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Research Article

Uniformly Asymptotic Stability of Positive Almost Periodic Solutions for a Discrete Competitive System

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This paper is devoted to the study of almost periodic solutions of a discrete two-species competitive system. With the help of the methods of the Lyapunov function, some analysis techniques, and preliminary lemmas, we establish a criterion for the existence, uniqueness, and uniformly asymptotic stability of positive almost periodic solution of the system. Numerical simulations are presented to illustrate the analytical results.

1. Introduction

In recent years, many works have been done for the difference system (see [1–14] and the references cited therein) since the discrete time models governed by the difference equation are more appropriate than the continuous ones when the populations have a short life expectancy, nonoverlapping generations in the real world. In particular, Qin et al. [1] introduced the following discrete Lotka-Volterra competitive system:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\left[r_1(n) - a_1(n)x_1(n) - \frac{c_2(n)x_2(n)}{1+x_2(n)}\right], \\ x_2(n+1) &= x_2(n)\exp\left[r_2(n) - a_2(n)x_2(n) - \frac{c_1(n)x_1(n)}{1+x_1(n)}\right], \\ &\qquad n = 0, 1, 2\ldots, \end{aligned} \tag{1}$$

where $x_i(0) > 0$, $x_i(n)$ stand for the densities of species x_i at the nth generation, $r_i(n)$ represent the natural growth rates of species x_i at the nth generation, $a_i(n)$ are the intraspecific effects of the nth generation of species x_i on own population, and $c_i(n)$ measure the interspecific effects of the nth generation of species x_i on species x_j $(i, j = 1, 2; i \neq j)$. They investigated the permanence and global asymptotic stability of positive periodic solutions of system (1).

Notice that the investigation of almost periodic solutions for difference equations is one of most important topics in the qualitative theory of difference equations due to the applications in biology, ecology, neural network, and so forth (see [10–14] in detail), and few work has been done previously on an almost periodic version which is corresponding to periodic system (1). In this paper, we will further investigate the existence, uniqueness, and uniformly asymptotic stability of positive almost periodic solution of the above almost periodic version. To this end, we assume that the coefficients of system (1) $\{r_i(n)\}$, $\{a_i(n)\}$ and $\{c_i(n)\}$ are bounded nonnegative almost periodic sequences.

For the sake of simplicity and convenience in the following discussion, the notations below will be used throughout this paper:

$$f^U = \sup_{n\in\mathbb{Z}^+}\{f(n)\}, \qquad f^L = \inf_{n\in\mathbb{Z}^+}\{f(n)\}, \tag{2}$$

where $\{f(n)\}$ is a bounded sequence and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

The remaining part of this paper is organized as follows. In the next section, we introduce some notations, definitions, and lemmas which are available for our main results. In Section 3, sufficient conditions for the existence, uniqueness, and uniformly asymptotic stability of positive almost periodic solution of system (1) are given. Numerical simulations are

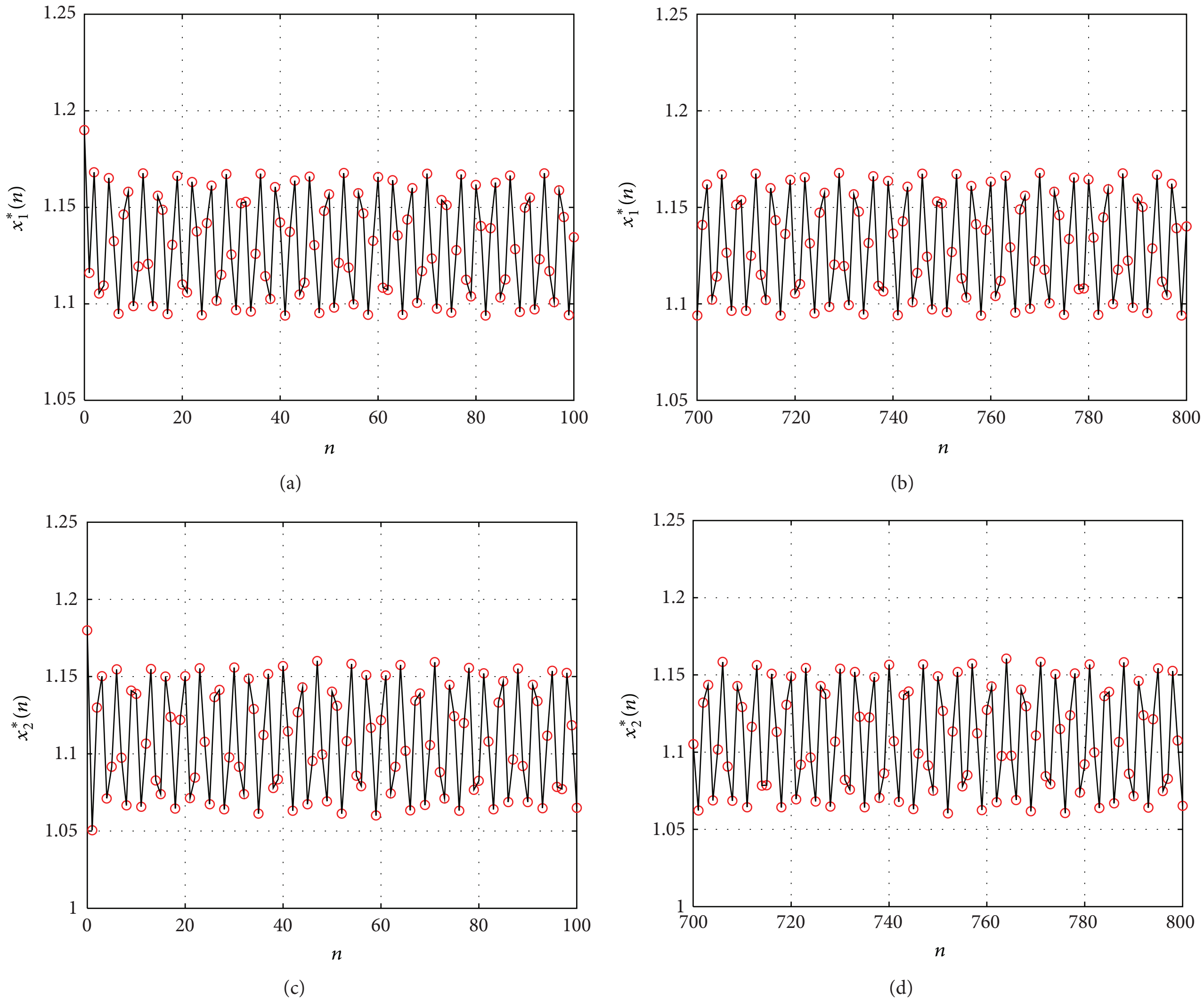

Figure 1: Positive almost periodic solution of system (13). (a), (c) Time-series $x_1^*(n)$ and $x_2^*(n)$ with initial values $x_1^*(0) = 1.19$, $x_2^*(0) = 1.18$ for $n \in [0, 100]$, respectively. (b), (d) Time-series $x_1^*(n)$ and $x_2^*(n)$ with the above initial values for $n \in [700, 800]$, respectively.

