Research Article

Continuous-Time Portfolio Selection and Option Pricing under Risk-Minimization Criterion in an Incomplete Market

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We study option pricing with risk-minimization criterion in an incomplete market where the dynamics of the risky underlying asset are governed by a jump diffusion equation. We obtain the Radon-Nikodym derivative in the minimal martingale measure and a partial integrodifferential equation (PIDE) of European call option. In a special case, we get the exact solution for European call option by Fourier transformation methods. Finally, we employ the pricing kernel to calculate the optimal portfolio selection by martingale methods.

1. Introduction

Option pricing problem is one of the predominant concerns in the financial market. Since the advent of the justly celebrated Black-Scholes option pricing formula in [1], there has been an increasing amount of literature describing the theory and its practice. Due to drawbacks of the Black-Scholes model which cannot explain numerous empirical facts: large, sudden movements in prices, heavy tails, and the incompleteness of markets, the concentration of losses in a few large downward moves, many option valuation models have been proposed and tested to fit those empirical facts. The jump diffusion models which have well computational and statistical features are successful to solve those drawbacks of the Black-Scholes model in [2–9]. Therefore, in this paper, with those nice properties, we choose this jump diffusion model.

Different from the Black-Scholes framework, we use jump diffusion to describe the price dynamics of underlying assets and let the market of our model be incomplete; that is, it is not possible to replicate the payoff of every contingent claim by a portfolio, and there are several equivalent martingale measures. How to choose a consistent pricing measure from the set of equivalent martingale measures becomes an important problem. That means we need to find some criteria to determine one from the set of equivalent

martingale measures in some economically or mathematically motivated fashion. Föllmer and Leukert (2000), Kallsen (1999), Cvitanić et al. (2001), and Bielecki and Jeanblanc (2008) in [10–13] identified a unique equivalent martingale measure by utility maximization. Then the option valuation under the minimal martingale measure was further developed by several researchers. Schweizer (1991), and Föllmer and Schweizer (1991) in [14, 15] found that in the minimal martingale measure, a unique risk-minimizing (or optimal) strategy hedging of contingent claims in incomplete market exists. In our paper, we name the criterion in the minimal martingale measure as *risk-minimization criterion*. Thus, in the incomplete market, option pricing is approximately possible with risk-minimization criterion. As presented in this paper, our work is based on the task of Föllmer and Schweizer in [14, 15], and the purpose of this paper is to find the minimal martingale measure and the measure switch of asset prices processes with jump diffusion. By using the minimal martingale measure, we obtain the Radon-Nikodym derivative which links the physical and risk-neutral densities and a PIDE of the European option. In a special case, we get the exact solution for European call option by Fourier transformation methods. Recently, Ruan et al. (2013) in [16] studied option pricing with risk-minimization criterion in an incomplete market when the dynamics of the risky

underlying asset are governed by a jump diffusion equation with stochastic volatility.

General equilibrium framework is also a popular method to deal with the option pricing in an incomplete market. Pan (2002), Liu and Pan (2003), and Liu et al. (2005) in [17–19] derived the pricing kernel with some restrictions of jump sizes in a general equilibrium setting. Recently, Zhang et al. (2012) in [9] presented an analytical form for the pricing kernel without any distributional assumption on the jumps, the moments in physical and risk-neutral measure, respectively. In this paper, we get pricing kernel and the physical moments in and the risk-neutral physical moments without any distributional assumption on the jumps which are similar to the results of Zhang et al. (2012) [9].

Anther contribution of our paper is finding the optimal portfolio selection and terminal wealth by martingale methods. The popular methods to solve the optimal portfolio problem are stochastic control methods, which derive some complex partial differential equations (PDE). For example, see [20–23]. However, martingale methods, built around the ideas of equivalent martingale measure in complete markets, began with Harrison and Kreps (1979) in [24] and were further developed by Harrison and Pliska (1983) in [25]. They claimed that an optimal terminal wealth was first identified by solving a static optimization problem and then an efficient portfolio was obtained by replicating the optimal terminal wealth. Karatzas et al. (1991) in [26] and He and Pearson (1991) in [27] considered a Brownian model in which the number of stocks was strictly less than the dimension of the driving Brownian motion. In all these papers, it was assumed that the underlying assets satisfy just diffusion processes. In this paper, considering a jump diffusion model, we employ the pricing kernel in minimal martingale measure to degenerate a stochastic optimal terminal wealth problem into a static optimization problem.

The rest of the paper is organized as follows. In Section 2, we present the model for the underlying market and Doob-Meyer decomposition of the risky asset. In Section 3, we investigate an explicit representation of the density process of the minimal martingale measure. In Section 4, we derive a PIDE with respect to the European option and present an exact solution of European call option by Fourier transformation methods in a special case. The optimal portfolio selection problem by martingale methods is studied in Section 5 and conclusions are given in Section 6.

2. The Model

In this paper, we consider the financial market with the following two basic assets:

(i) *a bond* whose price B_t at time t is given by

$$dB_t = rB_t dt, \qquad B_0 = 1; \tag{1}$$

(ii) *a stock* whose price S_t at time t is given by

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + \int_{R_0} (e^x - 1) \overline{N}(dt, dx), \quad S_0 > 0. \tag{2}$$

Here $t \in [0, T]$, $x \in R_0 \subset R$, constants $\mu > r > 0$, $\sigma > 0$, and $T > 0$; on the filtered complete space $(\Omega, F, \{F_t\}_{t \geq 0}, P)$, there are W_t which is 1-dimensional Brownian motion and $\overline{N}(dt, dx) = N(dt, dx) - v(dx)dt$ which is the compensated Poisson random measure. $N(dt, dx)$ is a Poisson random measure with compensator $v(dx)$. Additionally, $\int_{R_0} (e^x - 1)^2 v(dx) < \infty$.

In this model, the jump process $\eta(t) = \int_0^t \int_{R_0} (e^x - 1) \overline{N}(du, dx)$ built from a compensated Poisson random measure $\overline{N}(dt, dx)$ is the sum of its i.i.d. jumps, where the jumps occur according to a Poisson process $N(dt, dx)$. In particular, if we denote $R_0 = (-\infty, +\infty)$, $\overline{N}(dt, dx) = dN(t) - v(dx)dt$, $v(dx) = \lambda f(x)dx$ where λ is the jump intensity and $f(x)$ is distribution of the jump size x, then the compensated Poisson integral $\eta(t)$ can be rewritten as $\eta(t) = \sum_{i=1}^{N(t)} (e^{x_i} - 1) - t \int_{-\infty}^{+\infty} (e^x - 1) f(x)dx$.

