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Research Article

The Order Continuity of the Regular Norm on Regular Operator Spaces

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We present here some sufficient conditions for the regular norm on $\mathscr{L}^r(E,F)$ to be order continuous, and for $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ to be a KB-space. In particular we deduce a characterization of the order continuity of the regular norm using L- and M-weak compactness of regular operators. Also we characterize when the space $\mathscr{L}^r(E,F)$ is an L^p-space and is lattice isomorphic to an L^p-space for $1 < P < \infty$. Some related results are also obtained.

1. Introduction

For Banach lattices E and F, we use $\mathscr{L}(E,F)$ to denote the space of all continuous linear operators from E into F, and $\mathscr{L}^r(E,F)$ to denote the space of all regular operators from E into F, which is the linear span of the set $\mathscr{L}_+(E,F)$ of all positive operators from E into F. With respect to the operator norm $\|\cdot\|$ the space $\mathscr{L}^r(E,F)$ is not complete in general (see, e.g., [1]), but there exists a natural norm on $\mathscr{L}^r(E,F)$, the *regular norm* $\|\cdot\|_r$, which turns $\mathscr{L}^r(E,F)$ into a Banach space (see [2] for details). Namely,

$$\|T\|_r = \inf\{\|S\| : S \in \mathscr{L}_+(E,F),\ \pm T \le S\}. \tag{1}$$

In particular, $\|T\| \le \|T\|_r$. If $\mathscr{L}^r(E,F)$ is a vector lattice; then $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is a Banach lattice and $\|T\|_r = \||T|\|$ for all $T \in \mathscr{L}^r(E,F)$. For instance, if F is Dedekind complete, then $\mathscr{L}^r(E,F)$ is a Dedekind complete Banach lattice under the regular norm.

The natural and important questions are: if $\mathscr{L}^r(E,F)$ is a vector lattice (i.e., a Banach lattice), when is the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ order continuous? When is $\mathscr{L}^r(E,F)$ a KB-space with respect to the regular norm?

Wickstead showed in [3] some characterizations of the space $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ being (lattice isomorphic to) an AL- or AM-space. It is natural to ask that when $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is an L^p-space or lattice isomorphic to an L^p-space for $1 < p < \infty$.

The purpose of this work is to present some results involving the order continuity of the regular norm on $\mathscr{L}^r(E,F)$ and $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ being a KB-space. Furthermore we will also present a complete description for $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ being (lattice isomorphic to) an L^p-space with $1 < p < \infty$. Some related results are included as well.

Recall that an operator $T : E \to F$ is called L-*weakly compact* if Tball(E) is an L-weakly compact set in F; that is, $\|y_n\| \to 0$ for each disjoint sequence $(y_n)_1^\infty$ contained in the solid hull of Tball(E). Also T is called M-*weakly compact* if $\|Tx_n\| \to 0$ for each disjoint sequence $(x_n)_1^\infty \subset$ ball(E), where ball(E) denotes the unit ball of E. See, for example, [2].

We refer to [2, 4] for any unexplained terms from the theory of Banach lattices and operators.

2. Some General Results

We start with a necessary condition for the order continuity of the regular norm on spaces of regular operators.

Proposition 1. *Let E and F be Banach lattices. If the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous, then the norms both on E' and F are order continuous.*

Proof. If the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous, for each increasing sequence $(y_n)_1^\infty \subset [0,y] \subset F$, taking $x' \in E'_+$ with $\|x'\| = 1$ and defining $S, S_n : E \to F$ by

$$S_n x = x'(x)\,y_n, \quad Sx = x'(x)\,y \quad \text{for } x \in E \tag{2}$$

then $S_n, S \in \mathscr{L}_+(E,F)$ and $0 \le S_n \uparrow \le S$.

The order continuity of the regular norm implies that there is $U \in \mathscr{L}^r(E,F)$ such that $\|S_n - U\|_r \to 0$; thus $\|S_n - U\| \to 0$. Choosing $x_0 \in E$ with $x'(x_0) = 1$ we have

$$\|y_n - Ux_0\| = \|S_n x_0 - Ux_0\| \leq \|S_n - U\| \|x_0\| \longrightarrow 0. \quad (3)$$

It follows from Theorem 2.4.2 of [2] that the norm on F is order continuous.

Similarly, for each increasing sequence $(x_n')_1^\infty \subset [0, x'] \subset E'$, taking $y \in F_+$ with $\|y\| = 1$ and defining $T, T_n : E \to F$ by

$$T_n x = x_n'(x)\, y, \quad Tx = x'(x)\, y \quad \text{for } x \in E \quad (4)$$

then $T_n, T \in \mathscr{L}_+(E,F)$ and $0 \leq T_n \uparrow \leq T$.