carried out to substantiate the above analytical results in Section 4. Finally, we give some proofs of theorems in the appendices for convenience in reading this paper.

2. Preliminaries

In this section, we will need some preparations and give some notations, definitions, and lemmas which will be useful for our main results.

Denote by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{Z}^+$ the sets of real numbers, nonnegative real numbers, integers, and nonnegative integers, respectively. $\mathbb{R}^2$ and $\mathbb{R}^k$ denote the cone of 2-dimensional and k-dimensional real Euclidean space, respectively.

Definition 1 (see [13]). A sequence $x : \mathbb{Z} \to \mathbb{R}^k$ is called an almost periodic sequence if the following ε-translation set of x

$$E\{\varepsilon, x\} := \{\tau \in \mathbb{Z} : |x(n+\tau) - x(n)| < \varepsilon,\ \forall n \in \mathbb{Z}\} \quad (3)$$

is a relatively dense set in $\mathbb{Z}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each discrete interval of length $l(\varepsilon)$ contains a $\tau = \tau(\varepsilon) \in E\{\varepsilon, x\}$ such that

$$|x(n+\tau) - x(n)| < \varepsilon, \quad \forall n \in \mathbb{Z}. \quad (4)$$

τ is called the ε-translation number of $x(n)$.

Definition 2 (see [13]). Let $f : \mathbb{Z} \times \mathbb{D} \to \mathbb{R}^k$, where $\mathbb{D}$ is an open set in $\mathbb{R}^k$. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in \mathbb{D}$, or uniformly almost periodic for short, if for any $\varepsilon > 0$ and any compact set $\mathbb{S}$ in $\mathbb{D}$ there exists a positive integer $l(\varepsilon, \mathbb{S})$ such that any interval of length $l(\varepsilon, \mathbb{S})$ contains an integer τ for which

$$|f(n+\tau, x) - f(n, x)| < \varepsilon, \quad (5)$$

for all $n \in \mathbb{Z}$ and all $x \in \mathbb{S}$. τ is called the ε-translation number of $f(n, x)$.