Now, we want to find a unique optimal strategy hedging of contingent claims under risk-minimization criterion. Based on [15], it is equivalent to find the minimal martingale measure from the set of equivalent martingale measures and then obtain approximate prices of contingent claims.

With the Doob-Meyer decomposition, the discounted risky asset price process, $\widehat{S}_t = e^{-rt}S_t$, is a special semimartingale and can be written as

$$\widehat{S}_t = \widehat{S}_0 + M_t + A_t \tag{3}$$

with

$$M_t = \int_0^t \widehat{S}_{u^-} \sigma dW_u^S + \int_0^t \int_{R_0} \widehat{S}_{u^-} (e^x - 1) \overline{N}(du, dx), \tag{4}$$

$$A_t = \int_0^t \widehat{S}_{u^-} (\mu - r) du, \tag{5}$$

where M_t is the martingale part of $\widehat{S}_t$ and A_t is the predictable process of finite variation.

3. Minimal Martingale Measure

We introduce the notions of minimal martingale measure in this section. Föllmer and Schweizer (1991) [15] noticed that the optimal hedging strategy can be computed in terms of the minimal martingale measure. Furthermore, it is uniquely determined. Hence, in the minimal martingale measure, the Radon-Nikodym derivative can be found and computed. Before that, we define the minimal martingale measure.

Definition 1 (see [15]). A local martingale measure Q, equivalent to the original measure P, is called minimal if $Q = P$ on F_t and if any square-integrable P-martingale L which is P orthogonal to M remains a local martingale under Q.

Theorem 2 (see [15]). *(i) The minimal martingale measure Q is uniquely determined.*

(ii) $\widetilde{P}$ exists if and only if there exists a predictable process β_t that satisfies

$$Z_t = \frac{dQ}{dP} = 1 + \int_0^t \beta_t \, dM_t. \tag{6}$$

Using Theorem 2, we obtain the following theorem for computing the Radon-Nikodym derivative.

Theorem 3. *The Radon-Nikodym derivative in the minimal martingale measure Q is*

$$Z_t = \exp\left\{ -\theta\sigma W_t - \frac{1}{2}\theta^2\sigma^2 t \right.$$
$$+ \int_0^t \int_{R_0} \ln\left(1 - \theta\left(e^x - 1\right)\right) N\left(du, dx\right) \tag{7}$$
$$\left. + t \int_{R_0} \theta\left(e^x - 1\right) v\left(dx\right) \right\},$$

where

$$\theta = \frac{\mu - r}{\sigma^2 + \int_{R_0} \left(e^x - 1\right)^2 v\left(dx\right)}. \tag{8}$$

Proof. The theory of the Girsanov transformation shows that the predictable process of bounded variation can also be computed in terms of Z_t:

$$-dA_t = \frac{1}{Z_{t^-}} d\langle M, Z\rangle_t. \tag{9}$$

Throughout this paper, we make use of the notations that $\langle A, B\rangle_t$ defined quadratic variation process between A and B and denote $\langle A\rangle_t \doteq \langle A, A\rangle_t$. Under Q, the predictable process of bounded variation in the Doob-Meyer decomposition of M is given by

$$\frac{1}{Z_{t^-}} d\langle M, Z\rangle_t = \frac{\beta_t}{Z_{t^-}} d\langle M\rangle_t = -dA_t. \tag{10}$$

Using Theorem 2 and (10), we have

$$Z_t = 1 - \int_0^t Z_{u^-} \frac{dA_u}{d\langle M\rangle_u} dM_u. \tag{11}$$

Denote $dY_u = -(dA_u/d\langle M\rangle_u)dM_u$; then (11) can be written as

$$Z_t = 1 + \int_0^t Z_{u^-} dY_u. \tag{12}$$

From (4), we get

$$\langle M\rangle_t = \left\langle \int_0^t \widehat{S}_{u^-}\sigma dW_u^S + \int_0^t \int_{R_0} \widehat{S}_{u^-}\left(e^x - 1\right)\overline{N}\left(du, dx\right) \right\rangle$$
$$= \int_0^t \widehat{S}_{u^-}^2\sigma^2 du + \int_0^t \int_{R_0} \widehat{S}_{u^-}^2\left(e^x - 1\right)^2 v\left(dx\right) du$$
$$= \int_0^t \widehat{S}_{u^-}^2\left(\sigma^2 + \int_{R_0}\left(e^x - 1\right)^2 v\left(dx\right)\right) du. \tag{13}$$

Hence

$$dY_u = -\frac{dA_u}{d\langle M\rangle_u} dM_u$$
$$= -\frac{\widehat{S}_{u^-}\left(\mu - r\right) du}{\widehat{S}_{u^-}^2\left(\sigma^2 + \int_{R_0}\left(e^x - 1\right)^2 v\left(dx\right)\right) du}$$
$$\times \left[\widehat{S}_{u^-}\left(\sigma dW_u + \int_{R_0}\left(e^x - 1\right)\overline{N}\left(du, dx\right)\right)\right]$$
$$= -\frac{\left(\mu - r\right)\left(\sigma dW_u + \int_{R_0}\left(e^x - 1\right)\overline{N}\left(du, dx\right)\right)}{\sigma^2 + \int_{R_0}\left(e^x - 1\right)^2 v\left(dx\right)}. \tag{14}$$

From (12), we know that Z_t is the Doleans-Dade exponential. Thus, we obtain

$$Z_t = 1 + \int_0^t Z_{u^-} dY_u, \quad Z_0 = 1,$$
$$dY_t = -\theta\left(\sigma dW_t + \int_{R_0}\left(e^x - 1\right)\overline{N}\left(dt, dx\right)\right), \tag{15}$$

where

$$\theta = \frac{\mu - r}{\sigma^2 + \int_{R_0}\left(e^x - 1\right)^2 v\left(dx\right)}. \tag{16}$$

Solving (15), we obtain Z_t in Theorem 3. □

Remark 4. In the minimal martingale measure Q, we obtain unique Radon-Nikodym derivative which links the physical and minimal martingale measure. The Radon-Nikodym derivative is very important to get the PDE with respect to option price and to determine a pricing kernel which enables the portfolio selection problem to be solved by martingale methods. More details are presented in Section 5.