Again there is $V \in \mathscr{L}^r(E,F)$ such that $\|T_n - V\|_r \to 0$; thus $\|T_n - V\| \to 0$. Choosing $y' \in F'$ with $y'(y) = 1$, it is easy to verify that

$$\begin{aligned} \|x_n' - V'y'\| &= \|T_n'y' - V'y'\| \leq \|T_n' - V'\| \|y'\| \\ &= \|T_n - V\| \|y'\| \longrightarrow 0. \end{aligned} \quad (5)$$

Theorem 2.4.2 of [2] yields that the norm on E' is order continuous. □

Next result is a characterization of the order continuity of the regular norm on spaces of regular operators.

Theorem 2. *For Banach lattices E and F, the following statements are equivalent.*

(1) *$\mathscr{L}^r(E,F)$ is a vector lattice and the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous.*

(2) *Every positive operator $T : E \to F$ is L- and M-weakly compact.*

Proof. (1) $\Rightarrow$ (2). If the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous, then by the proposition above, the norms both on E' and F are order continuous.

For $0 \leq T : E \to F$, it suffices to show that T is M-weakly compact (see Theorem 3.6.17 of [2]). Otherwise, there is a disjoint sequence $(x_n)_1^\infty \subset \text{ball}(E)$ such that $\|Tx_n\| \geq \delta > 0$ for all $n \in \mathbb{N}$. Note that $0 \leq T|x_n| \to 0$ weakly as $|x_n| \to 0$ weakly (see Theorem 2.4.14 of [2]).

By Corollary 2.3.5 of [2] there exists a sequence of naturals (k_n) and a disjoint sequence $(y_n) \subset F_+$ such that $0 \leq y_n \leq T|x_{k_n}|$ and $\|y_n\| \geq c$, where c is any fixed number from $(0, \delta)$. Let $P_n F :\to \{y_n\}^{dd}$ be the band projection; hereby $\{y_n\}^{dd}$ denotes the band generated by y_n in F. It is easy to verify that $P_i \perp P_j$ and $P_i \leq I_F - P_j$ ($\forall\, i \neq j$); it follows that $P_1 + \cdots + P_n \uparrow \leq I_F$, and $(P_1 + \cdots + P_n)T \uparrow \leq T$, where I_F is the identity operator on F. Now the order continuity of the regular norm implies that $((P_1 + \cdots + P_n)T)_1^\infty$ is a $\|\cdot\|_r$-Cauchy sequence; in particular, $\|P_n T\| \to 0$ as $n \to \infty$. Therefore

$$0 < c \leq \|y_n\| = \|P_n y_n\| \leq \|P_n T |x_n|\| \leq \|P_n T\| \longrightarrow 0 \quad (6)$$

This is impossible, so (1) $\Rightarrow$ (2) holds.

(2) $\Rightarrow$ (1). For any $0 < y \in F_+$ and $0 < x' \in E_+'$, let $T : E \to F$ by $Tx = x'(x)y$. Clearly $T \geq 0$ and the L- and M-weak compactness of T yield the relatively weak compactness of both $[-y, y]$ and $[-x', x']$. It follows from Theorem 2.4.2 of [2] that the norms both on E' and F are order continuous; $\mathscr{L}^r(E,F)$ is certainly a (Dedekind complete) vector lattice.

For any decreasing sequence $T_n \in \mathscr{L}_+(E,F)$ with $\inf\{T_n : n \in \mathbb{N}\} = 0$, Proposition 3.6.19 of [2] yields that the operator norm, and hence the regular norm, on order interval $[0, T_1]$ is order continuous, which implies that $\|T_n\|_r = \|T_n\| \to 0$. Now the order continuity of the regular norm is following from Theorem 2.4.2 of [2]. □

It is clear that the identity operator on a Banach lattice E is M-weakly compact if and only if E is finite dimensional. The next result should be no surprise.

Corollary 3. *Let E be a Banach lattice. Then $\mathscr{L}^r(E)$ is a vector lattice and the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E)$ is order continuous if and only if* $\dim E < \infty$.

