m}\right]\right\}\overline{z}^m\end{aligned}$$

$$\begin{aligned}\left\|\widetilde{T}^*_{f+\overline{g}}h\right\|^2&=\left|\int_{\mathbb{D}}\widetilde{T}^*_{f+\overline{g}}h\,dA\right|^2+\int_{\mathbb{D}}\left|\frac{\partial\left(\widetilde{T}^*_{f+\overline{g}}h\right)}{\partial z}dA\right|^2\\&\quad+\int_{\mathbb{D}}\left|\frac{\partial\left(\widetilde{T}^*_{f+\overline{g}}h\right)}{\partial\overline{z}}\right|^2dA=\left|\sum_{k=1}^{\infty}k\left(a_k\overline{f}_k+b_kg_k\right)\right|^2\\&\quad+\left|\frac{a_0}{2}g_1+\sum_{k=1}^{\infty}\left[kb_kg_{k+1}+(k+1)a_{k+1}\overline{f_k}\right]\right|^2\\&\quad+\sum_{m=2}^{\infty}\frac{1}{m}\left|\sum_{k=1}^{m-1}ka_kg_{m-k}+\frac{a_0}{m+1}g_m+\sum_{k=1}^{\infty}\left[(m+k)a_{k+m}\overline{f}_k+kb_kg_{k+m}\right]\right|^2\\&\quad+\left|\frac{a_0}{2}\overline{f}_1+\sum_{k=1}^{\infty}\left[ka_k\overline{f}_{k+1}+(k+1)b_{k+1}g_k\right]\right|^2\\&\quad+\sum_{m=2}^{\infty}\frac{1}{m}\left|\sum_{k=1}^{m-1}kb_k\overline{f}_{m-k}+\frac{a_0}{m+1}\overline{f}_m+\sum_{k=1}^{\infty}\left[(m+k)b_{k+m}g_k+ka_k\overline{f}_{k+m}\right]\right|^2.\end{aligned} \tag{22}$$

For $i\geq 2$, let $a_i=1/i$ and $a_j=0$ for $j\neq i$. It follows that

$$\begin{aligned}&\frac{1}{i^2}\left(\sum_{k=1}^{\infty}\left|f_k\right|^2+\sum_{k=1}^{i-1}\left|g_k\right|^2+\left|\frac{1}{i+1}g_i\right|^2+\sum_{k=i+1}^{\infty}\left|g_k\right|^2\right)\\&\quad\geq\sum_{k=1}^{\infty}\frac{1}{i+k}\left|g_k\right|^2+\left|f_i\right|^2+\sum_{k=1}^{i-1}\frac{1}{k}\left|f_{i-k}\right|^2+\sum_{k=1}^{\infty}\frac{1}{k}\left|f_{i+k}\right|^2.\end{aligned} \tag{23}$$

Therefore,

$$\frac{1}{i^2}\left(\sum_{k=1}^{\infty}\left|f_k\right|^2+\sum_{k=1}^{\infty}\left|g_k\right|^2\right)\geq\sum_{k=1}^{\infty}\frac{1}{i+k}\left(\left|f_k\right|^2+\left|g_k\right|^2\right). \tag{24}$$

For every $k\geq 1$, we have

$$\left(\sum_{i=2}^{N}\frac{1}{i^2}\right)\left(\sum_{l=1}^{\infty}\left|f_l\right|^2+\sum_{l=1}^{\infty}\left|g_l\right|^2\right)\geq\left(\sum_{i=2}^{N}\frac{1}{i+k}\right)\left(\left|f_k\right|^2+\left|g_k\right|^2\right), \tag{25}$$

where $N\geq 2$. Letting $N\to\infty$, Since $\sum_{i=2}^{N}(1/i^2)$ and $(\sum_{l=1}^{\infty}|f_l|^2+\sum_{l=1}^{\infty}|g_l|^2)$ are convergent and $\sum_{i=2}^{N}(1/(i+k))$ ($k\geq 1$ is fixed) is disconvergent, we get $|f_k|=|g_k|=0$ for $k\geq 1$. The proof is finished. □

The following corollary generalizes Theorem 3 in [10].

Corollary 5. *Suppose that* $u=f+\overline{g}\in W^{1,\infty}(\mathbb{D})$ *with* $f,g\in H(\mathbb{D})$. *Then* T_u *is hyponormal on* $\mathscr{D}_h$ *if and only if* u *is a constant function.*

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