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Research Article

Strong Convergence for a Strongly Quasi-Nonexpansive Sequence in Hilbert Spaces

Satit Saejung and Kanokwan Wongchan

Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

Correspondence should be addressed to Satit Saejung; saejung@kku.ac.th

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We prove a strong convergence theorem for strongly quasi-nonexpansive sequence of mappings in Hilbert spaces. Moreover, we can improve the recent results of Tian and Jin (2011). We also give a simple proof of Marino-Xu's result (2006).

1. Introduction

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Recall that a mapping $T : H \to H$ is said to be *L-Lipschitzian* where $L > 0$ if $\|Tx - Ty\| \le L\|x - y\|$ for all $x, y \in H$. In this paper, we are interested in *nonexpansive mappings* (that is, 1-Lipschitzian ones) and *contractions* (that is, L-Lipschitzian ones with $L < 1$). The problem of finding a fixed point of such mappings plays an important role in many nonlinear equations appearing in both pure and applied sciences. The celebrated Banach's contraction principle is probably known as the major tool for the case of contraction mappings. However, for nonexpansive mappings, the situation is more difficult and different.

In 2000, Moudafi [1] introduced the viscosity approximation method, starting with an arbitrary initial $x_1 \in H$, and defined a sequence $\{x_n\}$ by

$$x_{n+1} = \frac{\varepsilon_n}{1+\varepsilon_n} f(x_n) + \frac{1}{1+\varepsilon_n} Tx_n \quad (n \ge 1), \qquad (1)$$

where T is a nonexpansive mapping, $f : H \to H$ is a contraction, and $\{\varepsilon_n\}$ is a sequence in $(0, 1)$ satisfying

(M1) $\lim_{n\to\infty}\varepsilon_n = 0$;

(M2) $\sum_{n=1}^{\infty} \varepsilon_n = \infty$;

(M3) $\lim_{n\to\infty}(1/\varepsilon_n) - (1/\varepsilon_{n+1}) = 0$.

It was proved that the sequence $\{x_n\}$ generated by (1) converges to a fixed point z of T and the following inequality holds:

$$\langle f(z) - z, q - z\rangle \le 0 \quad \forall q \in \operatorname{Fix}(T) := \{x \in H : x = Tx\}. \qquad (2)$$

In the literature, Moudafi's scheme has been widely studied and extended (see [2, 3]). It should be noted that the convergence of Moudafi's scheme is equivalent to that of its special setting with a constant contraction f (see [4]). In fact, this follows from the role of the nonexpansiveness of T.

In the earlier result, the following scheme was studied by Halpern [5]; starting with an arbitrary initial $x_1 \in H$ and a given $u \in H$, he defined a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) Tx_n \quad (n \ge 1), \qquad (3)$$

where $\{\alpha_n\}$ is a certain sequence in $(0, 1)$. In fact, Halpern proved in 1967 the convergence of the iterative sequence $\{x_n\}$ where $\alpha_n = n^{-\theta}$ and $\theta \in (0, 1)$. Many researchers (see, e.g., [6, 7]) have improved Halpern's result from Hilbert spaces to certain Banach spaces with the following conditions on $\{\alpha_n\}$:

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$ or $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Halpern also showed that conditions (C1) and (C2) are necessary for the convergence of the sequence generated by (3) for any given $x_1, u \in H$.

On the other hand, Chidume-Chidume [8] and Suzuki [9] independently discovered that together just conditions (C1) and (C2) are sufficient for the convergence of the following iterative sequence:

$$x_1, u \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n) T_\lambda x_n \quad (n \ge 1), \qquad (4)$$

where $T_\lambda = \lambda I + (1-\lambda)T$ and $\lambda \in (0,1)$. Recently, Saejung [10] proved that the conclusion remains true if T is a strongly nonexpansive mapping. It is noted that in Hilbert spaces the mapping T_λ is strongly nonexpansive whenever $\lambda \in (0,1)$. Recall that a mapping $T : H \to H$ is *strongly nonexpansive* (see [11, 12]) if it is nonexpansive and $\lim_{n\to\infty}\|(x_n - y_n) - (Tx_n - Ty_n)\| = 0$ whenever $\{x_n\}$, $\{y_n\}$ are sequences in H such that $\{x_n - y_n\}$ is bounded and $\lim_{n\to\infty}(\|x_n - y_n\| - \|Tx_n - Ty_n\|) = 0$.