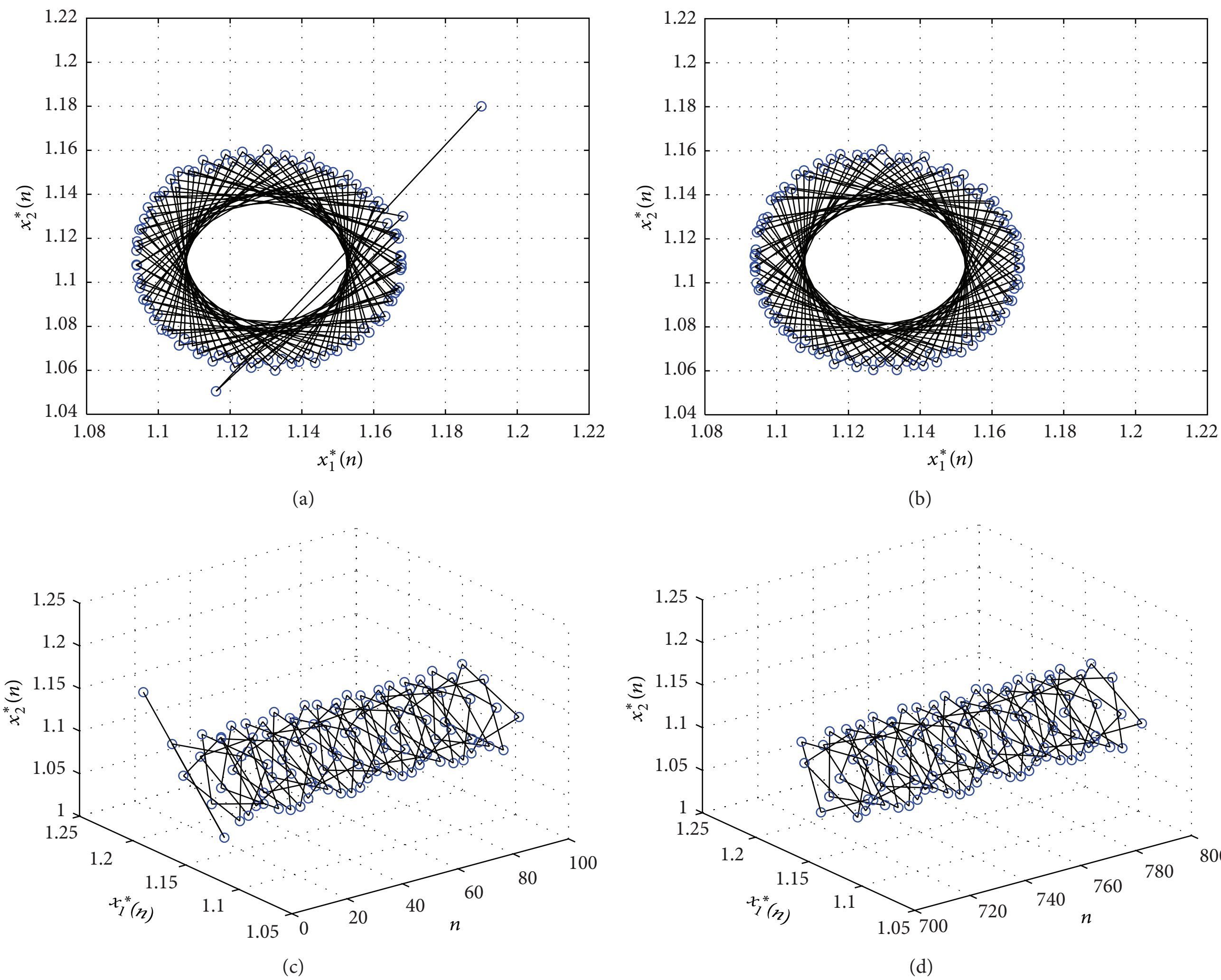

FIGURE 2: Phase portrait. (a), (b) 2-dimensional phase portrait of almost periodic system (13). Time-series $x_1^*(n)$ and $x_2^*(n)$ with initial values $x_1^*(0) = 1.19$, $x_2^*(0) = 1.18$ for $n \in [0, 100]$ and $n \in [700, 800]$, respectively. (c), (d) 3-dimensional phase portrait of almost periodic system (13). Time-series $x_1^*(n)$ and $x_2^*(n)$ with the above initial values for $n \in [0, 100]$ and $n \in [700, 800]$, respectively.

Lemma 3 (see [13]). $\{x(n)\}$ *is an almost periodic sequence if and only if for any sequence* $\{h'_k\} \subset \mathbb{Z}$ *there exists a subsequence* $\{h_k\} \subset \{h'_k\}$ *such that* $x(n + h_k)$ *converges uniformly on* $n \in \mathbb{Z}$ *as* $k \to \infty$. *Furthermore, the limit sequence is also an almost periodic sequence.*

Consider the following almost periodic difference system:

$$x(n+1) = f(n, x(n)), \quad n \in \mathbb{Z}^+, \tag{6}$$

where $f : \mathbb{Z}^+ \times \mathbb{S}_B \to \mathbb{R}^k$, $\mathbb{S}_B = \{x \in \mathbb{R}^k : \|x\| < B\}$, and $f(n, x)$ is almost periodic in n uniformly for $x \in \mathbb{S}_B$ and is continuous in x. The product system of (6) is the following system:

$$x(n+1) = f(n, x(n)), \qquad y(n+1) = f(n, y(n)), \tag{7}$$

and Zhang [14] obtained the following lemma.

Lemma 4 (see [14]). *Suppose that there exists a Lyapunov function* $V(n, x, y)$ *defined for* $n \in \mathbb{Z}^+$, $\|x\| < B$, $\|y\| < B$ *satisfying the following conditions:*

(i) $a(\|x - y\|) \le V(n, x, y) \le b(\|x - y\|)$, *where* $a, b \in K$ *with* $K = \{a \in C(\mathbb{R}^+, \mathbb{R}^+) : a(0) = 0$ *and* a *is increasing*$\}$;

(ii) $|V(n, x_1, y_1) - V(n, x_2, y_2)| \le L(\|x_1 - x_2\| + \|y_1 - y_2\|)$, *where* $L > 0$ *is a constant;*

(iii) $\Delta V_{(2.2)}(n, x, y) \le -\beta V(n, x, y)$, *where* $0 < \beta < 1$ *is a constant and*

$$\Delta V_{(2.2)}(n, x, y) = V(n + 1, f(n, x), f(n, y)) - V(n, x, y). \tag{8}$$

Moreover, if there exists a solution $\varphi(n)$ *of system* (6) *such that* $\|\varphi(n)\| \le B^* < B$ *for* $n \in \mathbb{Z}^+$, *then there exists a unique uniformly asymptotically stable almost periodic solution* $p(n)$ *of system* (6) *which satisfies* $\|p(n)\| \le B^*$. *In particular, if* $f(n, x)$

