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Research Article

Some Properties of the q-Extension of the p-Adic Gamma Function

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We study the q-extension of the p-adic gamma function $\Gamma_{p,q}$. We give a new identity for the q-extension of the p-adic gamma $\Gamma_{p,q}$ in the case $p = 2$. Also, we derive some properties and new representations of the q-extension of the p-adic gamma $\Gamma_{p,q}$ in general case.

1. Introduction

Let p be a prime number and let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. It is well known that the analogous of the classical gamma function Γ in p-adic context depends on modifying the factorial function $n!$ [1]. The factorial function $(n!)_p$ in $\mathbb{Q}_p$ is defined as

$$(n!)_p = \prod_{\substack{j<n \\ (p,j)=1}} j. \tag{1}$$

The p-adic gamma function Γ_p is defined by Morita [2] as the continuous extension to $\mathbb{Z}_p$ of the function $n \to (-1)^n (n!)_p$. That is, $\Gamma_p(x)$ is defined by the formula

$$\Gamma_p(x) = \lim_{n \to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j \tag{2}$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. The p-adic gamma function $\Gamma_p(x)$ had been studied by Diamond [3], Barsky [4], and others. The relationship between some special functions and the p-adic gamma function $\Gamma_p(x)$ were investigated by Gross and Koblitz [5], Cohen and Friedman [6]. and Shapiro [7].

The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ is defined by Koblitz as follows.

Definition 1 (see [8]). Let $q \in \mathbb{C}_p$, $|q-1|_p < 1$, $q \neq 1$. The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ is defined by formula

$$\Gamma_{p,q}(x) = \lim_{n \to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} \frac{1-q^j}{1-q} \tag{3}$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. We recall that $\lim_{q \to 1} \Gamma_{p,q} = \Gamma_p$.

The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ was studied by Koblitz [8, 9], Nakazato [10], Kim et al. [11], and Kim [12].

2. Main Results

In the present work, we give a new identity for the q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ in special case $p = 2$. Also, we derive some properties and representations for the q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$.

Theorem 2. *If $p = 2$, then for all $x \in \mathbb{Z}_2$*

$$\Gamma_{2,q}(x)\,\Gamma_{2,q}(1-x) = (-1)^{1+\sigma_1(x)} \lim_{n \to x} \prod_{\substack{j<n \\ (2,j)=1}} q^j, \tag{4}$$

where σ_1 *is defined by the formula*

$$\sigma_1\left(\sum_{j=0}^{\infty} a_j 2^j\right) = a_1. \tag{5}$$

Proof. Let $p = 2$ and $n \in \mathbb{N}$. From Proposition 3 in [12] we known that

$$\Gamma_{2,q}(n+1)\Gamma_{2,q}(-n) = (-1)^{n+1-[n/2]} \prod_{\substack{j<n+1\\(2,j)=1}} q^j. \tag{6}$$

Here, $[\cdot]$ is the greatest integer function. Taking $n-1$ in place of n, the relation becomes

$$\Gamma_{2,q}(n)\Gamma_{2,q}(1-n) = (-1)^{n-[(n-1)/2]} \prod_{\substack{j<n\\(2,j)=1}} q^j. \tag{7}$$

Now, let $n = a_0 + a_1 2 + a_2 2^2 + \cdots$ in base 2. If $a_0 \neq 0$, then $a_1 = 1$ in base 2 and

$$\begin{aligned}\left[\frac{n-1}{2}\right] &= \left[\frac{\left(a_0 - 1 + a_1 2 + a_2 2^2 + \cdots\right)}{2}\right] \\ &= \left[a_1 + a_2 2 + \cdots\right] \equiv a_1 \pmod{2}.\end{aligned} \tag{8}$$

Thus, we get

$$\begin{aligned}(-1)^{n-[(n-1)/2]} &= (-1)^n(-1)^{-[(n-1)/2]} = (-1)^1(-1)^{-a_1} \\ &= (-1)^{1-a_1} = (-1)^{1+a_1} = (-1)^{1+\sigma_1}.\end{aligned} \tag{9}$$

If $a_0 = 0$, then

$$\begin{aligned}\left[\frac{n-1}{2}\right] &= \left[\frac{\left(-1 + a_1 2 + a_2 2^2 + \cdots\right)}{2}\right] \\ &= \left[\frac{\left(1 + (a_1 - 1)2 + a_2 2^2 + \cdots\right)}{2}\right] \\ &\equiv a_1 - 1 \pmod{2}.\end{aligned} \tag{10}$$