In the aforementioned results, it was assumed that T has a fixed point; that is, $\mathrm{Fix}(T) \neq \emptyset$. Now we consider the following more general settings. A mapping $T : H \to H$ is

(i) *quasi-nonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - q\| \le \|x - q\|$ for all $x \in H$ and $q \in \mathrm{Fix}(T)$;

(ii) *strongly quasi-nonexpansive* if it is quasi-nonexpansive and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ whenever $\{x_n\}$ is a bounded sequence in H such that $\lim_{n\to\infty}(\|x_n - q\| - \|Tx_n - q\|) = 0$ for some $q \in \mathrm{Fix}(T)$.

In 2010, Maingé [2] proved the convergence of the sequence $\{x_n\}$ defined by $x_1 \in H$ and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T_\omega x_n, \qquad (5)$$

where $T_\omega = (1-\omega)I + \omega T$, $\omega \in (0, 1/2)$ and T is a quasi-nonexpansive mapping under the conditions (C1) and (C2). In 2011, Wongchan and Saejung [13] improved Maingé's result by replacing T_ω with a strongly nonexpansive mapping T. Hence, the restriction $\omega \in (0, 1/2)$ can be extended to $\omega \in (0,1)$.

There are also some other iterative schemes closely related to the schemes above studied by many authors. For example, inspired by the scheme studied by Yamada [14], Tian and Jin [15, 16] recently proposed the following iterative scheme, starting with an arbitrary initial $x_1 \in H$ and

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_\omega x_n \quad (n \ge 1), \qquad (6)$$

where f and T_ω are the same as Maingé's result but $F : H \to H$ is strongly monotone and Lipschitzian.

A careful reading shows that there are some connections between them. We will discuss and consolidate them into the following scheme: Started with an arbitrary initial $x_1 \in H$ and

$$x_{n+1} = \alpha_n (f(x_n) + g(T_n x_n)) + (1-\alpha_n) T_n x_n \quad (n \ge 1), \qquad (7)$$

where f, g are Lipschitzian and $\{T_n\}$ is a certain sequence of quasi-nonexpansive mappings.

2. Preliminaries

In this section, we collect together some known lemmas which are our main tool in proving our results. Let C be a closed and convex subset of H. Recall that the *metric projection* $P_C : H \to C$ is defined as follows: for $x \in H$, $P_C x$ is the only one point in C satisfying

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}. \qquad (8)$$

Lemma 1 (see [17]). *Let C be a nonempty closed convex subset of a Hilbert space H. Then for $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, y - z\rangle \ge 0$ for all $z \in C$.*

Lemma 2. *Let H be a Hilbert space. Then*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle \qquad (9)$$

for all $x, y \in H$.

We also need the following lemma.

Lemma 3 (see [18, Lemma 2.5]). *Let $\{a_n\} \subset [0,\infty)$, $\{\alpha_n\} \subset [0,1)$, and $\{b_n\} \subset (-\infty,\infty)$, $\hat{\alpha} \in [0,1)$ be such that*

(i) *$\{a_n\}$ is a bounded sequence;*

(ii) *$a_{n+1} \le (1-\alpha_n)^2 a_n + 2\alpha_n \hat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}} + \alpha_n b_n$ for all $n \in \mathbb{N}$;*

(iii) *whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0$, it follows that $\limsup_{k\to\infty} b_{n_k} \le 0$;*

(iv) *$\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.*

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (see [19, Lemma 2.3]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, and $\{t_n\}$ a sequence of real numbers with $\limsup_{n\to\infty} t_n \le 0$. Suppose that*

$$s_{n+1} \le (1-\alpha_n) s_n + \alpha_n t_n + u_n \quad \forall n \in \mathbb{N}. \qquad (10)$$

Then $\lim_{n\to\infty} s_n = 0$.

3. Main Results

Recall that $\{T_n : H \to H\}$ is a *strongly quasi-nonexpansive sequence* if it satisfies the following conditions:

(1) $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n) \neq \emptyset$;

(2) $\|T_n x - p\| \le \|x - p\|$ for all $x \in H$ and $p \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ and for all $n \in \mathbb{N}$;

(3) $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$ whenever $\{x_n\}$ is a bounded sequence in H such that $\lim_{n\to\infty}(\|x_n - q\| - \|T_n x_n - q\|) = 0$ for some $q \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

We also say that $\{T_n\}$ satisfies the *NST-condition* if whenever $\{z_n\}$ is a bounded sequence in H such that $\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$ it follows that every weak cluster point of $\{z_n\}$ belongs to $\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$.