Remark 5. From the Girsanov theorem, the Brown motion in the minimal martingale measure Q is

$$\widetilde{W}_t = W_t + \theta\sigma t, \tag{17}$$

and the compensatory of $N(du, dy)$ is

$$\widetilde{v}\left(dx\right) dt = \left(1 - \theta\left(e^x - 1\right)\right) v\left(dx\right) dt,$$
$$\widetilde{N}\left(dt, dx\right) = N\left(dt, dx\right) - \widetilde{v}\left(dx\right) dt. \tag{18}$$

Then (2) under the minimal martingale measure $\widetilde{P}$ is written as

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma d\widetilde{W}_t^S + \int_{R_0}\left(e^x - 1\right)\widetilde{N}\left(dt, dx\right)$$
$$- \theta\sigma^2 dt - \int_{R_0}\theta\left(e^x - 1\right)^2 v\left(dx\right) dt, \quad S_0 > 0. \tag{19}$$

To guarantee that $\widehat{S}_t$ is a martingale under the minimal martingale measure Q, the following corollary is necessary.

Corollary 6. *In the minimal martingale measure Q, $\widehat{S}_t$ is a martingale if and only if*

$$\mu - r - \theta\sigma^2 - \int_{R_0} \theta(e^x - 1)^2 v(dx) = 0. \tag{20}$$

Proof. Substituting $S_t = e^{-rt}\widehat{S}_t$ into (19), since $\widehat{S}_t$ is a martingale, the drift term must be identical to zero. Then we can get (20). □

Remark 7. Using (20), (19) in the minimal martingale measure Q is written as

$$\frac{dS_t}{S_{t^-}} = r_t dt + \sigma d\widetilde{W}_t^S + \int_{R_0} (e^x - 1)\widetilde{N}(dt, dx), \quad S_0 > 0. \tag{21}$$

The minimal martingale measure Q is a risk-neutral measure.

4. The Pricing Formula for European Call Option

In the minimal martingale measure Q, the price of the European call option $C(t, S_t)$ at time t with strike price K and maturity date T is given by

$$C(t, S_t) = E^Q\left[e^{-r(T-t)}(S_T - K)^+ \mid F_t\right], \tag{22}$$

and $C(T, S_T) = (S_T - K)^+$.

By the fact that the discounted price of the European call option is a martingale under Q, we can obtain the following theorem.

Theorem 8. *In the minimal martingale measure Q, the price of the European call option satisfies the following PIDE:*

$$0 = -rC(t, S_t) + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}rS_{t^-} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_{t^-}^2$$
$$+ \int_{R_0}\left(C(t, S_{t^-}e^x) - C(t, S_{t^-}) - (e^x - 1)\frac{\partial C}{\partial S}S_{t^-}\right)\widetilde{v}(dx) \tag{23}$$

and $C(T, S_T) = (S_T - K)^+$.

Proof. The total derivative of the discounted option price is

$$d\left(e^{-rt}C(t, S_t)\right)$$
$$= -r_t e^{-rt}C(t, S_t)dt + e^{-rt}\frac{\partial C}{\partial t}dt$$
$$+ e^{-rt}\frac{\partial C}{\partial S}dS^C + \frac{1}{2}e^{-rt}\frac{\partial^2 C}{\partial S^2}dS^C dS^C$$
$$+ e^{-rt}\int_{R_0}(C(t, S_{t^-}y) - C(t, S_{t^-}))N(dt, dx)$$
$$= e^{-rt}\left\{-r_t C(t, S_t)dt + \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}S_{t^-}\right.$$
$$\left. \times \left(rdt + \sigma d\widetilde{W}_t - \int_{R_0}(e^x - 1)\widetilde{v}(dx)dt\right)\right.$$

$$+ \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_{t^-}^2 dt$$
$$+ \int_{R_0}(C(t, S_{t^-}e^x) - C(t, S_{t^-}))\widetilde{N}(dt, dx)$$
$$+ \int_{R_0}(C(t, S_{t^-}e^x) - C(t, S_{t^-}))\widetilde{v}(dx)dt\Big\}$$
$$= e^{-rt}\left\{-rC(t, S_t) + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}r_t S_{t^-} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma_t^2 S_{t^-}^2\right.$$
$$+ \int_{R_0}\left(C(t, S_{t^-}e^x) - C(t, S_{t^-})\right.$$
$$\left.\left. - (e^x - 1)\frac{\partial C}{\partial S}S_{t^-}\right)\widetilde{v}(dx)\right\}dt$$
$$+ e^{-rt}\left\{\frac{\partial C}{\partial S}S_{t^-}\sigma d\widetilde{W}_t^S\right.$$
$$\left. + \int_{R_0}(C(t, S_{t^-}e^x) - C(t, S_{t^-}))\widetilde{N}(dt, dx)\right\}. \tag{24}$$

We make the drift term be zero, since the discounted price of the European put option is a martingale. Then we obtain the equation in Theorem 8. □

It is difficult to get the solution of European call option $C(t, S_t)$ from (23). Therefore, we degenerate the model (2) into a special case with following assumption to get an exact solution of the option price.

Assumption 9. (i) In physical measure P, we denote $R_0 = (-\infty, +\infty)$, $\overline{N}(dt, dx) = dN(t) - \lambda f(x)dx$, where λ is the jump intensity and $f(x)$ is distribution of the jump size x. Moreover, we denote $E[\cdot]$ as expectation operator in physical measure P.

(ii) In the minimal martingale measure Q, using Remark 4, we have $\widetilde{N}(dt, dx) = dN(t) - \lambda\widetilde{f}(x)dx$, where $\widetilde{f}(x) = (1 - \theta(e^x - 1))f(x)$ and denote $E^Q[\cdot]$ as expectation operator in this measure.

Then the stock price process (1) in physical measure can be written as

$$\frac{dS_t}{S_{t^-}} = \mu dt - \lambda E\left[(e^x - 1)\right]dt + \sigma dW_t$$
$$+ (e^x - 1)dN(t), \quad S_0 > 0, \tag{25}$$

and the Radon-Nikodym derivative can be written as

$$Z_t = \exp\left\{-\theta\sigma W_t + \theta\lambda E\left[(e^x - 1)\right]t\right.$$
$$\left. - \frac{1}{2}\theta^2\sigma^2 t + \sum_{i=1}^{N(t)}\ln\left(1 - \theta\left(e^{x_i} - 1\right)\right)\right\}, \tag{26}$$

where $\theta = (\mu - r)/(\sigma^2 + \lambda E[(e^x - 1)^2])$.