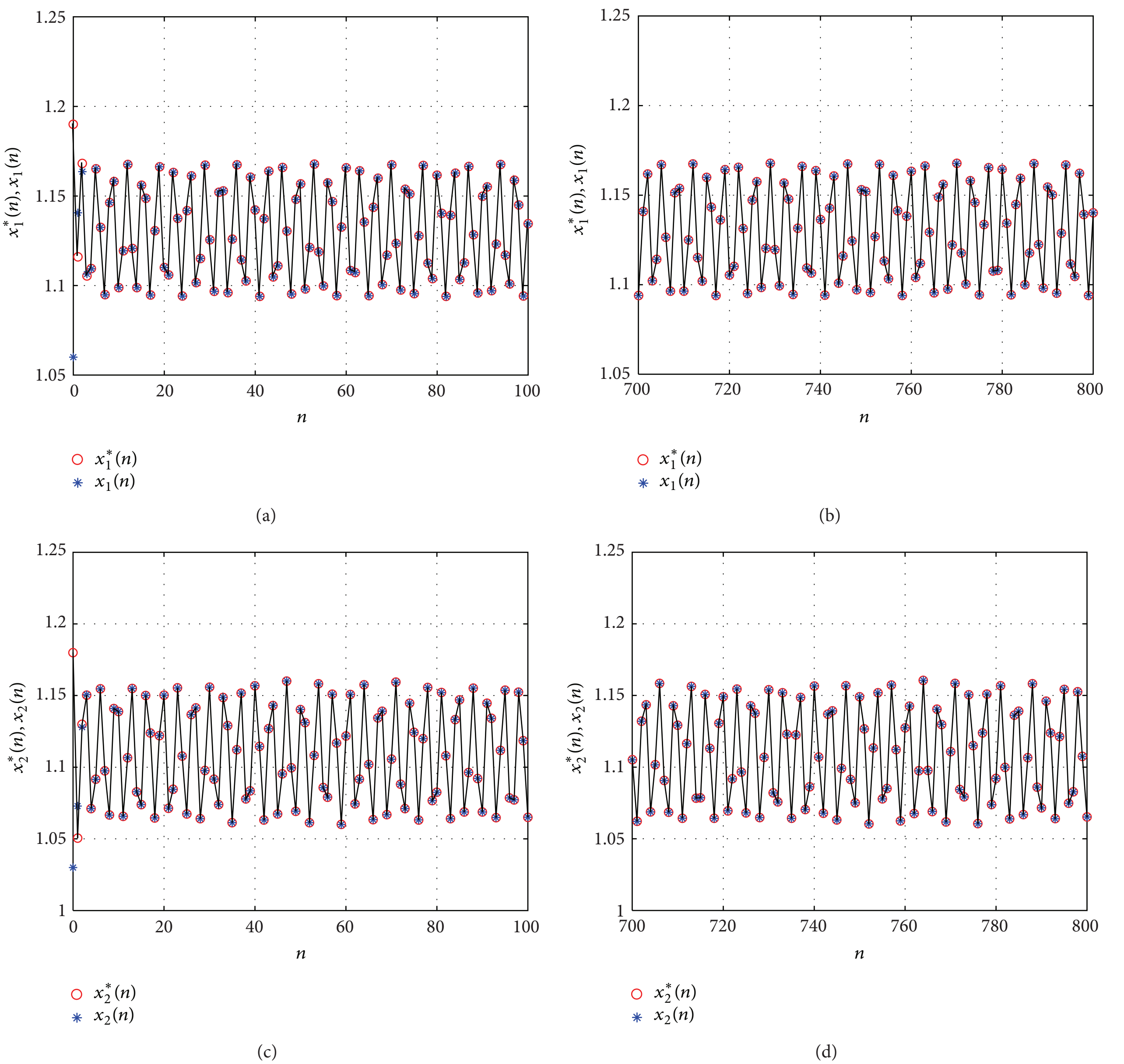

FIGURE 3: Uniformly asymptotic stability. (a), (c) Time-series $x_1^*(n)$ and $x_2^*(n)$ with initial values $x_1^*(0) = 1.19$, $x_2^*(0) = 1.18$ and $x_1(n)$ and $x_2(n)$ with initial values $x_1(0) = 1.06$, $x_2(0) = 1.03$ for $n \in [0, 100]$, respectively. (b), (d) Time-series $x_1^*(n)$, $x_2^*(n)$, $x_1(n)$, and $x_2(n)$ with the above initial values for $n \in [700, 800]$, respectively.

is periodic of period ω, then there exists a unique uniformly asymptotically stable periodic solution of system (6) *of periodic ω.*

Lemma 5 (see [1]). *Any positive solution* $(x_1(n), x_2(n))$ *of system* (1) *satisfies*

$$\limsup_{n\to+\infty} x_i(n) \le M_i \stackrel{\text{def}}{=} \frac{\left[\exp\left(r_i^U - 1\right)\right]}{a_i^L}, \quad i = 1, 2. \tag{9}$$

Lemma 6 (see [1]). *Suppose that system* (1) *satisfies the following assumptions:*

$$r_1^L > c_2^U, \qquad r_2^L > c_1^U. \tag{10}$$

Then, any positive solution $(x_1(n), x_2(n))$ *of system* (1) *satisfies*

$$\liminf_{n\to+\infty} x_i(n) \ge m_i \stackrel{\text{def}}{=} \frac{r_i^L - c_j^U}{a_i^U} \exp\left(r_i^L - a_i^U M_i - c_j^U\right), \quad i \ne j; i, j = 1, 2. \tag{11}$$

3. Main Result

From (9) and (11), we denote by Ω the set of all solutions $(x_1(n), x_2(n))$ of system (1) satisfying $m_i \le x_i(n) \le M_i$, $i = 1, 2$ for all $n \in \mathbb{Z}^+$. According to Lemma 4, we first prove that there is a bounded solution of system (1), and then structure a suitable Lyapunov function for system (1).

Theorem 7. *If the assumptions in* (10) *hold, then* $\Omega \neq \Phi$.

The proof of Theorem 7 is given in Appendix A.

Theorem 8. *If the assumptions in* (10) *are satisfied, furthermore,* $0 < \beta < 1$, *where* $\beta = \min\{s_1, s_2\}$, *and*

$$\begin{aligned} s_1 &= 2a_1^L m_1 - a_1^{U2} M_1^2 - \frac{\left(1 + a_1^U M_1\right) c_2^U M_2}{\left(1 + m_2\right)^2} \\ &\quad - \frac{\left(1 + a_2^U M_2\right) c_1^U M_1}{\left(1 + m_1\right)^2} - \frac{c_1^{U2} M_1^2}{\left(1 + m_1\right)^4}, \\ s_2 &= 2a_2^L m_2 - a_2^{U2} M_2^2 - \frac{\left(1 + a_2^U M_2\right) c_1^U M_1}{\left(1 + m_1\right)^2} \\ &\quad - \frac{\left(1 + a_1^U M_1\right) c_2^U M_2}{\left(1 + m_2\right)^2} - \frac{c_2^{U2} M_2^2}{\left(1 + m_2\right)^4}, \end{aligned} \tag{12}$$

then there exists a unique uniformly asymptotically stable almost periodic solution of system (1) *which is bounded by* Ω *for all* $n \in \mathbb{Z}^+$.

The proof of Theorem 8 is given in Appendix B.

4. Numerical Simulations

In this section, we give the following example to check the feasibility of the assumptions of Theorem 8.

Example 9. Consider the following discrete system:

$$\begin{aligned} &x_1(n+1) \\ &\quad = x_1(n) \exp\left[1.20 - 0.02 \sin\left(\sqrt{2} n\pi\right)\right. \\ &\qquad\qquad - \left(1.05 + 0.01 \sin\left(\sqrt{2} n\pi\right)\right) x_1(n) \\ &\qquad\qquad \left. - \frac{\left(0.025 + 0.002 \cos\left(\sqrt{2} n\pi\right)\right) x_2(n)}{1 + x_2(n)}\right], \\ &x_2(n+1) \\ &\quad = x_2(n) \exp\left[1.15 - 0.02 \cos\left(\sqrt{2} n\pi\right)\right. \\ &\qquad\qquad - \left(1.02 + 0.02 \cos\left(\sqrt{2} n\pi\right)\right) x_2(n) \\ &\qquad\qquad \left. - \frac{\left(0.035 + 0.005 \sin\left(\sqrt{3} n\pi\right)\right) x_1(n)}{1 + x_1(n)}\right]. \end{aligned} \tag{13}$$

A computation shows that

$$r_1^L - c_2^U = 1.1530 > 0, \qquad r_2^L - c_1^U = 1.0900 > 0, \tag{14}$$

and moreover, we have

$$s_1 = 0.3515, \qquad s_2 = 0.2502, \tag{15}$$

that is, $0 < \beta = \min\{s_1, s_2\} = 0.2502 < 1$. It is easy to see that the assumptions of Theorem 8 are satisfied. Hence, in system (13) there exists a unique uniformly asymptotically stable positive almost periodic solution. From Figure 1, it is easy to see that there exists a positive almost periodic solution $(x_1^*(t), x_2^*(t))$, and the 2-dimensional and 3-dimensional phase portraits of almost periodic system (13) are revealed in Figure 2, respectively. Figure 3 shows that any positive solution $(x_1(n), x_2(n))$ tends to the almost periodic solution $(x_1^*(n), x_2^*(n))$.