Remark 5.

(1) Being strongly nonexpansive the sequence and NST-condition are apparently inherited by subsequences.

(2) Suppose that $T_n = T : H \to H$ for all $n \geq 1$.

(i) If T is a strongly nonexpansive mapping, then $\{T_n\}$ is a strongly nonexpansive sequence.

(ii) If $I - T$ is demiclosed at zero, then $\{T_n\}$ satisfies NST-condition.

Recall that $I - T : H \to H$ is *demiclosed at zero* if $\{x_n\}$ is a sequence in H such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $w-\lim_{n\to\infty} x_n = p$; then $p \in \mathrm{Fix}(T)$.

We now state our main theorem.

Theorem 6. *Let $\{T_n : H \to H\}$ be a strongly quasi-nonexpansive sequence satisfying the NST-condition. Let $f, g : H \to H$ be α- and β-Lipschitzian, respectively. Suppose that $\{x_n\}$ is given by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \left(f(x_n) + g(T_n x_n) \right) + (1 - \alpha_n) T_n x_n \quad (n \geq 1), \tag{11}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions (C1) and (C2). Suppose that $\alpha + \beta < 1$. Then $\{x_n\}$ converges strongly to $p = P_{\bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)}(f+g)(p)$.

Before we give the proof, we note that $F := \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$ is closed and convex. It follows from $\alpha + \beta < 1$ that $f + g$ is an $(\alpha+\beta)$-contraction. Then the mapping $P_F(f+g) : F \to F$ is a contraction. By Banach's contraction principle, there exists a unique element $p \in F$ such that $p = P_F(f+g)(p)$. It follows then from Lemma 1 that $\langle (f+g)(p) - p, z - p\rangle \leq 0$ for all $z \in F$.

Let us consider the following three lemmas first.

Lemma 7. *The sequence $\{x_n\}$ is bounded. Hence, so are the sequences $\{f(x_n)\}$, $\{T_n x_n\}$, and $\{g(T_n x_n)\}$.*

Proof. We consider the following inequality:

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) + g(T_n x_n) - p\| \\ &\quad + (1-\alpha_n)\|T_n x_n - p\|.\end{aligned} \tag{12}$$

Since each T_n is quasi-nonexpansive and $p \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$, we have

$$\|T_n x_n - p\| \leq \|x_n - p\|. \tag{13}$$

It follows from the Lipschitzian conditions of f and g, respectively that,

$$\begin{aligned}&\alpha_n \|f(x_n) + g(T_n x_n) - p\| \\ &\quad\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|g(T_n x_n) - g(p)\| \\ &\qquad + \alpha_n \|f(p) + g(p) - p\| \\ &\quad\leq \alpha\alpha_n \|x_n - p\| + \beta\alpha_n \|x_n - p\| \\ &\qquad + \alpha_n \|f(p) + g(p) - p\|.\end{aligned} \tag{14}$$

Then, we have

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\quad\leq (1 - \alpha_n(1-(\alpha+\beta)))\|x_n - p\| \\ &\qquad + \alpha_n(1-(\alpha+\beta))\frac{\|f(p)+g(p)-p\|}{1-(\alpha+\beta)} \\ &\quad\leq \max\left\{\|x_n - p\|, \frac{\|f(p)+g(p)-p\|}{1-(\alpha+\beta)}\right\}.\end{aligned} \tag{15}$$

By induction, for all $n \geq 1$, we have

$$\|x_{n+1} - p\| \leq \max\left\{\|x_1 - p\|, \frac{\|f(p)+g(p)-p\|}{1-(\alpha+\beta)}\right\}. \tag{16}$$

In particular, the sequence $\{x_n\}$ is bounded. □

Lemma 8. *The following inequality holds for all $n \geq 1$:*

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad\leq (1-\alpha_n)^2\|x_n - p\|^2 + 2(\alpha+\beta)\alpha_n\|x_n - p\| \\ &\qquad \times \|x_{n+1} - p\| + 2\alpha_n\langle f(p) + g(p) - p, x_{n+1} - p\rangle.\end{aligned} \tag{17}$$