Hence,

$$\begin{aligned}(-1)^{n-[(n-1)/2]} &= (-1)^n(-1)^{-[(n-1)/2]} \\ &= (-1)^2(-1)^{-(a_1-1)} \\ &= (-1)^{2+a_1-1} \\ &= (-1)^{1+a_1} \\ &= (-1)^{1+\sigma_1(n)}.\end{aligned} \tag{11}$$

Thus, we have

$$\Gamma_{2,q}(n)\Gamma_{2,q}(1-n) = (-1)^{1+\sigma_1(n)} \prod_{\substack{j<n\\(2,j)=1}} q^j \tag{12}$$

and thus, we obtain

$$\Gamma_{2,q}(x)\Gamma_{2,q}(1-x) = (-1)^{1+\sigma_1(x)} \lim_{n\to x} \prod_{\substack{j<n\\(2,j)=1}} q^j. \tag{13}$$

□

We recall that the q-factorial $[n;q]!$ is defined in [13] by the formula

$$[n;q]! = [n;q][n-1;q]\cdots[2;q][1;q] \tag{14}$$

for $n \geq 1$, where

$$[x;q] = \frac{1-q^x}{1-q}. \tag{15}$$

Note that for $n = 0$, we can define $[0;q]! = 1$.

We use the following theorem to prove our results.

Theorem 3 (see [12]). *Let $n \in \mathbb{N}$. Then,*

$$\Gamma_{p,q}(n+1) = (-1)^{n+1} \frac{[n;q]!}{[p;q]^{[n/p]}\left[[n/p];q^p\right]!}, \tag{16}$$

where $[\cdot]$ is the greatest integer function. In particular,

$$\left[p^n - 1;q\right]! = (-1)^p[p;q]^{p^{n-1}-1}\left[p^{n-1}-1;q^p\right]!\Gamma_{p,q}(p^n). \tag{17}$$

Theorem 4. *Let $n \in \mathbb{N}$ and let s_n be the sum of the digits of $n = \sum_{j=0}^{s} a_j p^j$ $(a_s \neq 0)$ in base p. Then*

(a) $[[n/p^s];q]! = (-1)^{n+1-s}(-[p;q])^{(n-s_n)/(p-1)}$
$\prod_{j=0}^{s-1}[[n/p^{j+1}];q^p]!/[[n/p^j];q]!\prod_{j=0}^{s}\Gamma_{p,q}([n/p^j]+1)$

(b) $[n;q]! = (-1)^{n+1-s}(-[p;q])^{(n-s_n)/(p-1)}[[n/p];q^p]!$
$\prod_{j=1}^{s}[[n/p^{j+1}];q^p]!/[[n/p^j];q]!\prod_{j=0}^{s}\Gamma_{p,q}([n/p^j]+1)$.

Proof. From the Theorem 3 we know that

$$[n;q]! = (-1)^{n+1}[p;q]^{[n/p]}\left[\left[\frac{n}{p}\right];q^p\right]!\Gamma_{p,q}(n+1). \tag{18}$$

By taking $[n/p^0], [n/p^1], \ldots, [n/p^s]$ instead of n, respectively, we get the relations

$$\begin{aligned}\left[\left[\frac{n}{p^0}\right];q\right]! &= (-1)^{[n/p^0]+1}[p;q]^{[n/p^1]} \\ &\quad\times\left[\left[\frac{n}{p^1}\right];q^p\right]!\Gamma_{p,q}\left(\left[\frac{n}{p^0}\right]+1\right), \\ \left[\left[\frac{n}{p^1}\right];q\right]! &= (-1)^{[n/p^1]+1}[p;q]^{[n/p^2]} \\ &\quad\times\left[\left[\frac{n}{p^2}\right];q^p\right]!\Gamma_{p,q}\left(\left[\frac{n}{p^1}\right]+1\right), \\ &\vdots \\ \left[\left[\frac{n}{p^s}\right];q\right]! &= (-1)^{[n/p^s]+1}[p;q]^{[n/p^{s+1}]} \\ &\quad\times\left[\left[\frac{n}{p^{s+1}}\right];q^p\right]!\Gamma_{p,q}\left(\left[\frac{n}{p^s}\right]+1\right).\end{aligned} \tag{19}$$