The stock price process (21) in the minimal martingale measure Q can be written as

$$\frac{dS_t}{S_{t^-}} = r_t dt - \lambda E^Q\left[(e^x - 1)\right] dt + \sigma d\widetilde{W}_t$$

$$+ (e^x - 1) dN(t), \quad S_0 > 0, \tag{27}$$

and the price of the European call option satisfies the following equation:

$$0 = -rC(t, S) + \frac{\partial C}{\partial t} + \left(r - \lambda E^Q\left[(e^x - 1)\right]\right)\frac{\partial C}{\partial S}S$$

$$+ \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + \lambda\left\{E^Q\left[(C(t, Se^x))\right] - C(t, S)\right\} \tag{28}$$

and $C(T, S_T) = (S_T - K)^+$.

Using Fourier transformation methods, it is easy to obtain the solution of (28). The solution is shown in the following theorem.

Theorem 10. *The pricing formula of European call option under Assumption 9 is given by*

$$C(S, \tau) = -\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ik\ln S}\frac{K^{ik+1}}{k^2 - ik}e^{A\tau}dk, \tag{29}$$

where

$$A = -(1 + ik)r + ik\lambda E^Q(e^x - 1) - \frac{1}{2}\sigma^2(k^2 - ik)$$

$$+ \lambda E^Q\left[e^{-ikx} - 1\right], \quad \tau = T - t. \tag{30}$$

Proof. First, we denote $X = \ln S$ and $g(X, t) := c(e^X, t) = C(S, t)$; then, the PDE (28) can be rewritten as

$$-g_t = -rg + \frac{1}{2}\sigma^2 g_{XX} + \left[r - \lambda E^Q(e^x - 1) - \frac{1}{2}\sigma^2\right]g_X$$

$$+ \lambda\left\{E^Q\left[g(X + x, t)\right] - g(X, t)\right\},$$

$$g(X, T) = \left(e^X - K\right)^+. \tag{31}$$

Let $G(k, V, t)$ be Fourier transform of $g(X, V, t)$,

$$G(k, t) = \int_{-\infty}^{+\infty} e^{ikX}g(X, t)dX \tag{32}$$

and $G(k, T) = -K^{ik+1}/(k^2 - ik)$.

Denote $k = k_r + ik_i$, $\tau = T - t$; the inverse Fourier transform is given by

$$g(X, t) = \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}G(k, t)dk. \tag{33}$$

Then, the PDE (31) can be rewritten as

$$G_\tau = -(1 + ik)rG + ik\lambda E^Q(e^x - 1)G$$

$$-\frac{1}{2}\sigma^2(k^2 - ik)G + \lambda E^Q\left[e^{-ikx} - 1\right]G, \tag{34}$$

$$G(k, 0) = -\frac{K^{ik+1}}{k^2 - ik}.$$

The solution of (34) is given by

$$G(k, \tau) = -\frac{K^{ik+1}}{k^2 - ik}e^{A\tau}, \tag{35}$$

where $A = -(1 + ik)r + ik\lambda E^Q(e^x - 1) - (1/2)\sigma^2(k^2 - ik) + \lambda E^Q[e^{-ikx} - 1]$.

Hence,

$$g(X, \tau) = -\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\frac{K^{ik+1}}{k^2 - ik}e^{A\tau}dk. \tag{36}$$

Then

$$C(S, \tau) = -\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ik\ln S}\frac{K^{ik+1}}{k^2 - ik}e^{A\tau}dk. \tag{37}$$

$\square$

Note 1. Although the pricing formula (29) contains a complex integral, the result is real.

Then, we try to find the relationship of central moments between the physical measure and risk-neutral measure which can help us to study the negative variance risk premium, the implied volatility smirk, and the prediction of realized skewness. The typical literature is Bakshi et al. (2003) in [28], Carr and Wu (2009) in [29], and Neuberger (2012) in [30].

However, to the best of our knowledge, except Zhang et al. (2012) in [9], there is no literature that studies this relationship. First we present a lemma from Zhang et al. (2012) in [9] to prove Proposition 12 which presents the relationship of central moments between the physical measure and risk-neutral measure Q.

Lemma 11. *The first moment, and the second, third, and fourth central moments of the continuously compounded return within τ, $R_\tau = \ln S_T/S_t$, in the risk-neutral measure Q are given by*

$$E_t^Q(R_\tau) = \left[\mu - \frac{1}{2}\sigma^2 - \lambda^Q E^Q(e^x - 1 - x)\right]\tau,$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^2 = \tau\sigma^2 + \lambda^Q\tau\left[\left(\mu_x^Q\right)^2 + \text{Var}^Q(x)\right],$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^3$$

$$= \lambda^Q\tau\left[\left(\mu_x^Q\right)^3 + 3\mu_x^Q\text{Var}^Q(x) + 3rd^Q(x)\right],$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^4$$

$$= 3\left(\lambda^Q\tau\left[\left(\mu_x^Q\right)^2 + \text{Var}^Q(x)\right] + \tau\sigma^2\right)^2$$

$$+ \lambda^Q\tau\left[\left(\mu_x^Q\right)^4 + 6\left(\mu_x^Q\right)^2\text{Var}^Q(x)\right.$$

$$\left. + 4\mu_x^Q 3rd^Q(x) + 4th^Q(x)\right], \tag{38}$$

where $\mu_x^Q = E^Q(x)$ is the first moment and $\text{Var}^Q(x) = E^Q[(x - \mu_x^Q)^2]$, $3rd^Q(x) = E^Q[(x - \mu_x^Q)^3]$, and $4th^Q(x) = E^Q[(x - \mu_x^Q)^4]$ are the second, third, and fourth central moments in the risk-neutral measure of random number x. λ^Q is the jump intensity in the risk-neutral measure.