Proof. It follows from Lemma 2 that

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &\quad= \|\alpha_n(f(x_n) + g(T_n x_n) - p) + (1-\alpha_n)(T_n x_n - p)\|^2 \\ &\quad\leq (1-\alpha_n)^2\|T_n x_n - p\|^2 \\ &\qquad + 2\alpha_n\langle f(x_n) + g(T_n x_n) - p, x_{n+1} - p\rangle.\end{aligned} \tag{18}$$

Since each T_n is quasi-nonexpansive and $p \in \bigcap_{n=1}^{\infty}\mathrm{Fix}(T_n)$,

$$\|T_n x_n - p\|^2 \leq \|x_n - p\|^2. \tag{19}$$

Next, we consider

$$\begin{aligned}&\langle f(x_n) + g(T_n x_n) - p, x_{n+1} - p\rangle \\ &\quad= \langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\qquad + \langle g(T_n x_n) - g(p), x_{n+1} - p\rangle \\ &\qquad + \langle f(p) + g(p) - p, x_{n+1} - p\rangle \\ &\quad\leq \alpha\|x_n - p\|\|x_{n+1} - p\| + \beta\|x_n - p\| \\ &\qquad \times \|x_{n+1} - p\| + \langle f(p) + g(p) - p, x_{n+1} - p\rangle \\ &\quad= (\alpha+\beta)\|x_n - p\|\|x_{n+1} - p\| \\ &\qquad + \langle f(p) + g(p) - p, x_{n+1} - p\rangle.\end{aligned} \tag{20}$$

Hence, the result follows. □

Lemma 9. *If there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\liminf_{k\to\infty}(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|)\ge 0$, then*

$$\limsup_{k\to\infty}\left\langle f(p)+g(p)-p, x_{n_k+1}-p\right\rangle \le 0. \tag{21}$$

Proof. We note that $\lim_{k\to\infty}\alpha_{n_k}=0$. We consider the following inequality:

$$\begin{aligned}0 &\le \liminf_{k\to\infty}\left(\left\|x_{n_k+1}-p\right\|-\left\|x_{n_k}-p\right\|\right)\\ &\le \liminf_{k\to\infty}\left(\alpha_{n_k}\left\|f\left(x_{n_k}\right)-g\left(T_{n_k}x_{n_k}\right)-p\right\|\right.\\ &\qquad\left.+\left(1-\alpha_{n_k}\right)\left\|T_{n_k}x_{n_k}-p\right\|-\left\|x_{n_k}-p\right\|\right)\\ &\le \liminf_{k\to\infty}\left(\left\|T_{n_k}x_{n_k}-p\right\|-\left\|x_{n_k}-p\right\|\right)\\ &\le \limsup_{k\to\infty}\left(\left\|T_{n_k}x_{n_k}-p\right\|-\left\|x_{n_k}-p\right\|\right)\le 0.\end{aligned} \tag{22}$$

Then $\lim_{k\to\infty}(\|T_{n_k}x_{n_k}-p\|-\|x_{n_k}-p\|)=0$. Since $\{T_n\}$ is strongly quasi-nonexpansive, so is $\{T_{n_k}\}$. This implies that $\lim_{k\to\infty}\|x_{n_k}-T_{n_k}x_{n_k}\|=0$. Moreover,

$$\begin{aligned}&\left\|x_{n_k+1}-x_{n_k}\right\|\\ &\quad\le \left\|x_{n_k+1}-T_{n_k}x_{n_k}\right\|+\left\|T_{n_k}x_{n_k}-x_{n_k}\right\|\\ &\quad= \alpha_{n_k}\left\|f\left(x_{n_k}\right)+g\left(T_nx_{n_k}\right)-T_{n_k}x_{n_k}\right\|\\ &\qquad+\left\|T_{n_k}x_{n_k}-x_{n_k}\right\|.\end{aligned} \tag{23}$$

Then $\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $w-\lim_{l\to\infty}x_{n_{k_l}}=q$ and

$$\begin{aligned}&\limsup_{k\to\infty}\left\langle f(p)+g(p)-p, x_{n_k}-p\right\rangle\\ &\qquad=\lim_{l\to\infty}\left\langle f(p)+g(p)-p, x_{n_{k_l}}-p\right\rangle.\end{aligned} \tag{24}$$