Appendices

A. Proof of Theorem 7

Clearly, by an inductive argument we have from system (1) that

$$\begin{aligned} x_1(n) &= x_1(0) \exp \sum_{l=0}^{n-1} \left[r_1(l) - a_1(l) x_1(l) - \frac{c_2(l) x_2(l)}{1 + x_2(l)}\right], \\ x_2(n) &= x_2(0) \exp \sum_{l=0}^{n-1} \left[r_2(l) - a_2(l) x_2(l) - \frac{c_1(l) x_1(l)}{1 + x_1(l)}\right]. \end{aligned} \tag{A.1}$$

According to Lemmas 5 and 6, for any solution $(x_1(n), x_2(n))$ of system (1) and an arbitrarily small constant $\varepsilon > 0$, there exists n_0 sufficiently large such that

$$m_1 - \varepsilon \le x_1(n) \le M_1 + \varepsilon, \qquad m_2 - \varepsilon \le x_2(n) \le M_2 + \varepsilon, \qquad \forall n \ge n_0. \tag{A.2}$$

Set $\{\tau_k\}$ be any positive integer sequence such that $\tau_k \to +\infty$ as $k \to +\infty$, we can show that there exists a subsequence of $\{\tau_k\}$ still denoted by $\{\tau_k\}$, such that $x_i(n+\tau_k) \to x_i^*(n), i = 1, 2$ uniformly in n on any finite subset C of $\mathbb{Z}^+$ as $k \to +\infty$, where $C = \{a_1, a_2 \cdots a_m\}$, $a_h \in \mathbb{Z}^+$ $(h = 1, 2 \cdots m)$, and m is a finite number.

As a matter of fact, for any finite subset $C \subset \mathbb{Z}^+$, $\tau_k + a_h > n_0$, $h = 1, 2 \cdots m$, when k is large enough. Therefore, $m_i - \varepsilon \le x_i(n + \tau_k) \le M_i + \varepsilon$, $i = 1, 2$; that is, $\{x_i(n + \tau_k)\}$ are uniformly bounded for k large enough.

Now, for $a_1 \in C$, we can choose a subsequence $\{\tau_k^{(1)}\}$ of $\{\tau_k\}$ such that $\{x_1(a_1 + \tau_k^{(1)})\}$ and $\{x_2(a_1 + \tau_k^{(1)})\}$ uniformly converge on $\mathbb{Z}^+$ for k large enough.

Analogously, for $a_2 \in C$, we can also choose a subsequence $\{\tau_k^{(2)}\}$ of $\{\tau_k^{(1)}\}$ such that $\{x_1(a_2 + \tau_k^{(2)})\}$ and $\{x_2(a_2 + \tau_k^{(2)})\}$ uniformly converge on $\mathbb{Z}^+$ for k large enough.

Repeating the above process, for $a_m \in C$, we get a subsequence $\{\tau_k^{(m)}\}$ of $\{\tau_k^{(m-1)}\}$ such that $\{x_1(a_m+\tau_k^{(m)})\}$ and $\{x_2(a_m+\tau_k^{(m)})\}$ uniformly converge on $\mathbb{Z}^+$ for k large enough.

Now, we choose the sequence $\{\tau_k^{(m)}\}$ which is a subsequence of $\{\tau_k\}$ denoted by $\{\tau_k\}$; then, for all $n \in C$, we obtain that $x_i(n+\tau_k) \to x_i^*(n)$, $i = 1,2$ uniformly in $n \in C$ as $k \to +\infty$. Hence, the conclusion is valid by the arbitrary of C.

Recall the almost periodicity of $\{r_i(n)\}$, $\{a_i(n)\}$ and $\{c_i(n)\}$, $i = 1,2$, for the above sequence $\{\tau_k\}$, $\tau_k \to +\infty$ as $k \to +\infty$, there exists a subsequence denoted by $\{\tau_k\}$ such that

$$\begin{aligned} r_i(n+\tau_k) &\longrightarrow r_i(n), \\ a_i(n+\tau_k) &\longrightarrow a_i(n), \\ c_i(n+\tau_k) &\longrightarrow c_i(n), \end{aligned} \tag{A.3}$$

as $k \to +\infty$ uniformly on $\mathbb{Z}^+$.

For any $\alpha \in \mathbb{Z}^+$, we can assume that $\tau_k + \alpha \geq n_0$ for k large enough. Let $n \in \mathbb{Z}^+$, by an inductive argument of system (1) from $\tau_k + \alpha$ to $n + \tau_k + \alpha$, we obtain

$$\begin{aligned} &x_1(n+\tau_k+\alpha) \\ &\quad = x_1(\tau_k+\alpha)\exp\sum_{l=\tau_k+\alpha}^{n+\tau_k+\alpha-1}\left[r_1(l)-a_1(l)x_1(l)-\frac{c_2(l)x_2(l)}{1+x_2(l)}\right], \\ &x_2(n+\tau_k+\alpha) \\ &\quad = x_2(\tau_k+\alpha)\exp\sum_{l=\tau_k+\alpha}^{n+\tau_k+\alpha-1}\left[r_2(l)-a_2(l)x_2(l)-\frac{c_1(l)x_1(l)}{1+x_1(l)}\right]. \end{aligned} \tag{A.4}$$

Thus, it derives that

$$\begin{aligned} &x_1(n+\tau_k+\alpha) \\ &\quad = x_1(\tau_k+\alpha)\exp\sum_{l=\alpha}^{n+\alpha-1}\left[r_1(l+\tau_k)-a_1(l+\tau_k)x_1(l+\tau_k)\right. \\ &\qquad\qquad \left. -\frac{c_2(l+\tau_k)x_2(l+\tau_k)}{1+x_2(l+\tau_k)}\right], \\ &x_2(n+\tau_k+\alpha) \\ &\quad = x_2(\tau_k+\alpha)\exp\sum_{l=\alpha}^{n+\alpha-1}\left[r_2(l+\tau_k)-a_2(l+\tau_k)x_2(l+\tau_k)\right. \\ &\qquad\qquad \left. -\frac{c_1(l+\tau_k)x_1(l+\tau_k)}{1+x_1(l+\tau_k)}\right]. \end{aligned} \tag{A.5}$$

