Research Article

Periodic Solutions for Circular Restricted $N + 1$-Body Problems

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For circular restricted $N + 1$-body problems, we study the motion of a sufficiently small mass point (called the zero mass point) in the plane of N equal masses located at the vertices of a regular polygon. By using variational minimizing methods, for some N, we prove the existence of the noncollision periodic solution for the zero mass point with some fixed wingding number.

1. Introduction and Main Results

In this paper, we study the planar circular restricted $N + 1$-body problems. Suppose that points of positive masses $m_1 = \cdots = m_N = 1$ move on a circular orbit around the center of masses, and that the sufficiently small mass point, called the zero mass point, moves on the moving plane of the given N equal masses and does not influence the motion of $m_1, \ldots, m_N$, but the motion of the zero mass point is affected by the given N equal mass points. Let $\rho_i = e^{\sqrt{-1}(2\pi i/N)}$ and we denote the position vectors of the given N bodies by $q_i(t)$, $i = 1, \ldots, N$, then

$$q_1(t) = re^{\sqrt{-1}2\pi t}\rho_1, \ldots, \qquad q_i(t) = re^{\sqrt{-1}2\pi t}\rho_i, \ldots,$$
$$q_N(t) = re^{\sqrt{-1}2\pi t}, \tag{1}$$

where the radius $r > 0$. It is known that $q_1(t), \ldots, q_N(t)$ satisfy the following Newtonian equations:

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i = 1, \ldots, N, \tag{2}$$

where

$$U = \sum_{1 \le i < j \le N} \frac{m_i m_j}{|q_i - q_j|}. \tag{3}$$

We also assume that

$$\sum_{i=1}^{N} m_i q_i = \sum_{i=1}^{N} q_i = 0. \tag{4}$$

The orbit $q(t) \in R^2$ for the zero mass point is governed by the gravitational forces of $m_1, \ldots, m_N$ and therefore it satisfies the following equation:

$$\ddot{q} = \sum_{i=1}^{N} \frac{m_i (q_i - q)}{|q_i - q|^3}. \tag{5}$$

For N-body problems, there are many papers concerned with the periodic solutions by using variational methods; see [1–15] and the references therein. In [1], Chenciner and Montgomery proved the existence of the remarkable figure-"8" type periodic solution for planar Newtonian 3-body problems with equal masses. Marchal [4] studied the fixed end problem for Newtonian n-body problems and proved that the minimizer for the Lagrangian action has no interior collision. In [6], Simó used computer to discover many new periodic solutions for Newtonian n-body problems. Zhang and Zhou [10–12] decomposed the Lagrangian action for n-body problems into some sum for two body problems and [11, 12] avoided collisions by comparing the lower bound for the Lagrangian action on the symmetry collision orbits and the upper bound for the Lagrangian action on test orbits in some cases.

Motivated by the above works, we use variational methods to study the circular restricted $N + 1$-body problem with some fixed wingding numbers and N equal masses.

For the readers' conveniences, we recall the definition of the winding number, which can be found in many books on the classical differential geometry.

Definition 1. Let $\Gamma : x(t), t \in [a,b]$ be a given oriented continuous closed curve, and p be a point on the plane not on the curve. Then, the mapping $\varphi : \Gamma \to S^1$ given by

$$\varphi(x(t)) = \frac{x(t) - p}{|x(t) - p|}, \quad t \in [a,b] \tag{6}$$

is defined to be the position mapping of the curve Γ relative to p, and when the point on Γ goes around the curve once, its image point $\varphi(x(t))$ will go around S^1 a number of times; this number is called the winding number of the curve Γ relative to p, and we denote it by $\deg(\Gamma, p)$. If p is the origin, we write $\deg \Gamma$.

Let

$$R(n) = \begin{pmatrix} \cos \dfrac{2\pi}{n} & -\sin \dfrac{2\pi}{n} \\ \sin \dfrac{2\pi}{n} & \cos \dfrac{2\pi}{n} \end{pmatrix} \in SO(2) \tag{7}$$

be a counter-clockwise rotation of angle $2\pi/n$ in R^2.