As $\lim_{k\to\infty}\|x_{n_k}-x_{n_k+1}\|=0$, we have $\limsup_{k\to\infty}\langle f(p)+g(p)-p, x_{n_k+1}-p\rangle=\langle f(p)+g(p)-p,q-p\rangle$. Since $\{T_n\}$ satisfies NST-condition, we have $q\in F$ and hence $\langle f(p)+g(p)-p,q-p\rangle\le 0$. Therefore,

$$\limsup_{k\to\infty}\left\langle f(p)+g(p)-p, x_{n_k+1}-p\right\rangle \le 0, \tag{25}$$

as desired. □

Proof of Theorem 6. We are ready to apply Lemma 3. Set

$$\begin{aligned}a_n &:= \left\|x_n-p\right\|^2,\\ b_n &:= \left\langle f(p)+g(p)-p, x_{n+1}-p\right\rangle,\\ \widehat{\alpha} &:= \alpha+\beta.\end{aligned} \tag{26}$$

It follows that

(i) $\{a_n\}$ is a bounded sequence (by Lemma 7);

(ii) $a_{n+1}\le(1-\alpha_n)^2a_n+2\alpha_n\widehat{\alpha}\sqrt{a_n}\sqrt{a_{n+1}}+\alpha_nb_n$ for all $n\ge 1$ (by Lemma 8);

(iii) whenever $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ satisfying $\liminf_{k\to\infty}(a_{n_k+1}-a_{n_k})\ge 0$, it follows that $\limsup_{k\to\infty}b_{n_k}\le 0$ (by Lemma 9).

Hence, $\lim_{n\to\infty}\|x_n-p\|=\lim_{n\to\infty}a_n=0$. This completes the proof. □

4. Deduced Results

4.1. Wongchan and Saejung's Result. Setting $g\equiv 0$ and $T_n\equiv T$ for all $n\in\mathbb{N}$ in the proof of Theorem 6, we immediately have the following result of Wongchan and Saejung ([13, Theorem 6]).

Corollary 10. *Let C be a closed convex subset of a Hilbert space H and $T: C\to C$ a strongly quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Suppose that $f: C\to C$ is a contraction and a sequence $\{x_n\}$ is generated by $x_1\in C$ and*

$$x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n)Tx_n, \tag{27}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions (C1) and (C2). Then $\{x_n\}$ converges strongly to $p=P_{\mathrm{Fix}(T)}f(p)$.

4.2. Tian and Jin's Result I. Recall that a mapping $F: H\to H$ is *η-strongly monotone* if $\langle x-y, Fx-Fy\rangle\ge\eta\|x-y\|^2$ for all $x,y\in H$.

Lemma 11. *Let $F: H\to H$ be an η-strongly monotone and κ-Lipschitzian mapping. Then $\|(I-\mu F)x-(I-\mu F)y\|\le\sqrt{1-2\tau}\|x-y\|$ where $\tau=\mu(\eta-(\mu\kappa^2/2))$ for all $x,y\in H$. In particular, if $0<\mu<2\eta/\kappa^2$, then $I-\mu F$ is a contraction.*

Proof. Let $x,y\in H$. Then

$$\begin{aligned}&\left\|(I-\mu F)x-(I-\mu F)y\right\|^2\\ &\quad=\left\|(x-y)-\mu(Fx-Fy)\right\|^2\\ &\quad=\left\|x-y\right\|^2-2\mu\langle x-y, Fx-Fy\rangle\\ &\qquad+\mu^2\left\|Fx-Fy\right\|^2\\ &\quad\le\left\|x-y\right\|^2-2\mu\eta\|x-y\|^2+\mu^2\kappa^2\|x-y\|^2\\ &\quad=\left(1-2\mu\left(\eta-\frac{\mu\kappa^2}{2}\right)\right)\left\|x-y\right\|^2\\ &\quad=(1-2\tau)\left\|x-y\right\|^2.\end{aligned} \tag{28}$$

□

Theorem 12. *Let $T : H \to H$ be a strongly quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Let $F : H \to H$ be an η-strongly monotone and κ-Lipschitzian mapping. Let $f : H \to H$ be an L-Lipschitzian mapping and let a sequence $\{x_n\}$ be generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T x_n \quad (n \geq 1), \tag{29}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the conditions (C1) and (C2). Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma L < 1 - \sqrt{1-2\tau}$, where $\tau = \mu(\eta - (\mu\kappa^2/2))$. Then $\{x_n\}$ converges to $p = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)p$.