Theorem 4. *For Banach lattices E and F, the following statements are equivalent.*

(1) *$(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is a KB-space.*

(2) *F is a KB-space and $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous.*

(3) *F is a KB-space and every positive operator $T : E \to F$ is M-weakly compact.*

Proof. (1) $\Rightarrow$ (2). If $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is a KB-space, then $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ certainly is order continuous. For a norm bounded increasing sequence $(y_n)_1^\infty \subset F_+$, taking $x' \in E_+'$ with $\|x'\| = 1$ and defining $S_n : E \to F$ by $S_n x = x'(x) y_n$ for $x \in E$, then $(S_n)_1^\infty \subset \mathscr{L}_+(E,F)$ also is increasing and $\|\cdot\|_r$-bounded, so there is $U \in \mathscr{L}^r(E,F)$ such that $\|S_n - U\|_r \to 0$; thus $\|S_n - U\| \to 0$. Choosing $x_0 \in E$ with $x'(x_0) = 1$ we have

$$\|y_n - Ux_0\| = \|S_n x_0 - Ux_0\| \leq \|S_n - U\| \|x_0\| \longrightarrow 0. \quad (7)$$

It follows that F is a KB-space.

(2) $\Rightarrow$ (3) is a consequence of Theorem 2. Now we show that (3) $\Rightarrow$ (1). Clearly $\mathscr{L}^r(E,F)$ is a Banach lattice under the regular norm as F is a KB-space. If $(T_n)_1^\infty \subset \mathscr{L}_+(E,F)$ is a $\|\cdot\|_r$-bounded increasing sequence, then for each $x \in E_+$, $T_n x$ is norm convergent as it is a norm bounded increasing sequence in F. It is easy to see that there is a $T \in \mathscr{L}_+(E,F)$ such that $T_n \to T$ with respect to the strong operator topology; it follows that $T_n \uparrow T$ and by hypothesis T is M-weakly compact. Proposition 3.6.19 of [2] yields that $\|T - T_n\|_r = \|T - T_n\| \to 0$ which implies that $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is a KB-space. □

It is obvious that if $T : E \to F$ is regular then T' is also regular, and the converse is false in general. For example, let $T : L^2[0,1] \to c_0$ defined by $Tf = (\int_0^1 f(t) r_n(t) dt)$, where $r_n(t) = \text{sgn}(\sin 2^n \pi t)$ is the nth Rademacher function on $[0,1]$. Then $T' : \ell_1 \to L^2[0,1]$, $T'(\lambda_n) = \sum_{n=1}^\infty \lambda_n r_n$, is regular (as it is order bounded) but T is not regular. The following results will show some relationships between the order continuity of the regular norms in $\mathscr{L}^r(E,F)$, $\mathscr{L}^r(E,F'')$ and $\mathscr{L}^r(F',E')$.

Theorem 5. *For Banach lattices E and F, the following assertions are equivalent.*

(1) *The regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F'')$ is order continuous.*

(2) *$(\mathscr{L}^r(E,F''),\|\cdot\|_r)$ is a KB-space.*

(3) *The regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(F',E')$ is order continuous.*

(4) *$(\mathscr{L}^r(F',E'),\|\cdot\|_r)$ is a KB-space.*

Proof. Let $\Phi : \mathscr{L}(E,F'') \to \mathscr{L}(F',E')$ by $\Phi(T) = T'j$ for $T \in \mathscr{L}(E,F'')$, where $j : F' \to F'''$ is the natural embedding. According to Theorem 5.6 of [5] the operator $T \in \mathscr{L}(E,F'')$ is regular if and only if $\Phi(T)$ is regular and $\|\Phi(T)\|_r = \|T\|_r$.

Moreover Φ is an order continuous isometric lattice isomorphism from $(\mathscr{L}(E,F''),\|\cdot\|_r)$ onto $(\mathscr{L}(F',E'),\|\cdot\|_r)$. Thus (1) $\Leftrightarrow$ (3) is a simple consequence of these facts. Also the equivalences of (1) and (2), (3) and (4) easily follow from Theorem 4 and the proof of Theorem 2 (remembering that the norm on E' is order continuous if and only if E' is a KB-space; compare Theorem 2.4.14 of [2]). □

Corollary 6. *Let E and F be Banach lattices such that F is reflexive. Then the following statements are equivalent.*

(1) *The regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous.*

(2) *$(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is a KB-space.*

(3) *The regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(F',E')$ is order continuous.*

(4) *$(\mathscr{L}^r(F',E'),\|\cdot\|_r)$ is a KB-space.*

Theorem 7. *Let E and F be Banach lattices, $H \subset E$ and $G \subset F$ closed sublattices. Supposing that there is a positive projection P from E onto H then the following statements hold.*

(1) *If $\mathscr{L}^r(E,F)$ is a vector lattice and the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous, then $\mathscr{L}^r(H,G)$ also is a vector lattice and $\|\cdot\|_r$ on $\mathscr{L}^r(H,G)$ is order continuous.*