By multiplying of the equalities above, we can easily obtain

$$\begin{aligned}\left[\left[\frac{n}{p^s}\right];q\right]! &= (-1)^{[n/p^0]+\cdots+[n/p^s]+s+1}[p;q]^{[n/p^1]+\cdots+[n/p^{s+1}]}\\ &\quad\times\left[\left[\frac{n}{p^{s+1}}\right];q^p\right]!\prod_{j=0}^{s-1}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\\ &\quad\times\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right)\\ &= (-1)^{(n-s_n)/(p-1)}(-1)^{n+1-s}[p;q]^{(n-s_n)/(p-1)}\\ &\quad\times\left[\left[\frac{n}{p^{s+1}}\right];q^p\right]!\prod_{j=0}^{s-1}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\\ &\quad\times\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right).\end{aligned}\tag{20}$$

Therefore, we get the relation (a)

$$\begin{aligned}\left[\left[\frac{n}{p^s}\right];q\right]! &= (-1)^{n+1-s}\left(-[p;q]^{(n-s_n)/(p-1)}\right)\\ &\quad\times\prod_{j=0}^{s-1}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right),\\ [n;q]! &= (-1)^{[n/p^0]+\cdots+[n/p^s]+s+1}[p;q]^{[n/p^1]+\cdots+[n/p^{s+1}]}\\ &\quad\times\left[\left[\frac{n}{p}\right];q^p\right]!\prod_{j=1}^{s}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\\ &\quad\times\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right)\\ &= (-1)^{(n-s_n)/(p-1)}(-1)^{n+1-s}[p;q]^{(n-s_n)/(p-1)}\\ &\quad\times\left[\left[\frac{n}{p}\right];q^p\right]!\prod_{j=1}^{s}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\\ &\quad\times\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right).\end{aligned}\tag{21}$$

Therefore, we get the relation (b)

$$\begin{aligned}[n;q]! &= (-1)^{n+1-s}(-[p;q])^{(n-s_n)/(p-1)}\left[\left[\frac{n}{p}\right];q^p\right]!\\ &\quad\times\prod_{j=1}^{s}\frac{\left[\left[n/p^{j+1}\right];q^p\right]!}{\left[\left[n/p^j\right];q\right]!}\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right).\end{aligned}\tag{22}$$

□

Theorem 5. *Let $n\in\mathbb{N}$ and let $n=\sum_{j=0}^{s}a_jp^j$ ($a_s\neq 0$). Then*

$$\begin{aligned}\left[p^n-1;q\right]! &= (-1)^p(-[p;q])^{(p^n-1)/(p-1)}\\ &\quad\times[p;q]^{-n}\left[p^{n-1}-1;q^p\right]!\\ &\quad\times\prod_{j=0}^{n-2}\frac{\left[p^j-1;q^p\right]!}{\left[p^{j+1}-1;q\right]!}\prod_{j=0}^{n}\Gamma_{p,q}\left(p^j\right).\end{aligned}\tag{23}$$

Proof. From Theorem 3 it follows that

$$\left[p^j-1;q\right]! = (-1)^p[p;q]^{p^{j-1}-1}\left[p^{j-1}-1;q^p\right]!\Gamma_{p,q}\left(p^j\right).\tag{24}$$

Taking of $0,1,\ldots,n$ instead of j, respectively, we have the equalities

$$\begin{aligned}\left[p^0-1;q\right]! &= 1 = (-1)\,\Gamma_{p,q}\left(p^0\right),\\ \left[p^1-1;q\right]! &= (-1)^p[p;q]^{p^0-1}\left[p^0-1;q^p\right]!\Gamma_{p,q}\left(p^1\right),\\ \left[p^2-1;q\right]! &= (-1)^p[p;q]^{p^1-1}\left[p^1-1;q^p\right]!\Gamma_{p,q}\left(p^2\right),\\ &\vdots\\ \left[p^n-1;q\right]! &= (-1)^p[p;q]^{p^{n-1}-1}\left[p^{n-1}-1;q^p\right]!\Gamma_{p,q}\left(p^n\right).\end{aligned}\tag{25}$$