Proof. First we rewrite the iteration (29) as follows:

$$x_{n+1} = \alpha_n \left(\widehat{f}(x_n) + \widehat{g}(Tx_n)\right) + (1 - \alpha_n) T x_n, \tag{30}$$

where $\widehat{f} = \gamma f$ and $\widehat{g} = I - \mu F$. Note that $\widehat{f}$ is a γL-Lipschitzian and $\widehat{g}$ is a $\sqrt{1-2\tau}$-Lipschitzian. Using $\gamma L + \sqrt{1-2L} < 1$ and putting $T_n = T$ for all $n \in \mathbb{N}$ in Theorem 6 imply that $\{x_n\}$ converges to $p \in \mathrm{Fix}(T)$, where

$$p = P_{\mathrm{Fix}(T)}\left(\widehat{f} + \widehat{g}\right)(p) = P_{\mathrm{Fix}(T)}\left(I - \mu F + \gamma f\right)(p). \tag{31}$$

□

Lemma 13 (see [12]). *If $T : H \to H$ is a quasi-nonexpansive mapping, then the mapping $T_\omega := (1-\omega)I + \omega T$ is strongly quasi-nonexpansive wherever $\omega \in (0,1)$.*

Using Theorem 12 and Lemma 13, we immediately have the following result which is an improvement of Tian and Jin's result ([15, Theorem 3.1]).

Theorem 14. *Let $T : H \to H$ be a quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Let $F : H \to H$ be an η-strongly monotone and κ-Lipschitzian mapping. Let $f : H \to H$ be an L-Lipschitzian mapping and let the sequence $\{x_n\}$ be generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F) T_\omega x_n \quad (n \geq 1), \tag{32}$$

where $T_\omega = (1-\omega)I + \omega T$, $\omega \in (0,1)$ and the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the conditions (C1) and (C2). Suppose that $0 < \mu < 2\eta/\kappa^2$ and $0 < \gamma L < 1 - \sqrt{1-2\tau}$ where $\tau = \mu(\eta - (\mu\kappa^2/2))$. Then $\{x_n\}$ converges to $p = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)(p)$.

Remark 15. Theorem 14 improves the result of Tian and Jin ([15, Theorem 3.1]) in the following ways.

(i) We assume that $\gamma L < 1 - \sqrt{1-2\tau}$ while [15, Theorem 3.1] is proved under the assumptions $\gamma L < \tau$. We note that $\tau < 1 - \sqrt{1-2\tau}$.

(ii) Our result allows us to choose ω in the wider interval $(0,1)$ while [15, Theorem 3.1] is proved under the assumptions $\omega \in (0, 1/2)$.

4.3. Tian and Jin's Result II. Recall that a mapping $A : H \to H$ is *strongly positive* with the coefficient $\overline{\gamma} > 0$ if

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \tag{33}$$

for all $x \in H$.

Lemma 16 (see [20]). *Let A be a strongly positive self-adjoint linear bounded operator with coefficient $\overline{\gamma} > 0$ on H and $0 < \rho \leq \|A\|^{-1}$. Then $\| I - \rho A \| \leq 1 - \rho\overline{\gamma}$.*

Theorem 17. *Let $T : H \to H$ be a strongly quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Let $A : H \to H$ be a bounded linear self-adjoint operator and strongly positive with the coefficient $\overline{\gamma}$. Let $f : H \to H$ be an α-contraction mapping and let a sequence $\{x_n\}$ be generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n \quad (n \geq 1), \tag{34}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the conditions (C1) and (C2). Suppose that $0 < \gamma\alpha < \overline{\gamma}$. Then $\{x_n\}$ converges to $p = P_{Fix(T)}(I - A + \gamma f)p$.

Proof. By Lemma 16, we can choose $t \in (0,1)$ such that $\| I - tA \| \leq 1 - t\overline{\gamma}$. Rewrite the iteration (34) as follows:

$$x_{n+1} = \widehat{\alpha}_n \left(\widehat{f}(x_n) + \widehat{g}(Tx_n)\right) + (1 - \widehat{\alpha}_n) T x_n, \tag{35}$$

where $\widehat{f} := t\gamma f$, $\widehat{g} := I - tA$ and $\widehat{\alpha}_n \equiv \alpha_n / t$ for all $n \in \mathbb{N}$. Note that $\widehat{f}$ is $t\gamma\alpha$-Lipschitzian and $\widehat{g}$ is $(1-t\overline{\gamma})$-Lipschitzian. It follows from $0 < \gamma\alpha < \overline{\gamma}$ that

$$t\gamma\alpha + 1 - t\overline{\gamma} = 1 - t\left(\overline{\gamma} - \alpha\gamma\right) < 1. \tag{36}$$