The first moment and the second, third, and fourth central moments of the continuously compounded return in the physical measure are given by

$$E_t(R_\tau) = \left[\mu - \frac{1}{2}\sigma^2 - \lambda E(e^x - 1 - x)\right]\tau,$$

$$E_t[R_\tau - E_t(R_\tau)]^2 = \tau\sigma^2 + \lambda\tau\left[\mu_x^2 + \text{Var}(x)\right],$$

$$E_t[R_\tau - E_t(R_\tau)]^3 = \lambda\tau\left[\mu_x^3 + 3\mu_x\text{Var}(x) + 3rd(x)\right],$$

$$E_t[R_\tau - E_t(R_\tau)]^4$$

$$= 3\left(\lambda\tau\left[\mu_x^2 + \text{Var}(x)\right] + \tau\sigma^2\right)^2$$

$$+ \lambda\tau\left[\mu_x^4 + 6\mu_x^2\text{Var}(x) + 4\mu_x 3rd(x) + 4th(x)\right], \tag{39}$$

where $\mu_x = E(x)$ is the first moment and $\text{Var}(x) = E[(x - \mu_x)^2]$, $3rd(x) = E[(x - \mu_x)^3]$, and $4th(x) = E[(x - \mu_x)^4]$ are second, third, and fourth central moments in the physical measure of random number x. λ is the jump intensity in the physical measure.

Proof. See Zhang et al. (2012) in [9]. □

Proposition 12. *The first moment and the second, third, and fourth central moments of the continuously compounded return within τ, $R_\tau = \ln S_T/S_t$, in the physical measure are given by*

$$E_t(R_\tau) = \mu - \frac{1}{2}\sigma^2 - \lambda E[(e^x - 1 - x)],$$

$$E_t[R_\tau - E_t(R_\tau)]^2 = \sigma^2\tau + \lambda\tau E[x^2],$$

$$E_t[R_\tau - E_t(R_\tau)]^3 = \lambda\tau E[x^3], \tag{40}$$

$$E_t[R_\tau - E_t(R_\tau)]^4 = \lambda\tau E[x^4] + 3\left(\lambda\tau E[x^2] + \tau\sigma^2\right)^2.$$

The first moment and the second, third, and fourth central moments of the continuously compounded return in the minimal martingale measure Q are given by

$$E_t^Q(R_\tau) = \mu - \frac{1}{2}\sigma^2 - \lambda E[(1 - \theta(e^x - 1))(e^x - 1 - x)],$$

$$E_t^Q[R_\tau - E_t^Q(R_\tau)]^2 = \sigma^2\tau + \lambda\tau E[(1 - \theta(e^x - 1))x^2],$$

$$E_t^Q[R_\tau - E_t^Q(R_\tau)]^3 = \lambda\tau E[(1 - \theta(e^x - 1))x^3],$$

$$E_t^Q[R_\tau - E_t^Q(R_\tau)]^4$$

$$= \lambda\tau E[(1 - \theta(e^x - 1))x^4]$$

$$+ 3\left(\tau\sigma^2 + \lambda\tau E[(1 - \theta(e^x - 1))x^2]\right)^2. \tag{41}$$

Proof. To compute central moments in the physical measure, we have

$$\mu_x = E(x),$$

$$\text{Var}(x) = E\left[(x - \mu_x)^2\right] = E[x^2] - \mu_x^2,$$

$$3rd(x) = E\left[(x - \mu_x)^3\right]$$

$$= E\left[x^3 - 3x^2\mu_x + 3x\mu_x^2 - \mu_x^3\right]$$

$$= E[x^3] - 3\mu_x^2 E[x^2] + 2\mu_x^3, \tag{42}$$

$$4th(x) = E\left[(x - \mu_x)^4\right]$$

$$= E\left[x^4 - 4x^3\mu_x + 6x^2\mu_x^2 - 4x\mu_x^3 + \mu_x^4\right]$$

$$= E[x^4] - 4\mu_x E[x^3] + 6\mu_x^2 E[x^2] - 3\mu_x^4.$$

Then, from Lemma 11, we have

$$E_t(R_\tau) = \mu - \frac{1}{2}\sigma^2 - \lambda E[(e^x - 1 - x)],$$

$$E_t[R_\tau - E_t(R_\tau)]^2 = \sigma^2\tau + \lambda\tau\left[\mu_x^2 + \text{Var}(x)\right]$$

$$= \sigma^2\tau + \lambda\tau E[x^2],$$

$$E_t[R_\tau - E_t(R_\tau)]^3 = \lambda\tau\left[\mu_x^3 + 3\mu_x\text{Var}(x) + 3rd(x)\right]$$

$$= \lambda\tau E[x^3],$$

$$E_t[R_\tau - E_t(R_\tau)]^4 = 3\left(\lambda\tau\left[\mu_x^2 + \text{Var}(x)\right] + \tau\sigma^2\right)^2$$

$$+ \lambda\tau\left[\mu_x^4 + 6\mu_x^2\text{Var}(x)\right.$$

$$\left. + 4\mu_x 3rd(x) + 4th(x)\right]$$

$$= \lambda\tau E[x^4] + 3\left(\lambda\tau E[x^2] + \tau\sigma^2\right)^2. \tag{43}$$

To compute central moments in the physical measure, we get

$$\mu_x^Q = E^Q(x) = E[(1 - \theta(e^x - 1))x],$$

$$\text{Var}^Q(x) = E^Q\left[\left(x - \mu_x^Q\right)^2\right]$$

$$= E\left[(1 - \theta(e^x - 1))\left(x - \mu_x^Q\right)^2\right]$$

$$= \frac{E\left[(1 - \theta(e^x - 1))\left(x^2 - 2\mu_x^Q x + \left(\mu_x^Q\right)^2\right)\right]}{E(e^{-\gamma x})}$$

$$= E\left[(1 - \theta(e^x - 1))x^2\right] - \left(\mu_x^Q\right)^2,$$

$$3rd^Q(x) = E^Q\left[\left(x - \mu_x^Q\right)^3\right]$$

$$= E\left[\left(1 - \theta\left(e^x - 1\right)\right)\left(x - \mu_x^Q\right)^3\right]$$

$$= E\Big[\left(1 - \theta\left(e^x - 1\right)\right)$$

$$\times\left(x^3 - 3x^2\mu_x^Q + 3x\left(\mu_x^Q\right)^2 - \left(\mu_x^Q\right)^3\right)\Big]$$

$$= E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^3\right] - 3\mu_x^Q E$$

$$\times\left[\left(1 - \theta\left(e^x - 1\right)\right)x^2\right] + 2\left(\mu_x^Q\right)^3,$$

$$4th^Q(x) = E^Q\left[\left(x - \mu_x^Q\right)^4\right]$$

$$= E\left[\left(1 - \theta\left(e^x - 1\right)\right)\left(x - \mu_x^Q\right)^4\right]$$