(2) *If $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is a KB-space then $(\mathscr{L}^r(H,G),\|\cdot\|_r)$ also is a KB-space.*

Proof. Suppose that $\mathscr{L}^r(E,F)$ is a vector lattice and the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous. For $0 \le T : H \to G$, then $0 \le TP : E \to G \subset F$, Theorem 2 yields that TP is L- and M-weakly compact. For any disjoint sequence $(y_n)_1^\infty$ contained in the solid hull of $T\text{ball}(H)$ in G, then $(y_n)_1^\infty$ is a disjoint sequence in F as G is a sublattice of F, which is contained in the solid hull of $(TP)\text{ball}(E)$ as $T\text{ball}(H) \subset (TP)\text{ball}(E)$, so that $\|y_n\| \to 0$; that is, T is L-weakly compact. Also for each disjoint sequence $(x_n)_1^\infty \subset \text{ball}(H)$, $(x_n)_1^\infty \subset \text{ball}(E)$ is disjoint as H is a sublattice of E; it follows that $\|Tx_n\| = \|TPx_n\| \to 0$, which implies that T is M-weakly compact. Again by Theorem 2 $\mathscr{L}^r(H,G)$ is a vector lattice and the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(H,G)$ is order continuous; that is, (1) holds.

If $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is a KB-space then it follows from Theorem 4 and (1) that F is a KB-space, and hence G, as a closed sublattice of a KB-space, also is a KB-space, and that $(\mathscr{L}^r(H,G),\|\cdot\|_r)$ is a Banach lattice with an order continuous norm. Again Theorem 4 yields that $(\mathscr{L}^r(H,G),\|\cdot\|_r)$ is a KB-space, so (2) holds. □

Note that each Banach lattice F can be identified with a closed sublattice of F'', and so, as a consequence of Theorems 4 and 7 we have the following result.

Corollary 8. *Let E and F be Banach lattices. If $(\mathscr{L}^r(F',E'),\|\cdot\|_r)$ is a KB-space (equivalently the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(F',E')$ is order continuous), then so is $(\mathscr{L}^r(E,F),\|\cdot\|_r)$.*

Remark 9. The regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(F',E')$ may fail to be order continuous even if $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is a KB-space.

For example, let $E = c_0$ and $F = \ell_1(\ell_\infty^n) = (c_0(\ell_1^n))'$. Clearly F is a KB-space and $F' = \ell_\infty(\ell_1^n)$. Define $T : \ell_1 \to F'$ by

$$T(\lambda_n) = ((\lambda_1),(\lambda_1,\lambda_2),(\lambda_1,\lambda_2,\lambda_3),\ldots), \quad \forall (\lambda_n) \in \ell_1, \tag{8}$$

it is easy to see that T is an isometric lattice homomorphism; that is, F' contains a closed sublattice isometrically lattice isomorphic to ℓ_1. Thus Theorem 2.4.14 of [2] implies that F'' fails to be a KB-space (i.e., the norm on F'' is not order continuous).

Now it follows from Theorem 12 (see next) that $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is a KB-space, but the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F'')$, and hence on $\mathscr{L}^r(F',E')$, is not order continuous as the norm on F'' is not order continuous (see the proof of Theorem 2).

3. Some Concrete Sufficient Conditions

In this section we will present some sufficient conditions on Banach lattices E and F such that the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous, or $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is a KB-space.

Proposition 10. *Let E be an AM-space with a strong order unit and F a Banach lattice with an order continuous norm. Then the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous.*

Proof. We may assume that E is equipped with the strong order unit norm and also the norm on F is order continuous; clearly $(\mathscr{L}^r(E,F),\|\cdot\|_r)$ is a Banach lattice. For $0 \le T_n \uparrow \le T$ in $\mathscr{L}^r(E,F)$ then $0 \le T_n x \uparrow \le Tx$ for each $x \in E_+$. It follows from the order continuity of the norm on F that $(T_n x)_1^\infty$ is norm convergent. So there is $S \in \mathscr{L}_+(E,F)$ such that $T_n \to S$

with respect to the strong operator topology and obviously $T_n \uparrow S$. In particular

$$\|S - T_n\|_r = \|S - T_n\| = \|Se - T_n e\| \longrightarrow 0, \quad (9)$$

where e is a strong order unit of E. Therefore Theorem 2.4.2 of [2] yields that the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is order continuous. □

Remark 11. If E fails to possess a strong order unit the above result is false even if E is an AM-space; E, E' and F are atomic with an order continuous norm. For example, let $E = F = c_0$ and then the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E)$ is not order continuous, compare also Corollary 3.