Setting $T_n \equiv T$ for all $n \in \mathbb{N}$ in Theorem 6 implies that $\{x_n\}$ converges to $p \in \mathrm{Fix}(T)$ such that $p = P_{\mathrm{Fix}(T)}(\widehat{f} + \widehat{g})p = P_{\mathrm{Fix}(T)}(t\gamma f + I - tA)p$; that is, $\langle t\gamma f(p) + p - tAp - p, p - w\rangle \geq 0$ for all $w \in \mathrm{Fix}(T)$. This implies that $\langle \gamma f(p) - Ap, p - w\rangle \geq 0$ for all $w \in \mathrm{Fix}(T)$; that is, $p = P_{\mathrm{Fix}(T)}(\gamma f + I - A)p$. This completes the proof. □

Using Lemma 13 and Theorem 17, we immediately have the following result which is an improvement of Tian and Jin's result ([16, Theorem 3.1]).

Theorem 18. *Let $T : H \to H$ be a quasi-nonexpansive mapping such that $I-T$ is demiclosed at zero. Let $A : H \to H$ be a bounded linear self-adjoint operator and strongly positive with the coefficient $\overline{\gamma}$. Let $f : H \to H$ be an α-contraction mapping, and let the sequence $\{x_n\}$ be generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T_\omega x_n \quad (n \geq 1), \tag{37}$$

where $T_\omega = (1-\omega)I + \omega T$, $\omega \in (0,1)$ and the sequence $\{\alpha_n\} \subset (0,1)$ satisfies the conditions (C1) and (C2). Suppose that $0 < \gamma\alpha < \overline{\gamma}$. Then $\{x_n\}$ converges to $p = P_{\mathrm{Fix}(T)}(I - A + \gamma f)p$.

Remark 19. Theorem 18 improves the result of Tian and Jin ([16, Theorem 3.1]). In fact, their result was proved under the assumption $\omega \in (0, 1/2)$ while our result allows us to choose ω in the wider interval $(0,1)$.

5. A Discussion on Marino-Xu's Result

The following theorem is studied by many authors; for example, see [3].

Theorem 20. *Let C be a closed convex subset of a Hilbert space H. Suppose that*

(i) $T : C \to C$ *is a nonexpansive mapping and* $\mathrm{Fix}(T) \neq \emptyset$;

(ii) $\{\alpha_n\} \subset (0,1)$ *is a sequence satisfying the conditions (C1), (C2), and (C3).*

Define the following iterative sequence:

$$u, x_1 \in C, \tag{38}$$

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T x_n. \tag{39}$$

Then $\{x_n\}$ converges to $P_{\mathrm{Fix}(T)} u$.

Using the technique in [4], we can give a simple proof of the following result proved by Marino and Xu [20].

Theorem 21. *Suppose that*

(i) $A : H \to H$ *is a bounded linear self-adjoint operator and it is strongly positive with the coefficient* $\overline{\gamma}$;

(ii) $T : H \to H$ *is a nonexpansive mapping and* $\mathrm{Fix}(T) \neq \emptyset$;

(iii) $f : H \to H$ *is an α-contraction;*

(iv) γ *is a positive number such that* $0 < \gamma\alpha < \overline{\gamma}$;

(v) $\{\alpha_n\} \subset (0,1)$ *is a sequence satisfying the conditions (C1), (C2), and (C3).*

Define the following iterative sequence:

$$z_1 \in H \tag{40}$$

$$z_{n+1} = \alpha_n \gamma f(z_n) + (I - \alpha_n A) T z_n. \tag{41}$$

Then $\{z_n\}$ converges to $\widehat{z} \in \mathrm{Fix}(T)$ and $\langle A\widehat{z} - \gamma f(\widehat{z}), \widehat{z} - w\rangle \leq 0$ for all $w \in \mathrm{Fix}(T)$.