Let $k \to +\infty$, we have

$$\begin{aligned} &x_1^*(n+\alpha) \\ &\quad = x_1^*(\alpha)\exp\sum_{l=\alpha}^{n+\alpha-1}\left[r_1(l)-a_1(l)x_1^*(l)-\frac{c_2(l)x_2^*(l)}{1+x_2^*(l)}\right], \\ &x_2^*(n+\alpha) \\ &\quad = x_2^*(\alpha)\exp\sum_{l=\alpha}^{n+\alpha-1}\left[r_2(l)-a_2(l)x_2^*(l)-\frac{c_1(l)x_1^*(l)}{1+x_1^*(l)}\right]. \end{aligned} \tag{A.6}$$

Since α is arbitrary, we know that $(x_1^*(n), x_2^*(n))$ is a solution of system (1) on $\mathbb{Z}^+$, and

$$\begin{aligned} 0 < m_1 - \varepsilon \leq x_1^*(n) \leq M_1 + \varepsilon, \\ 0 < m_2 - \varepsilon \leq x_2^*(n) \leq M_2 + \varepsilon, \quad \forall n \in \mathbb{Z}^+. \end{aligned} \tag{A.7}$$

Notice that ε is an arbitrarily small positive constant; it follows that

$$0 < m_1 \leq x_1^*(n) \leq M_1, \qquad 0 < m_2 \leq x_2^*(n) \leq M_2, \qquad \forall n \in \mathbb{Z}^+. \tag{A.8}$$

Thus, $\Omega \neq \Phi$. This completes the proof.

B. Proof of Theorem 8

Denote $p_1(n) = \ln x_1(n)$, $p_2(n) = \ln x_2(n)$. It follows from system (1) that

$$\begin{aligned} p_1(n+1) &= p_1(n) + r_1(n) - a_1(n)e^{p_1(n)} - \frac{c_2(n)e^{p_2(n)}}{1+e^{p_2(n)}}, \\ p_2(n+1) &= p_2(n) + r_2(n) - a_2(n)e^{p_2(n)} - \frac{c_1(n)e^{p_1(n)}}{1+e^{p_1(n)}}. \end{aligned} \tag{B.1}$$

According to Theorem 7, we can see that the system (B.1) has a bounded solution $(p_1(n), p_2(n))$ satisfying

$$\begin{aligned} \ln m_1 \leq p_1(n) \leq \ln M_1, \\ \ln m_2 \leq p_2(n) \leq \ln M_2, \quad n \in \mathbb{Z}^+. \end{aligned} \tag{B.2}$$

Thus, $|p_1(n)| \leq A$, $|p_2(n)| \leq B$, where $A = \max\{|\ln m_1|, |\ln M_1|\}$, $B = \max\{|\ln m_2|, |\ln M_2|\}$. Define the norm

$\|(p_1(n), p_2(n))\| = |p_1(n)| + |p_2(n)|$, where $(p_1(n), p_2(n)) \in \mathbb{R}^2$. Consider the product system of system (B.1) as follow:

$$\begin{aligned}
p_1(n+1) &= p_1(n) + r_1(n) - a_1(n) e^{p_1(n)} - \frac{c_2(n) e^{p_2(n)}}{1+e^{p_2(n)}},\\
p_2(n+1) &= p_2(n) + r_2(n) - a_2(n) e^{p_2(n)} - \frac{c_1(n) e^{p_1(n)}}{1+e^{p_1(n)}},\\
q_1(n+1) &= q_1(n) + r_1(n) - a_1(n) e^{q_1(n)} - \frac{c_2(n) e^{q_2(n)}}{1+e^{q_2(n)}},\\
q_2(n+1) &= q_2(n) + r_2(n) - a_2(n) e^{q_2(n)} - \frac{c_1(n) e^{q_1(n)}}{1+e^{q_1(n)}}.
\end{aligned} \tag{B.3}$$

We assume that $Y = (p_1(n), p_2(n))$, $W = (q_1(n), q_2(n))$ are any two solutions of system (B.1) defined on $\mathbb{S}$; then, $\|Y\| \le D$, $\|W\| \le D$, where $D = A+B$, and $\mathbb{S} = \{(p_1(n), p_2(n)) \mid \ln m_i \le p_i(n) \le \ln M_i, i = 1, 2, n \in \mathbb{Z}^+\}$.

Let us construct a Lyapunov function defined on $\mathbb{Z}^+ \times \mathbb{S} \times \mathbb{S}$ as follows:

$$V(n, Y, W) = (p_1(n) - q_1(n))^2 + (p_2(n) - q_2(n))^2. \tag{B.4}$$

It is obvious that the norm $\|Y-W\| = |p_1(n)-q_1(n)| + |p_2(n)-q_2(n)|$ is equivalent to $\|Y-W\|_* = [(p_1(n)-q_1(n))^2 + (p_2(n)-q_2(n))^2]^{1/2}$; that is, there are two constants $C_1 > 0$, $C_2 > 0$, such that

$$C_1 \|Y-W\| \le \|Y-W\|_* \le C_2 \|Y-W\|, \tag{B.5}$$

then,

$$(C_1 \|Y-W\|)^2 \le V(n, Y, W) \le (C_2 \|Y-W\|)^2. \tag{B.6}$$

Let $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$, $a(x) = C_1^2 x^2$, $b(x) = C_2^2 x^2$; then, condition (i) of Lemma 4 is satisfied.