By multiplying of the equalities above, we can easily obtain

$$\begin{aligned}\left[p^n-1;q\right]! &= (-1)^{np+1}[p;q]^{p^0+p^1+\cdots+p^{n-1}-n}\left[p^{n-1}-1;q^p\right]!\\ &\quad\times\prod_{j=0}^{n-2}\frac{\left[p^j-1;q^p\right]!}{\left[p^{j+1}-1;q\right]!}\prod_{j=0}^{n}\Gamma_{p,q}\left(p^j\right).\end{aligned}\tag{26}$$

Thus,

$$\begin{aligned}\left[p^n-1;q\right]! &= (-1)^p(-[p;q])^{(p^n-1)/(p-1)}\\ &\quad\times[p;q]^{-n}\left[p^{n-1}-1;q^p\right]!\\ &\quad\times\prod_{j=0}^{n-2}\frac{\left[p^j-1;q^p\right]!}{\left[p^{j+1}-1;q\right]!}\prod_{j=0}^{n}\Gamma_{p,q}\left(p^j\right).\end{aligned}\tag{27}$$

□

Lemma 6. *Let $n\in\mathbb{Z}^+$, $n=\sum_{j=0}^{s}a_jp^j$ ($a_s\neq 0$), and let p be a prime number. Then, for $j=0,1,\ldots,s$*

$$\frac{\left[\left[n/p^j\right];q\right]!}{[p;q]^{[n/p^j]}\left[\left[n/p^j\right];q^p\right]!} = \prod_{k=1}^{[n/p^j]}\frac{1-q^k}{1-q^{kp}}\quad(0\le k\le s).\tag{28}$$

Proof. For $j = 0$

$$\frac{[n;q]!}{[p;q]^n [n;q^p]!} = \frac{[1;q][2;q]\cdots[n,q]}{[p;q]^n [1;q^p][2;q^p]\cdots[n;q^p]}$$
$$= \left(\frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^n}{1-q}\right) \times \left(\left(\frac{1-q^p}{1-q}\right)^n \frac{1-q^p}{1-q^p}\cdots\frac{1-q^{np}}{1-q^p}\right)^{-1}$$
$$= \left(\frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^n}{1-q}\right) \times \left(\frac{1-q^p}{1-q}\frac{1-q^{2p}}{1-q}\cdots\frac{1-q^{np}}{1-q}\right)^{-1}$$
$$= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q^p)(1-q^{2p})\cdots(1-q^{np})}. \tag{29}$$

For $1 \le j \le s$ it follows that

$$\frac{[[n/p^j];q]!}{[p;q]^{[n/p^j]}[[n/p^j];q^p]!} = \frac{[1;q][2;q]\cdots[[n/p^j],q]}{[p;q]^{[n/p^j]}[1;q^p][2;q^p]\cdots[[n/p^j];q^p]}$$
$$= \left(\frac{1-q}{1-q}\frac{1-q^2}{1-q}\cdots\frac{1-q^{[n/p^j]}}{1-q}\right) \times \left(\left(\frac{1-q^p}{1-q}\right)^{[n/p^j]}\frac{1-q^p}{1-q^p}\cdots\frac{1-q^{[n/p^j]p}}{1-q^p}\right)^{-1} \tag{30}$$
$$= \frac{(1-q)(1-q^2)\cdots\left(1-q^{[n/p^j]}\right)}{(1-q^p)(1-q^{2p})\cdots\left(1-q^{[n/p^j]p}\right)}.$$

Then, we obtain

$$\frac{[[n/p^j];q]!}{[p;q]^{[n/p^j]}[[n/p^j];q^p]!} = \prod_{k=1}^{[n/p^j]}\frac{1-q^k}{1-q^{kp}}. \tag{31}$$

□

Theorem 7. *Let $n \in \mathbb{N}$ and let s_n be the sum of the digits of $n = \sum_{j=0}^{s} a_j p^j$ $(a_s \neq 0)$ in base p. Then*

$$[n;q]! = (-1)^{((n-s_n)/(p-1))+n+1-s}\prod_{k=1}^{[n/p^1]}\frac{(1-q^{kp})}{(1-q^k)}\cdots \prod_{k=1}^{[n/p^s]}\frac{(1-q^{kp})}{(1-q^k)}\prod_{j=0}^{s}\Gamma_{p,q}\left(\left[\frac{n}{p^j}\right]+1\right). \tag{32}$$

Proof. This theorem can be proved by using Theorem 4 and Lemma 6. □

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