Define

$$W^{1,2}\left(\frac{R}{Z}, R^2\right)$$
$$= \left\{ x(t) \mid x(t), \dot{x}(t) \in L^2\left(R, R^2\right), x(t+1) = x(t) \right\}. \tag{8}$$

The norm of $W^{1,2}(R/Z, R^2)$ is

$$\|x\| = \left[\int_0^1 |x|^2 dt\right]^{1/2} + \left[\int_0^1 |\dot{x}|^2 dt\right]^{1/2}. \tag{9}$$

We consider the Lagrangian functional of (5) as

$$f(q) = \int_0^1 \left[\frac{1}{2}|\dot{q}|^2 + \sum_{i=1}^N \frac{1}{|q - q_i|}\right] dt \tag{10}$$

on $\Lambda_\pm$, where

$$\Lambda_- = \left\{ q \in W^{1,2}\left(\frac{R}{Z}, R^2\right) \,\middle|\, \begin{array}{l} q\left(t + \dfrac{1}{2}\right) = R(2)\, q(t), q(t) \neq q_i(t), \forall t \in [0,1], i = 1, \ldots, N, \\ \deg(q - q_N) = -1, \deg(q_j - q_N) = 1, j = 1, \ldots, N-1 \end{array} \right\},$$

$$\Lambda_+ = \left\{ q \in W^{1,2}\left(\frac{R}{Z}, R^2\right) \,\middle|\, \begin{array}{l} q\left(t + \dfrac{1}{N}\right) = R(N)\, q(t), q(t) \neq q_i(t), \forall t \in [0,1], i = 1, \ldots, N, \\ \deg(q - q_N) = 1, \deg(q_j - q_N) = 1, j = 1, \ldots, N-1 \end{array} \right\}. \tag{11}$$

Our main results are the following.

Theorem 2. *For $N = 2, 3$, the minimizer of $f(q)$ on the closure $\overline{\Lambda}_-$ of Λ_- is a noncollision 1-periodic solution of* (5).

Remark 3. In proving Theorem 2, we need to use test functions. We find that when $N = 3$, if the test functions are circular orbits, we cannot get the desired results on $\overline{\Lambda}_-$. Therefore, we select elliptic orbits as test functions.

Theorem 4. *For $2 \leq N \leq 32402$, the minimizer of $f(q)$ on the closure $\overline{\Lambda}_+$ of Λ_+ is a noncollision 1-periodic solution of* (5).

2. Preliminaries

In this section, we will list some basic lemmas and inequality to prove our theorems.

Lemma 5 (see [8]). *The radius r for the moving orbit of N equal mass points is*

$$r = \left(\frac{1}{4\pi}\right)^{2/3} \left[\sum_{1 \leq j \leq N-1} \csc\left(\frac{\pi}{N}j\right)\right]^{1/3}. \tag{12}$$

Lemma 6 (Zhang and Zhou [9]). *Suppose that $g \in SO(k)$ is an element of finite order s and it has no fixed point*

other than the origin (i.e., 1 is not an eigenvalue of g). Then $\int_0^T X(t)dt = 0$ for all $X(t) = (x_1(t), \ldots, x_N(t)) \in E_g^N$, where $E_g^N = \{(x_1, \ldots, x_N) \mid x_i \in H^1(R/TZ, R^K), x_i(t + T/s) = g(x_i(t)), \forall t \in [0, T], i = 1, \ldots, N\}$.

Lemma 7 (Poincare-Wirtinger inequality [16]). *Let $q \in W^{1,2}(R/TZ, R^K)$ and $\int_0^T q(t)dt = 0$, and then*

$$\int_0^T |q(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{q}(t)|^2 dt. \tag{13}$$

Lemma 8 (see [17]). *Let X be a reflexive Banach space, S be a weakly closed subset of X, and $f : S \to R \cup \{+\infty\}$. If $f \not\equiv +\infty$ is weakly lower semicontinuous and coercive ($f(x) \to +\infty$ as $\|x\| \to +\infty$), then f attains its infimum on S.*

Lemma 9 (Palais's symmetry principle [18]). *Let σ be an orthogonal representation of a finite or compact group G, let H be a real Hilbert space, and let $f : H \to R$ satisfy $f(\sigma \cdot x) = f(x), \forall \sigma \in G, \forall x \in H$.*

Set $F = \{x \in H \mid \sigma \cdot x = x, \forall \sigma \in G\}$. Then the critical point of f in F is also a critical point of f in H.

Remark 10. By Palais's symmetry principle and the perturbation invariance for wingding numbers, we know that

the critical point of $f(q)$ in $\Lambda_{\pm}$ is a periodic solution of Newtonian equation (5).