Recall that Banach lattice E possesses the *positive Schur property* if every weakly null sequence in E_+ is norm convergent to 0.

Theorem 12. *Let E be a Banach lattice such that E' possesses the positive Schur property, F a Banach lattice. Then $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is a KB-space if and only if F is a KB-space.*

Proof. The part of "only if" is obvious. If F is a KB-space, by Theorem 4 it suffices to show that each positive operator $T : E \to F$ is M-weakly compact. Indeed, if T is not M-weakly compact then there is a disjoint sequence $(x_n)_1^\infty \subset \text{ball}(E)$ with $x_n \geq 0$ and $\|Tx_n\| \geq 2\delta > 0$ for $n \in \mathbb{N}$. Note that $Tx_n \to 0$ weakly as $x_n \to 0$ weakly (see Theorem 2.4.14 of [2]); by Proposition 2.3.4 of [2] there exists a disjoint sequence $(y_n')_1^\infty \subset \text{ball}(F')$, $y_n \geq 0$, satisfying

$$\left(T' y_n'\right)(x_n) = y_n'(Tx_n) > \delta, \quad \forall n. \quad (10)$$

Also by Theorem 2.5.6 and 3.4.18 of [2], T is weakly compact and so is T' by Gantmacher's theorem (see Theorem 17.2 of [4]), so we may assume that $T' y_n'$ is weakly convergent (replacing by a subsequence if necessary); say $T' y_n' \to x'$ weakly, then for each $x \in E$

$$x'(x) = \lim_{n\to\infty} T' y_n'(x) = \lim_{n\to\infty} y_n'(Tx) = 0 \quad (11)$$

as $y_n' \to 0$ in $\sigma(F', F)$ (see Corollary 2.4.3 of [2]); that is, $T' y_n' \to 0$ weakly, so the positive Schur property of E' implies that $\|T' y_n'\| \to 0$ and it follows that

$$0 < \delta < y_n'(Tx_n) = \left(T' y_n'\right)(x_n) \leq \left\|T' y_n'\right\| \longrightarrow 0. \quad (12)$$

This is impossible, thus T is M-weakly compact. □

The following result is a dual version of Theorem 12.

Theorem 13. *Let F be a Banach lattice with the positive Schur property, E a Banach lattice. Then $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is a KB-space if and only if the norm on E' is order continuous.*

Proof. The part of "only if" easily follows from the proof of Theorem 2. If the norm on E' is order continuous, for $T \in \mathscr{L}_+(E,F)$ and each disjoint sequence $(x_n)_1^\infty \subset \text{ball}(E)$, then $|x_n| \to 0$ weakly, and $T|x_n| \to 0$ weakly. It follows from the positive Schur property of F that $\|Tx_n\| \to 0$ as $|Tx_n| \leq T|x_n|$; that is, T is M-weakly compact; Theorem 4 yields that $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is a KB-space. □

For a Banach lattice E and $1 \leq p \leq \infty$, recall that E has the *strong ℓ_p-decomposition property* if there exists a constant M such that for all disjoint elements $x_1, \ldots, x_n$ in E we have $(\sum_{i=1}^n \|x_i\|^p)^{1/p} \leq M\|\sum_{i=1}^n x_i\|$ for $p < \infty$ and $\max\{\|x_i\| : i = 1, \ldots, n\} \leq M\|\sum_{i=1}^n x_i\|$ in case $p = \infty$. The number $\sigma(E) = \inf\{p \geq 1 : E$ has the strong ℓ_p-decomposition property$\}$ is call the *upper index* of E.

Similarly E has the *strong ℓ_p-composition property* if there exists a constant M such that for all disjoint elements $x_1, \ldots, x_n$ in E we have $\|\sum_{i=1}^n x_i\| \leq M(\sum_{i=1}^n \|x_i\|^p)^{1/p}$ for $p < \infty$ and $\|\sum_{i=1}^n x_i\| \leq M \max\{\|x_i\| : i = 1, \ldots, n\}$ in case $p = \infty$. The number $s(E) = \sup\{p \geq 1 : E$ has the strong ℓ_p-composition property$\}$ is called *lower index* of E.

It is known that $1 \leq s(E) \leq \sigma(E) \leq \infty$ for any Banach lattice E. If $\sigma(E) < \infty$ then E has an order continuous norm. If $s(E) > 1$ then the norm on E' is order continuous. See [6] for details

Also recall that if the norm on a Banach lattice E is p-superadditive then $\sigma(E) \leq p$; and if E has a p-subadditive norm then $s(E) \geq p$; see Proposition 2.8.2 of [2].