$$= E\Big[\left(1 - \theta\left(e^x - 1\right)\right)\left(x^4 - 4x^3\mu_x^Q + 6x^2\left(\mu_x^Q\right)^2\right.$$

$$\left. - 4x\left(\mu_x^Q\right)^3 + \left(\mu_x^Q\right)^4\right)\Big]$$

$$= E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^4\right]$$

$$- 4E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^3\right]\mu_x^Q$$

$$+ 6E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^2\right]\left(\mu_x^Q\right)^2 - 3\left(\mu_x^Q\right)^4.$$

$$(44)$$

Then, from (25), we have

$$E_t^Q(R_\tau) = \mu - \frac{1}{2}\sigma^2 - \lambda E$$

$$\times\left[\left(1 - \theta\left(e^x - 1\right)\right)\left(e^x - 1 - x\right)\right],$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^2$$

$$= \sigma^2\tau + \lambda^Q\tau\left[\left(\mu_x^Q\right)^2 + \mathrm{Var}^Q(x)\right]$$

$$= \sigma^2\tau + \lambda\tau E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^2\right],$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^3$$

$$= \lambda^Q\tau\left[\left(\mu_x^Q\right)^3 + 3\mu_x^Q\mathrm{Var}^Q(x) + 3rd^Q(x)\right]$$

$$= \lambda\tau\left\{\left(\mu_x^Q\right)^3 + 3\mu_x^Q\right.$$

$$\times\left[E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^2\right] - \left(\mu_x^Q\right)^2\right]$$

$$+ \left[E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^3\right] - 3\mu_x^Q E\right.$$

$$\left.\left.\times\left[\left(1 - \theta\left(e^x - 1\right)\right)x^2\right] + 2\left(\mu_x^Q\right)^3\right]\right\}$$

$$= \lambda\tau E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^3\right],$$

$$E_t^Q\left[R_\tau - E_t^Q(R_\tau)\right]^4$$

$$= 3\left(\lambda\tau\left[\left(\mu_x^Q\right)^2 + \mathrm{Var}^Q(x)\right] + \sigma^2\tau\right)^2$$

$$+ \lambda^Q\tau\left[\left(\mu_x^Q\right)^4 + 6\left(\mu_x^Q\right)^2\mathrm{Var}^Q(x)\right.$$

$$\left. + 4\mu_x^Q 3rd^Q(x) + 4th^Q(x)\right]$$

$$= \lambda\tau E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^4\right]$$

$$+ 3\left(\lambda\tau E\left[\left(1 - \theta\left(e^x - 1\right)\right)x^2\right] + \sigma^2\tau\right)^2.$$

$$(45)$$

□

5. Portfolio Selection

Under Assumption 9, we study the portfolio selection problem by martingale methods in this section. We first suppose that the portfolio $(\pi(t), 1 - \pi(t))$ which is the proportion of money invested in the stock market and bond market, respectively, the wealth process $y(t) = y^\omega(t)$, and the initial wealth $y(0) = y_0 > 0$ satisfy the equation as follows:

$$\frac{dy(t)}{y(t)} = rdt + \pi(t)\left[(\mu - r)dt - \lambda E\left[\left(e^x - 1\right)\right]dt\right.$$

$$\left. + \sigma dW_t + \left(e^x - 1\right)dN(t)\right]$$

$$y(0) = y_0.$$

$$(46)$$

The investor maximizes his/her expected utility,

$$\max_{\pi(t)} E\left[U\left(y(T)\right)\right]. \quad (47)$$

We suppose $U(y(T)) = \gamma^{-1}y(T)^\gamma$, $\gamma < 1$, $\gamma \neq 0$; then the problem which is to find portfolio to maximize the expected utility is given by the following.

Problem 13. Consider

$$\max \quad E\left[\gamma^{-1}y(T)^\gamma\right]$$

$$\text{subject to} \quad (y(\cdot), \pi(\cdot)) \text{ satisfies } (46), \quad (48)$$

$$\pi(\cdot) \text{ is admissible and tame.}$$

We define the pricing kernel as follows:

$$H(t) = e^{-rt}Z_t$$

$$= \exp\left\{-\theta\sigma W_t - rt - \frac{1}{2}\theta^2\sigma^2 t\right.$$

$$\left. + \theta\lambda E\left[\left(e^x - 1\right)\right]t + \sum_{i=1}^{N(t)}\ln\left(1 - \theta\left(e^{x_i} - 1\right)\right)\right\}.$$

$$(49)$$

Then

$$dH(t) = -H(t)\left[\theta\sigma dW_t + rdt - \theta\lambda E\left[(e^x - 1)\right]dt \right.$$
$$\left. + \theta(e^x - 1)\,dN(t)\right]. \tag{50}$$

Using $\mu - r = \theta\sigma^2 + \theta\lambda E[(e^x - 1)^2]$ and (46), we have

$$d(y(t)H(t))$$
$$= y(t)\,dH(t) + H(t)\,dy(t) + dy(t)\,dH(t)$$
$$= -y(t)H(t)\left(\theta\sigma dW_t + rdt - \theta\lambda E\right.$$
$$\times\left[(e^x - 1)\right]dt$$
$$\left. + \theta(e^x - 1)\,dN(t)\right)$$
$$+ rH(t)y(t)\,dt + H(t)\pi(t)y(t)$$
$$\times\left(\left(\mu - r\right)dt - \lambda E\left[(e^x - 1)\right]dt + \sigma dW_t\right.$$
$$\left. + (e^x - 1)\,dN(t)\right)$$
$$- H(t)y(t)\theta\sigma^2 dt - \pi(t)y(t)$$
$$\times H(t)\theta\lambda E\left[(e^x - 1)^2\right]$$
$$= y(t)H(t)\left(\sigma\pi(t) - \sigma\theta\right)dW_t - y(t)H(t)$$
$$\times\left(-\theta\lambda E\left[(e^x - 1)\right]dt + \theta(e^x - 1)\,dN(t)\right)$$
$$+ H(t)\pi(t)y(t)$$
$$\times\left(-\lambda E\left[(e^x - 1)\right]dt + (e^x - 1)\,dN(t)\right). \tag{51}$$

It suggests that $y(t)H(t)$ is a martingale in physical measure.

In order to solve Problem 13 one needs only first to solve the following optimization problem in the terminal wealth. Y is predictable stochastic process.