Proof. Choose $t \in (0,1)$ such that $\| I - tA \| \leq 1 - t\overline{\gamma}$. First we show that $I - tA + t\gamma f$ is a contraction. To see this, let $x, y \in H$. Then

$$\begin{aligned} &\|(I - tA + t\gamma f)x - (I - tA + t\gamma f)y\| \\ &\quad \leq \|(I - tA)x - (I - tA)y\| + t\gamma \|f(x) - f(y)\| \\ &\quad \leq \|I - tA\| \|x - y\| + t\gamma \|f(x) - f(y)\| \\ &\quad \leq (1 - t\overline{\gamma}) \|x - y\| + t\gamma\alpha \|x - y\| \\ &\quad = (1 - t(\overline{\gamma} - \gamma\alpha)) \|x - y\|. \end{aligned} \tag{42}$$

It follows from $\gamma\alpha < \overline{\gamma}$ that $I - tA + t\gamma f$ is a contraction. Note that $P_{\mathrm{Fix}(T)}$ is nonexpansive and hence $P_{\mathrm{Fix}(T)}(I - tA + t\gamma f)$ is a contraction from $\mathrm{Fix}(T)$ into itself. It follows from the closedness of $\mathrm{Fix}(T)$ and the Banach's contraction principle that there exists a unique element $\widehat{z} \in \mathrm{Fix}(T)$ such that

$$\widehat{z} = P_{\mathrm{Fix}(T)} (I - tA + t\gamma f)(\widehat{z}). \tag{43}$$

Therefore,

$$\langle A\widehat{z} - \gamma f(\widehat{z}), \widehat{z} - w\rangle \leq 0 \quad \forall w \in \mathrm{Fix}(T). \tag{44}$$

Now we define the following iterative sequence:

$$\begin{aligned} x_1 &= z_1, \\ x_{n+1} &= \frac{\alpha_n}{t}((I - tA) T\widehat{z} + t\gamma f(\widehat{z})) + \left(1 - \frac{\alpha_n}{t}\right) T x_n. \end{aligned} \tag{45}$$

It follows from Theorem 20 that the sequence $\{x_n\}$ converges to $\widehat{z} = P_{\mathrm{Fix}(T)}(I - tA + t\gamma f)(\widehat{z})$. Observe that

$$z_{n+1} = \frac{\alpha_n}{t}((I - tA) T z_n + t\gamma f(z_n)) + \left(1 - \frac{\alpha_n}{t}\right) T z_n. \tag{46}$$

We next consider the following expression:

$$\begin{aligned} &\|z_{n+1} - x_{n+1}\| \\ &\quad = \left\| \left(1 - \frac{\alpha_n}{t}\right)(Tz_n - Tx_n) + \frac{\alpha_n}{t}(I - tA)(Tz_n - T\widehat{z}) \right. \\ &\qquad \left. + \frac{\alpha_n}{t} t\gamma (f(z_n) - f(\widehat{z})) \right\| \\ &\quad \leq \left(1 - \frac{\alpha_n}{t}\right) \|z_n - x_n\| + \frac{\alpha_n}{t}(1 - t\overline{\gamma}) \|z_n - \widehat{z}\| + \alpha_n \gamma\alpha \|z_n - \widehat{z}\| \\ &\quad = \left(1 - \frac{\alpha_n}{t}\right) \|z_n - x_n\| + \left(\frac{\alpha_n}{t} - \alpha_n(\overline{\gamma} - \gamma\alpha)\right) \|z_n - \widehat{z}\| \\ &\quad \leq (1 - \alpha_n(\overline{\gamma} - \gamma\alpha)) \|z_n - x_n\| + \left(\frac{\alpha_n}{t} - \alpha_n(\overline{\gamma} - \gamma\alpha)\right) \|x_n - \widehat{z}\| \\ &\quad = (1 - \alpha_n(\overline{\gamma} - \gamma\alpha)) \|z_n - x_n\| + \alpha_n(\overline{\gamma} - \gamma\alpha) \\ &\qquad \times \left(\frac{(1/t) - (\overline{\gamma} - \gamma\alpha)}{\overline{\gamma} - \gamma\alpha}\right) \|x_n - \widehat{z}\|. \end{aligned} \tag{47}$$

It follows from Lemma 4 that $\lim_{n\to\infty} \|z_n - x_n\| = 0$. Therefore, we conclude that $\{z_n\}$ converges to $\widehat{z} \in \mathrm{Fix}(T)$ and $\langle A\widehat{z} - \gamma f(\widehat{z}), \widehat{z} - w\rangle \leq 0$ for all $w \in \mathrm{Fix}(T)$. This completes the proof. □

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