Moreover, for any $(n, Y, W), (n, \widetilde{Y}, \widetilde{W}) \in \mathbb{Z}^+ \times \mathbb{S} \times \mathbb{S}$, we have

$$\begin{aligned}
&\left|V(n, Y, W) - V\left(n, \widetilde{Y}, \widetilde{W}\right)\right|\\
&\quad = \left|(p_1(n) - q_1(n))^2 + (p_2(n) - q_2(n))^2 \right.\\
&\qquad \left. -(\widetilde{p}_1(n) - \widetilde{q}_1(n))^2 - (\widetilde{p}_2(n) - \widetilde{q}_2(n))^2\right|\\
&\quad \le \left|(p_1(n) - q_1(n))^2 - (\widetilde{p}_1(n) - \widetilde{q}_1(n))^2\right|\\
&\qquad + \left|(p_2(n) - q_2(n))^2 - (\widetilde{p}_2(n) - \widetilde{q}_2(n))^2\right|\\
&\quad = \left|(p_1(n) - q_1(n)) + (\widetilde{p}_1(n) - \widetilde{q}_1(n))\right|\\
&\qquad \cdot \left|(p_1(n) - q_1(n)) - (\widetilde{p}_1(n) - \widetilde{q}_1(n))\right|\\
&\qquad + \left|(p_2(n) - q_2(n)) + (\widetilde{p}_2(n) - \widetilde{q}_2(n))\right|\\
&\qquad \cdot \left|(p_2(n) - q_2(n)) - (\widetilde{p}_2(n) - \widetilde{q}_2(n))\right|\\
&\quad \le \left(|p_1(n)| + |q_1(n)| + |\widetilde{p}_1(n)| + |\widetilde{q}_1(n)|\right)\\
&\qquad \cdot \left(|p_1(n) - \widetilde{p}_1(n)| + |q_1(n) - \widetilde{q}_1(n)|\right)\\
&\qquad + \left(|p_2(n)| + |q_2(n)| + |\widetilde{p}_2(n)| + |\widetilde{q}_2(n)|\right)\\
&\qquad \cdot \left(|p_2(n) - \widetilde{p}_2(n)| + |q_2(n) - \widetilde{q}_2(n)|\right)\\
&\quad \le L\left\{|p_1(n) - \widetilde{p}_1(n)| + |p_2(n) - \widetilde{p}_2(n)|\right.\\
&\qquad \left. + |q_1(n) - \widetilde{q}_1(n)| + |q_2(n) - \widetilde{q}_2(n)|\right\}\\
&\quad = L\left\{\left\|Y - \widetilde{Y}\right\| + \left\|W - \widetilde{W}\right\|\right\},
\end{aligned} \tag{B.7}$$

where $\widetilde{Y} = (\widetilde{p}_1(n), \widetilde{p}_2(n))$, $\widetilde{W} = (\widetilde{q}_1(n), \widetilde{q}_2(n))$, and $L = 4\max\{A, B\}$. Thus, condition (ii) of Lemma 4 is satisfied.

Finally, calculating the $\Delta V(n)$ of $V(n)$ along the solutions of system (B.3), we have

$$\begin{aligned}
&\Delta V_{(B.3)}(n)\\
&\quad = V(n+1) - V(n)\\
&\quad = (p_1(n+1) - q_1(n+1))^2\\
&\qquad + (p_2(n+1) - q_2(n+1))^2\\
&\qquad - (p_1(n) - q_1(n))^2 - (p_2(n) - q_2(n))^2\\
&\quad = \left[(p_1(n+1) - q_1(n+1))^2 - (p_1(n) - q_1(n))^2\right]\\
&\qquad + \left[(p_2(n+1) - q_2(n+1))^2 - (p_2(n) - q_2(n))^2\right]\\
&\quad = \left[(p_1(n) - q_1(n)) - a_1(n)\left(e^{p_1(n)} - e^{q_1(n)}\right)\right.\\
&\qquad \left. - c_2(n)\left(\frac{e^{p_2(n)}}{1+e^{p_2(n)}} - \frac{e^{q_2(n)}}{1+e^{q_2(n)}}\right)\right]^2 - (p_1(n) - q_1(n))^2\\
&\qquad + \left[(p_2(n) - q_2(n)) - a_2(n)\left(e^{p_2(n)} - e^{q_2(n)}\right)\right.\\
&\qquad \left. - c_1(n)\left(\frac{e^{p_1(n)}}{1+e^{p_1(n)}} - \frac{e^{q_1(n)}}{1+e^{q_1(n)}}\right)\right]^2 - (p_2(n) - q_2(n))^2\\
&\quad = -2a_1(n)(p_1(n) - q_1(n))\left(e^{p_1(n)} - e^{q_1(n)}\right)\\
&\qquad - 2c_2(n)(p_1(n) - q_1(n))\left(\frac{e^{p_2(n)}}{1+e^{p_2(n)}} - \frac{e^{q_2(n)}}{1+e^{q_2(n)}}\right)\\
&\qquad + 2a_1(n)c_2(n)\left(\frac{e^{p_2(n)}}{1+e^{p_2(n)}} - \frac{e^{q_2(n)}}{1+e^{q_2(n)}}\right)\left(e^{p_1(n)} - e^{q_1(n)}\right)\\
&\qquad + a_1^2(n)\left(e^{p_1(n)} - e^{q_1(n)}\right)^2\\
&\qquad + c_2^2(n)\left(\frac{e^{p_2(n)}}{1+e^{p_2(n)}} - \frac{e^{q_2(n)}}{1+e^{q_2(n)}}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&- 2a_2(n)(p_2(n) - q_2(n))\left(e^{p_2(n)} - e^{q_2(n)}\right) \\
&- 2c_1(n)(p_2(n) - q_2(n))\left(\frac{e^{p_1(n)}}{1+e^{p_1(n)}} - \frac{e^{q_1(n)}}{1+e^{q_1(n)}}\right) \\
&+ 2a_2(n)c_1(n)\left(\frac{e^{p_1(n)}}{1+e^{p_1(n)}} - \frac{e^{q_1(n)}}{1+e^{q_1(n)}}\right)\left(e^{p_2(n)} - e^{q_2(n)}\right) \\
&+ a_2^2(n)\left(e^{p_2(n)} - e^{q_2(n)}\right)^2 \\
&+ c_1^2(n)\left(\frac{e^{p_1(n)}}{1+e^{p_1(n)}} - \frac{e^{q_1(n)}}{1+e^{q_1(n)}}\right)^2.
\end{aligned} \tag{B.8}
$$

By the mean value theorem, it derives that

$$
\begin{aligned}
e^{p_i(n)} - e^{q_i(n)} &= \xi_i(n)(p_i(n) - q_i(n)), \\
\frac{e^{p_i(n)}}{1+e^{p_i(n)}} - \frac{e^{q_i(n)}}{1+e^{q_i(n)}} &= \frac{\eta_i(n)}{(1+\eta_i(n))^2}(p_i(n) - q_i(n)),
\end{aligned} \tag{B.9}
$$