Theorem 14. *Let E and F be Banach lattices. If $s(E) > \sigma(F)$ then $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is a KB-space.*

Proof. The norm on E' clearly is order continuous. Note that if $\sigma(F) < \infty$ then F is a KB-space. Indeed, if F is not a KB-space, then F contains a sublattice H lattice isomorphic to c_0, which implies that $\sigma(F) = \infty$ as $\sigma(c_0) = \infty$. Now the rest is a simple consequence of Theorem 4, Theorem 6.7 of [6], and Theorem 3.6.17 of [2]. □

Corollary 15. *Let E and F be Banach lattices. If the norm of E is p-subadditive, the norm of F is q-superadditive and $1 \leq q < p \leq \infty$, then $(\mathscr{L}^r(E,F), \|\cdot\|_r)$is a KB-space.*

Remark 16. It is worth to point out that $s(E) > \sigma(F)$ fails to be true in general even if $\mathscr{L}^r(E,F)$ is a KB-space, see [7, Example 3.6].

For E and F being L^p- and L^q-spaces, respectively, we have the following characterization.

Theorem 17. *Let E and F be infinite dimensional L^p-space, and L^q-space respectively, then $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is a KB-space if and only if $q < p$.*

Proof. The part of "if" is a simple consequence of Corollary 15. To see the part of "only if", we may first assume that

$$H_n = \begin{cases} \ell_p & \text{if } p < \infty \\ c_0 & \text{if } p = \infty \end{cases} \qquad G_q = \begin{cases} \ell_q & \text{if } q < \infty \\ c_0 & \text{if } q = \infty \end{cases} \quad (13)$$

are sublattices of E and F, respectively. Suppose that $p \leq q$ then $H_p \subset G_q$.

If $p < \infty$ there is a positive projection P from E onto H_p (the existence of P is following from Theorem 2.7.11 of [2]), then $P : E \to H_p \subset G_q \subset F$ is not M-weakly compact, which, by Theorem 4, implies that $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is not a KB-space.

Also if $p = \infty$, then $q = \infty$ and F is not a KB-space, Theorem 4 again yields that $(\mathscr{L}^r(E,F), \|\cdot\|_r)$ is not a KB-space. □

The next result shows that under the regular norm the spaces $\mathscr{L}^r(E,F)$ are rather rare to be L^p-spaces.

Theorem 18. *Let E and F be non-zero Banach lattices and $1 < p,\ q < \infty$ with $(1/p)+(1/q) = 1$. Then the following assertions are equivalent.*

(1) *The regular norm $\|\cdot\|_r$ is p-additive on $\mathscr{L}_+(E,F)$.*

(2) *One of following two conditions holds.*

  (a) $\dim E = 1$ *and the norm on F is p-additive (i.e., F is an L^p-space).*

  (b) $\dim F = 1$ *and the norm on E is q-additive (i.e., E is an L^q-space).*

Proof. (2) ⇒ (1) is obvious. To see that (1) ⇒ (2), we assume that $\|\cdot\|_r$ is p-additive on $\mathscr{L}_+(E,F)$. For any $y_1, y_2 \in F_+$, pick $x' \in E'_+$ with $\|x'\| = 1$; thus the p-additivity of the regular norm yields that

$$\begin{aligned} \|y_1 + y_2\|^p &= \left\|x' \bigotimes (y_1 + y_2)\right\|^p \\ &= \left\|x' \bigotimes y_1\right\|^p + \left\|x' \bigotimes y_2\right\|^p \\ &= \|y_1\|^p + \|y_2\|^p \end{aligned} \quad (14)$$

which means that F is an L^p-space. A similar argument involving a fixed element of F_+ and two elements of E'_+ shows that E' is an L^p-space; hence E is an L^q-space (compare with Theorem 2.7.1 of [2]), where $p^{-1} + q^{-1} = 1$.

Now if both $\dim(E) \geq 2$ and $\dim(F) \geq 2$ hold we will obtain a contradiction. In fact, we may assume that ℓ_2^q and ℓ_2^p are 2-dimensional sublattices of E and F, respectively; define $T_1, T_2 : \ell_2^q \to \ell_2^p$ by

$$\begin{aligned} T_1(\lambda_1, \lambda_2) &= (\lambda_1, 0), \\ T_2(\lambda_1, \lambda_2) &= (0, \lambda_2) \quad \text{for } (\lambda_1, \lambda_2) \in \ell_2^q; \end{aligned} \quad (15)$$

then $\|T_1\| = \|T_2\| = 1$. Let P be a positive contractive projection from E onto ℓ_q^2 (see Theorem 2.7.11 of [2]); it follows that

$$\|T_1 + T_2\|^p = \|PT_1 + PT_2\|^p = \|PT_1\|^p + \|PT_2\|^p = 2. \quad (16)$$