Problem 14. Consider

$$\max \quad E\left[\gamma^{-1}Y^\gamma\right]$$
$$\text{subject to} \quad E\left[H(T)Y\right] = y_0. \tag{52}$$

Solving Problem 14, we obtain the following theorem.

Theorem 15. *If Y^* is optimal for Problem 14, then Y^* is also optimal for Problem 13, where*

$$Y^* = y_0\exp\left\{-\frac{1}{\gamma - 1}\theta\sigma W_T + rT - \frac{(2\gamma - 1)\theta^2\sigma^2}{2(\gamma - 1)^2}T\right.$$
$$- \theta\lambda E\left[(e^x - 1)\right]T$$
$$\left. - \lambda E\left[(1 - \theta(e^x - 1))^{\gamma/(\gamma - 1)} - 1\right]T\right\} \tag{53}$$
$$\cdot\exp\left\{\sum_{i=1}^{N(T)}\frac{1}{\gamma - 1}\ln\left(1 - \theta(e^{x_i} - 1)\right)\right\}$$

and $\pi^*(t) = (1/(1 - \gamma))((\mu - r)/(\sigma^2 + \lambda E[(e^x - 1)^2]))$.

Proof. The Lagrangian function of Problem 14 is given by

$$L(Y, \lambda) = E\left[\gamma^{-1}Y^\gamma + \lambda(y_0 - H(T)Y)\right], \tag{54}$$

where λ is the Lagrangian multiplier. Equating the derivatives of the $L(Y, \lambda)$ with respect to Y and λ, respectively, to zero, we have

$$\frac{\partial L(Y, \lambda)}{\partial Y} = E\left[Y^{\gamma - 1} - \lambda H(T)\right] = 0, \tag{55}$$

$$L(Y, \lambda) = y_0 - E[H(T)Y] = 0. \tag{56}$$

From (55), we get

$$Y^* = \lambda^{1/(\gamma - 1)}H(T)^{1/(\gamma - 1)}. \tag{57}$$

Substituting (57) into (56), we obtain

$$\lambda^{1/(\gamma - 1)} = \frac{y_0}{E\left[H(T)^{\gamma/(\gamma - 1)}\right]}, \tag{58}$$

since

$$E\left[H(T)^{\gamma/(\gamma - 1)}\right]$$
$$= E\exp\left\{-\frac{\gamma}{\gamma - 1}\theta\sigma W_T - \frac{\gamma}{\gamma - 1}rT\right.$$
$$- \frac{1}{2}\frac{\gamma}{\gamma - 1}\theta^2\sigma^2 T + \frac{\gamma}{\gamma - 1}\theta\lambda E\left[(e^x - 1)\right]T$$
$$\left. + \sum_{i=1}^{N(T)}\frac{\gamma}{\gamma - 1}\ln\left(1 - \theta(e^{x_i} - 1)\right)\right\}$$
$$= E\exp\left\{\frac{\gamma\theta^2\sigma^2}{2(\gamma - 1)^2}T - \frac{\gamma rT}{\gamma - 1} + \frac{\gamma}{\gamma - 1}\theta\lambda E\left[(e^x - 1)\right]T\right\}$$
$$\cdot E\exp\left\{\sum_{i=1}^{N(T)}\frac{\gamma}{\gamma - 1}\ln\left(1 - \theta(e^{x_i} - 1)\right)\right\}$$
$$= \exp\left\{\frac{\gamma\theta^2\sigma^2}{2(\gamma - 1)^2}T - \frac{\gamma rT}{\gamma - 1} + \frac{\gamma}{\gamma - 1}\theta\lambda E\left[(e^x - 1)\right]T\right\}$$
$$\cdot\exp\left\{\lambda E\left[(1 - \theta(e^x - 1))^{\gamma/(\gamma - 1)} - 1\right]T\right\}$$
$$= \exp\left\{\frac{\gamma\theta^2\sigma^2}{2(\gamma - 1)^2}T - \frac{\gamma r}{\gamma - 1}T + \frac{\gamma}{\gamma - 1}\theta\lambda E\left[(e^x - 1)\right]T\right.$$
$$\left. + \lambda E\left[(1 - \theta(e^x - 1))^{\gamma/(\gamma - 1)} - 1\right]T\right\}. \tag{59}$$

From (58), we have

$$
\lambda^{1/(\gamma-1)} = y_0 \exp\left\{ -\frac{\gamma\theta^2\sigma^2}{2(\gamma-1)^2}T + \frac{\gamma r}{\gamma-1}T - \frac{\gamma}{\gamma-1} \right.
$$
$$
\times \theta\lambda E\left[(e^x - 1)\right]T
$$
$$
\left. -\lambda E\left[\left(1 - \theta\left(e^x - 1\right)\right)^{\gamma/(\gamma-1)} - 1\right]T \right\}.
$$
$$(60)$$

Then

$$
Y^* = \lambda^{1/(\gamma-1)}H(T)^{1/(\gamma-1)}
$$
$$
= y_0 \exp\left\{ -\frac{\gamma\theta^2\sigma^2}{2(\gamma-1)^2}T + \frac{\gamma r}{\gamma-1}T \right.
$$
$$
- \frac{\gamma}{\gamma-1}\theta\lambda E\left[(e^x - 1)\right]T
$$
$$
\left. -\lambda E\left[\left(1 - \theta\left(e^x - 1\right)\right)^{\gamma/(\gamma-1)} - 1\right]T \right\}
$$
$$
\cdot \exp\left\{ -\frac{1}{\gamma-1}\theta\sigma W_T - \frac{1}{\gamma-1}rT - \frac{1}{2}\frac{1}{\gamma-1} \right.
$$
$$
\times \theta^2\sigma^2 T + \frac{1}{\gamma-1}\theta\lambda E\left[(e^x - 1)\right]T
$$
$$
\left. + \sum_{i=1}^{N(T)}\frac{1}{\gamma-1}\ln\left(1 - \theta\left(e^{x_i} - 1\right)\right) \right\}
$$
$$
= y_0 \exp\left\{ -\frac{1}{\gamma-1}\theta\sigma W_T + rT \right.
$$
$$
- \frac{(2\gamma-1)\theta^2\sigma^2}{2(\gamma-1)^2}T - \theta\lambda E\left[(e^x - 1)\right]T
$$
$$
\left. -\lambda E\left[\left(1 - \theta\left(e^x - 1\right)\right)^{\gamma/(\gamma-1)} - 1\right]T \right\}
$$
$$
\cdot \exp\left\{ \sum_{i=1}^{N(T)}\frac{1}{\gamma-1}\ln\left(1 - \theta\left(e^{x_i} - 1\right)\right) \right\}.
$$
$$(61)$$