$i = 1, 2$, where $\xi_i(n)$ and $\eta_i(n)$ lie between $e^{p_i(n)}$ and $e^{q_i(n)}$, respectively. Substituting (B.9) into (B.8), we get

$$
\begin{aligned}
&\Delta V_{(B.3)}(n) \\
&\quad = -2a_1(n)\xi_1(n)(p_1(n) - q_1(n))^2 \\
&\quad + \frac{c_2^2(n)\eta_2^2(n)}{(1+\eta_2(n))^4}(p_2(n) - q_2(n))^2 \\
&\quad - \frac{2c_2(n)\eta_2(n)}{(1+\eta_2(n))^2}(p_1(n) - q_1(n))(p_2(n) - q_2(n)) \\
&\quad + a_1^2(n)\xi_1^2(n)(p_1(n) - q_1(n))^2 \\
&\quad + \frac{2a_1(n)c_2(n)\xi_1(n)\eta_2(n)}{(1+\eta_2(n))^2}(p_1(n) - q_1(n)) \\
&\quad \times (p_2(n) - q_2(n)) \\
&\quad - 2a_2(n)\xi_2(n)(p_2(n) - q_2(n))^2 \\
&\quad + \frac{c_1^2(n)\eta_1^2(n)}{(1+\eta_1(n))^4}(p_1(n) - q_1(n))^2 \\
&\quad - \frac{2c_1(n)\eta_1(n)}{(1+\eta_1(n))^2}(p_2(n) - q_2(n))(p_1(n) - q_1(n)) \\
&\quad + a_2^2(n)\xi_2^2(n)(p_2(n) - q_2(n))^2 \\
&\quad + \frac{2a_2(n)c_1(n)\xi_2(n)\eta_1(n)}{(1+\eta_1(n))^2}(p_2(n) - q_2(n)) \\
&\quad \times (p_1(n) - q_1(n)) \\
&\quad = \left[-2a_1(n)\xi_1(n) + a_1^2(n)\xi_1^2(n) + \frac{c_1^2(n)\eta_1^2(n)}{(1+\eta_1(n))^4}\right] \\
&\quad \times (p_1(n) - q_1(n))^2 \\
&\quad + \left[-2a_2(n)\xi_2(n) + a_2^2(n)\xi_2^2(n) + \frac{c_2^2(n)\eta_2^2(n)}{(1+\eta_2(n))^4}\right] \\
&\quad \times (p_2(n) - q_2(n))^2 \\
&\quad + 2\left[\frac{a_2(n)c_1(n)\xi_2(n)\eta_1(n)}{(1+\eta_1(n))^2} - \frac{c_1(n)\eta_1(n)}{(1+\eta_1(n))^2}\right. \\
&\qquad \left. + \frac{a_1(n)c_2(n)\xi_1(n)\eta_2(n)}{(1+\eta_2(n))^2} - \frac{c_2(n)\eta_2(n)}{(1+\eta_2(n))^2}\right] \\
&\quad \times (p_1(n) - q_1(n))(p_2(n) - q_2(n)) \\
&\quad \le \left[-2a_1^L m_1 + a_1^{U2}M_1^2 + \frac{c_1^{U2}M_1^2}{(1+m_1)^4}\right](p_1(n) - q_1(n))^2 \\
&\quad + \left[-2a_2^L m_2 + a_2^{U2}M_2^2 + \frac{c_2^{U2}M_2^2}{(1+m_2)^4}\right](p_2(n) - q_2(n))^2 \\
&\quad + \left[\frac{a_2^U c_1^U M_1 M_2}{(1+m_1)^2} + \frac{c_1^U M_1}{(1+m_1)^2} + \frac{a_1^U c_2^U M_1 M_2}{(1+m_2)^2} + \frac{c_2^U M_2}{(1+m_2)^2}\right] \\
&\quad \times \left\{(p_1(n) - q_1(n))^2 + (p_2(n) - q_2(n))^2\right\} \\
&\quad = -\left[2a_1^L m_1 - a_1^{U2}M_1^2 - \frac{c_1^{U2}M_1^2}{(1+m_1)^4} - \frac{a_2^U c_1^U M_1 M_2}{(1+m_1)^2}\right. \\
&\qquad \left. - \frac{c_1^U M_1}{(1+m_1)^2} - \frac{a_1^U c_2^U M_1 M_2}{(1+m_2)^2} - \frac{c_2^U M_2}{(1+m_2)^2}\right] \\
&\quad \times (p_1(n) - q_1(n))^2 \\
&\quad - \left[2a_2^L m_2 - a_2^{U2}M_2^2 - \frac{c_2^{U2}M_2^2}{(1+m_2)^4} - \frac{a_2^U c_1^U M_1 M_2}{(1+m_1)^2}\right. \\
&\qquad \left. - \frac{c_1^U M_1}{(1+m_1)^2} - \frac{a_1^U c_2^U M_1 M_2}{(1+m_2)^2} - \frac{c_2^U M_2}{(1+m_2)^2}\right] \\
&\quad \times (p_2(n) - q_2(n))^2 \\
&\quad = -\left\{s_1(p_1(n) - q_1(n))^2 + s_2(p_2(n) - q_2(n))^2\right\} \\
&\quad \le -\beta\left\{(p_1(n) - q_1(n))^2 + (p_2(n) - q_2(n))^2\right\} \\
&\quad = -\beta V(n),
\end{aligned} \tag{B.10}
$$

where $\beta = \min\{s_1, s_2\}$. By the conditions of Theorem 8, we have $0 < \beta < 1$, and hence, condition (iii) of Lemma 4 is satisfied. So, it follows from Lemma 4 that there exists a unique uniformly asymptotically stable almost periodic solution

$(p_1^*(n), p_2^*(n))$ of system (B.1) which is bounded by $\mathbb{S}$ for all $n \in \mathbb{Z}^+$; that is, there exists a unique uniformly asymptotically stable almost periodic solution $(x_1^*(n), x_2^*(n))$ of system (1) which is bounded by Ω for all $n \in \mathbb{Z}^+$. This completed the proof.

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