Also it is easy to calculate that $\|T_1 + T_2\| = 2^{1/p-1/q}$; this is impossible. (1) ⇒ (2) holds. □

Theorem 19. *Let E and F be non-zero Banach lattices and $1 < p,\ q < \infty$ with $(1/p)+(1/q) = 1$. Then the following assertions are equivalent.*

(1) *The regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is equivalent to a p-additive norm.*

(2) *One of following two conditions holds.*

  (a) $\dim E < \infty$ *and the norm on F is equivalent to a p-additive norm.*

  (b) $\dim F < \infty$ *and the norm on E is equivalent to a q-additive norm.*

Proof. (1) ⇒ (2). Suppose that the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is equivalent to a p-additive norm. We first show that the norms both on E' and F are equivalent to q-additive and p-additive norms, respectively, where $p^{-1} + q^{-1} = 1$.

For each disjoint sequence $(y_n)_1^\infty \subset F_+$ with $\|y_n\| = 1$, fix $x' \in E'_+$ with $\|x'\| = 1$. It is easy to verify that $(x' \bigotimes y_n)_1^\infty \subset \mathscr{L}_+(E,F)$ is a disjoint sequence with $\|x' \bigotimes y_n\| = 1$ for all $n \in \mathbb{N}$. Corollary 2.8.12 of [2] yields that $(x' \bigotimes y_n)_1^\infty$ is equivalent to the natural basis of ℓ^p. Note that

$$\left\|\sum_{i=1}^n \lambda_i \left(x' \bigotimes y_i\right)\right\| = \left\|x' \bigotimes \left(\sum_{i=1}^n \lambda_i y_i\right)\right\| = \left\|\sum_{i=1}^n \lambda_i y_i\right\| \quad (17)$$

for all $n \in \mathbb{N}$ and $\lambda_i \in \mathbb{R}$. It follows that $(y_n)_1^\infty$ is equivalent to the natural basis of ℓ^p, which by Corollary 2.8.12 of [2] implies that the norm on F is equivalent to a p-additive norm.

A similar argument involving a fixed element of F_+ and a disjoint sequence of elements of E'_+ shows that the norm on E' is equivalent to a p-additive norm; hence the norm on E is equivalent to a q-additive norm.

Now we show that either $\dim E < \infty$ or $\dim F < \infty$. Otherwise, both E and F are infinite dimensional. Renorming E and F with equivalent q-additive and p-additive norms, respectively, the regular norm on $\mathscr{L}^r(E,F)$ is still equivalent to a p-additive norm. Thus we may assume that the norms on E and F are q- and p-additive, and that $\ell^q \subset E$ and $\ell^p \subset F$ are sublattices, respectively. By Theorem 2.7.11 of [2] there is a positive contractive projection P from E onto ℓ^q. Consider the operators $T_n : \ell^q \to \ell^p$ by $T_n(\lambda_k) = \lambda_n e_n$, where e_n is the element in ℓ^q and ℓ^p with nth entry equals to 1 and all others are 0. Then it is easy to verify that $(T_n P)_1^\infty$ is a disjoint sequence in $\mathscr{L}_+(E,F)$ with $\|T_n P\| = 1$. Corollary 2.8.12 of [2] yields that $(T_n P)_1^\infty$ is equivalent to the natural basis of ℓ^p. In particular, we have

$$\begin{aligned} Bn^{1/p} &\leq \|T_1 + T_2 + \cdots + T_n\| \\ &= \|T_1 P + T_2 P + \cdots + T_n P\| \leq An^{1/p} \end{aligned} \quad (18)$$

for all $n \in \mathbb{N}$, where $A > 0$ and $B > 0$ are constants. But

$$\|T_1 + T_2 + \cdots + T_n\| = \sup\left\{\left\|(\lambda_i)_1^n\right\|_p : \left\|(\lambda_i)_1^n\right\|_q \leq 1\right\} \quad (19)$$

which easily shows that $\|T_1 + T_2 + \cdots + T_n\| \leq 1$ if $q \leq p$ and $\|T_1 + T_2 + \cdots + T_n\| \leq n^{1/p-1/q}$ for $q > p$. This is impossible for either $q \leq p$ or $q > p$. So (1) ⇒ (2) holds.