In order to calculate the $\pi^*(t)$, from (57) and (58), we have

$$
Y^* = y_0 \frac{H(T)^{1/(\gamma-1)}}{E\left[H(T)^{\gamma/(\gamma-1)}\right]}.
$$
$$(62)$$

Thanks to the facts $Y^* = y^*(T)$ and $E_t[y^*(T)H(T)] = E_t[Y^*H(T)]$, then

$$
y^*(t)H(t) = E_t\left[Y^*H(T)\right] = y_0 \frac{E_t\left[H(T)^{\gamma/(\gamma-1)}\right]}{E\left[H(T)^{\gamma/(\gamma-1)}\right]}.
$$
$$(63)$$

From (51), we have

$$
H(t)^{\gamma/(\gamma-1)}
$$
$$
= \exp\left\{ -\frac{\gamma}{\gamma-1}\theta\sigma W_t - \frac{\gamma}{\gamma-1}rt \right.
$$
$$
- \frac{1}{2}\frac{\gamma}{\gamma-1}\theta^2\sigma^2 t + \frac{\gamma}{\gamma-1}\theta\lambda E\left[(e^x - 1)\right]t
$$
$$
\left. + \sum_{i=1}^{N(t)}\frac{\gamma}{\gamma-1}\ln\left(1 - \theta\left(e^{x_i} - 1\right)\right) \right\}
$$
$$
= \exp\left\{ -\frac{\gamma}{\gamma-1}\theta\sigma W_t - \frac{\gamma^2}{2(\gamma-1)^2}\theta^2\sigma^2 t \right.
$$
$$
- \lambda E\left[\left(1 - \theta\left(e^x - 1\right)\right)^{\gamma/(\gamma-1)} - 1\right]t
$$
$$
\left. + \sum_{i=1}^{N(t)}\frac{\gamma}{\gamma-1}\ln\left(1 - \theta\left(e^{x_i} - 1\right)\right) \right\}
$$
$$
\cdot \exp\left\{ \frac{\gamma\theta^2\sigma^2}{2(\gamma-1)^2}t - \frac{\gamma r}{\gamma-1}t \right.
$$
$$
+ \frac{\gamma}{\gamma-1}\theta\lambda E\left[(e^x - 1)\right]t
$$
$$
\left. + \lambda E\left[\left(1 - \theta\left(e^x - 1\right)\right)^{\gamma/(\gamma-1)} - 1\right]t \right\}.
$$
$$(64)$$

By denoting the martingale

$$
z(t) := \exp\left\{ -\frac{\gamma}{\gamma-1}\theta\sigma W_t - \frac{\gamma^2}{2(\gamma-1)^2}\theta^2\sigma^2 t \right.
$$
$$
- \lambda E\left[\left(1 - \theta\left(e^x - 1\right)\right)^{\gamma/(\gamma-1)} - 1\right]t
$$
$$
\left. + \sum_{i=1}^{N(t)}\frac{\gamma}{\gamma-1}\ln\left(1 - \theta\left(e^{x_i} - 1\right)\right) \right\},
$$
$$(65)$$

and function

$$
f(x) := \exp\left\{ \frac{\gamma\theta^2\sigma^2}{2(\gamma-1)^2}t - \frac{\gamma r}{\gamma-1}t \right.
$$
$$
+ \frac{\gamma}{\gamma-1}\theta\lambda E\left[(e^x - 1)\right]t
$$
$$
\left. + \lambda E\left[\left(1 - \theta\left(e^x - 1\right)\right)^{\gamma/(\gamma-1)} - 1\right]t \right\}.
$$
$$(66)$$

Then $H(t)^{\gamma/(\gamma-1)}$ can be written as

$$
H(t)^{\gamma/(\gamma-1)} = z(t) \cdot f(t).
$$
$$(67)$$

From (63) and (67), we obtain

$$
\begin{aligned}
y^* (t) H (t) &= E_t \left[Y^* H (T) \right] \\
&= y_0 \frac{E_t \left[z (T) f (T) \right]}{E \left[z (T) f (T) \right]} = y_0 \frac{f (T) E_t \left[z (T) \right]}{f (T) E \left[z (T) \right]} \\
&= y_0 \frac{E_t \left[z (T) \right]}{E \left[z (T) \right]} = y_0 \frac{z (t)}{z (0)} = y_0 z (t) .
\end{aligned}
\tag{68}
$$

Hence,

$$
\begin{aligned}
d &\left(y^* (t) H (t) \right) \\
&= y_0 dz (t) \\
&= y_0 z (t) \left\{ -\frac{\gamma}{\gamma - 1} \theta \sigma dW_t \right. \\
&\qquad - \lambda E \left[\left(1 - \theta \left(e^x - 1 \right) \right)^{\gamma / (\gamma - 1)} - 1 \right] dt \\
&\qquad + \left. \left[\left(1 - \theta \left(e^x - 1 \right) \right)^{\gamma / (\gamma - 1)} - 1 \right] dN (t) \right\} \\
&= -y^* (t) H (t) \left\{ \frac{\gamma}{\gamma - 1} \theta \sigma dW_t \right. \\
&\qquad + \lambda E \left[\left(1 - \theta \left(e^x - 1 \right) \right)^{\gamma / (\gamma - 1)} - 1 \right] dt \\
&\qquad - \left. \left[\left(1 - \theta \left(e^x - 1 \right) \right)^{\gamma / (\gamma - 1)} - 1 \right] dN(t) \right\} .
\end{aligned}
\tag{69}
$$

Comparing the diffusion of (69) and (51), we have

$$
\pi (t) = \frac{\theta}{1 - \gamma} .
\tag{70}
$$

□

6. Conclusions

With risk-minimization criterion, we employ the minimal martingale measure to solve the pricing problem in an incomplete market. Then we obtain the Radon-Nikodym derivative in the minimal martingale measure and a PIDE with respect to the European option. In a special case, we get the exact solution of European call option by Fourier transformation methods. Finally, we employ the pricing kernel to calculate the optimal portfolio selection by martingale methods.

Acknowledgment

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