(2)(a) ⇒ (1). Let $E = \operatorname{span}\{e_1, e_2, \ldots, e_m\}$ with $\{e_1, e_2, \ldots, e_m\} \subset E_+$ pairwise disjoint and $\|e_i\| = 1$. Then each $T \in \mathscr{L}^r(E,F)$ corresponds to unique $(x_1(T), x_2(T), \ldots, x_m(T))$; moreover, $x_i(T) = Te_i \in F$, satisfying the following conditions.

(i) $T \geq 0 \Leftrightarrow x_i(T) \geq 0$ for $1 \leq i \leq m$.

(ii) $x_i(|T|) = |x_i(T)|$ for all $T \in \mathscr{L}^r(E,F)$ and $1 \leq i \leq m$.

(iii) $x_i(\lambda T + \mu S) = \lambda x_i(T) + \mu x_i(S)$ for all $T, S \in \mathscr{L}^r(E,F)$, $\lambda, \mu \in \mathbb{R}$ and $1 \le i \le m$.

(iv) $\max\{\|x_i(T)\| : 1 \le i \le m\} \le \|T\|_r = \||T|\| \le \sum_{i=1}^{m} \|x_i(T)\|$.

Since the norm on F is equivalent to a p-additive norm, for each disjoint sequence $(y_k)_1^\infty \subset F_+$, by Corollary 2.8.12 of [2] there exist constants $A > 0$, $B > 0$ such that

$$B\left\|\left(\lambda_k \|y_k\|\right)_1^n\right\|_p \le \left\|\sum_{k=1}^{n} \lambda_k y_k\right\| \le A\left\|\left(\lambda_k \|y_k\|\right)_1^n\right\|_p \qquad (*)$$

for all $\lambda_k \in \mathbb{R}$ and $n \in \mathbb{N}$. Now for any disjoint sequence $(T_n)_1^\infty \subset \mathscr{L}_+(E,F)$ with $\|T_n\| = 1$, note that the disjointness of $(x_i(T_n))_{n=1}^\infty \subset F_+$ for each $1 \le i \le m$; it follows from (iii), (iv), and $(*)$ that

$$\begin{aligned} \left\|\sum_{k=1}^{n} \lambda_k T_k\right\| &\le \sum_{i=1}^{m} \left\|\sum_{k=1}^{n} \lambda_k x_i(T_k)\right\| \\ &\le A \sum_{i=1}^{m} \left(\sum_{k=1}^{n} |\lambda_k|^p \|x_i(T_k)\|^p\right)^{1/p} \\ &\le mA \left\|(\lambda_k)_1^n\right\|_p \end{aligned} \qquad (20)$$

as $\|x_i(T_k)\| \le \|T_k\| = 1$. Also

$$\begin{aligned} \left\|\sum_{k=1}^{n} \lambda_k T_k\right\| &\ge \max\left\{\left\|\sum_{k=1}^{n} \lambda_k x_i(T_k)\right\| : 1 \le i \le m\right\} \\ &\ge B \max\left\{\left(\sum_{k=1}^{n} |\lambda_k|^p \|x_i(T_k)\|^p\right)^{1/p} : 1 \le i \le m\right\} \\ &\ge Bm^{-1/p}\left(\sum_{i=1}^{m}\sum_{k=1}^{n} |\lambda_k|^p \|x_i(T_k)\|^p\right)^{1/p} \\ &= Bm^{-1/p}\left(\sum_{k=1}^{n} |\lambda_k|^p \left(\sum_{i=1}^{m} \|x_i(T_k)\|^p\right)\right)^{1/p} \\ &\ge Bm^{-1}\left\|(\lambda_k)_1^n\right\|_p \end{aligned} \qquad (21)$$

as $1 \le \sum_{=1}^{m} \|x_i(T_k)\| \le \left(\sum_{i=1}^{m} \|x_i(T_k)\|^p\right)^{1/p} m^{1/q}$.

Therefore

$$Bm^{-1}\left\|(\lambda_k)_1^n\right\|_p \le \left\|\sum_{k=1}^{n} \lambda_k T_k\right\| \le mA\left\|(\lambda_k)_1^n\right\|_p \qquad (22)$$

for all $\lambda_k \in \mathbb{R}$ and $n \in \mathbb{N}$; that is, $(T_n)_1^\infty$ is equivalent to the natural basis of ℓ^p. Corollary 2.8.12 of [2] again shows that the regular norm $\|\cdot\|_r$ on $\mathscr{L}^r(E,F)$ is equivalent to a p-additive norm, so (2) (a) $\Rightarrow$ (1) holds.

The proof of (2) (b) $\Rightarrow$ (1) is similar with (2) (a) $\Rightarrow$ (1). This completes the proof. □

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