Lemma 11. (1) *(Gordon's theorem [19]) Let $x \in W^{1,2}([t_1, t_2], R^K)$ and $x(t_1) = x(t_2) = 0$. Then for any $a > 0$, one has*

$$\int_{t_1}^{t_2} \left(\frac{1}{2} |\dot{x}|^2 + \frac{a}{|x|} \right) dt \geq \frac{3}{2} (2\pi)^{2/3} a^{2/3} (t_2 - t_1)^{1/3}. \tag{14}$$

(2) *(Long and Zhang [20]) Let $x \in W^{1,2}(R/TZ, R^K)$, and $\int_0^T x \, dt = 0$, then for any $a > 0$, we have*

$$\int_0^T \left(\frac{1}{2} |\dot{x}|^2 + \frac{a}{|x|} \right) dt \geq \frac{3}{2} (2\pi)^{2/3} a^{2/3} T^{1/3}. \tag{15}$$

3. Proof of Theorems

In order to get our theorems, we need two steps to complete the proof.

Step 1. We will establish the existence of variational minimizers of $f(q)$ in (10) on $\overline{\Lambda}_{\pm}$.

Lemma 12. $f(q)$ *in (10) attains its infimum on* $\overline{\Lambda}_{\pm}$.

Proof. It is easy to check that the eigenvalue λ of $R(n)$ satisfies

$$\begin{cases} \lambda = \cos \dfrac{2\pi}{n}, \\ \sin \dfrac{2\pi}{n} = 0, \end{cases} \tag{16}$$

which implies $\lambda \neq 1$ for $n \geq 2$. Then by Lemma 6, we have

$$\int_0^1 q(t) = 0 \quad \forall q \in \overline{\Lambda}_{\pm}. \tag{17}$$

So by using Lemma 7, for all $q \in \overline{\Lambda}_{\pm}$, we see that an equivalent norm of (9) on $\overline{\Lambda}_{\pm}$ is

$$\|q\| \cong \left[\int_0^T |\dot{q}|^2 dt \right]^{1/2}. \tag{18}$$

Hence, by the definition of $f(q)$, f is coercive on $\overline{\Lambda}_{\pm}$.

Similar to the proof of Lemma 2.4 in [13], we can get the conclusion that f is weakly lower semi-continuous on $\overline{\Lambda}_{\pm}$.

Therefore, by using Lemma 8, it can be concluded that $f(q)$ in (10) attains its infimum on $\overline{\Lambda}_{\pm}$. $\square$

Step 2. We will prove that the variational minimizer in Lemma 12 is the noncollision 1-period solution of (5).

For any collision generalized solution q, we can estimate the lower bound for the value of Lagrangian action functional.

Lemma 13. *For* $\partial \Lambda_- = \{ q \in W^{1,2}(R/Z, R^2) \mid q(t + 1/2) = R(2)q(t), \exists 1 \leq i_0 \leq N, t_0 \in [0, 1] \ s.t. \ q_{i_0}(t_0) = q(t_0) \}$, *we have*

$$\inf_{q \in \partial \Lambda_-} f(q)$$

$$\geq \frac{3}{2} (4\pi)^{2/3} N^{-1/3} + \left(\frac{N-1}{N} \right) \frac{3}{2} (2\pi)^{2/3} N^{2/3} - \frac{1}{2} (2\pi)^2 r^2$$

$$\triangleq d_0. \tag{19}$$

Proof. It follows from (4) that

$$\sum_{i=1}^N \dot{q}_i = 0, \tag{20}$$

which implies

$$\sum_{i=1}^N |\dot{q} - \dot{q}_i|^2$$

$$= \sum_{i=1}^N \left(|\dot{q}|^2 + |\dot{q}_i|^2 - 2 \langle \dot{q}, \dot{q}_i \rangle \right)$$

$$= N |\dot{q}|^2 + \sum_{i=1}^N |\dot{q}_i|^2 - 2 \left\langle \dot{q}, \sum_{i=1}^N \dot{q}_i \right\rangle \tag{21}$$

$$= N |\dot{q}|^2 + \sum_{i=1}^N |\dot{q}_i|^2.$$

Therefore,

$$|\dot{q}|^2 = \frac{1}{N} \sum_{i=1}^N \left(|\dot{q} - \dot{q}_i|^2 - |\dot{q}_i|^2 \right). \tag{22}$$

Hence,

$$f(q) = \int_0^1 \left[\frac{1}{2} |\dot{q}|^2 + \sum_{i=1}^N \frac{1}{|q - q_i|} \right] dt$$

$$= \frac{1}{N} \int_0^1 \sum_{i=1}^N \left[\frac{1}{2} |\dot{q} - \dot{q}_i|^2 + \frac{N}{|q - q_i|} \right] dt \tag{23}$$

$$- \frac{1}{2N} \int_0^1 \sum_{i=1}^N |\dot{q}_i|^2 dt.$$

If $q \in \overline{\Lambda}_-$ is a collision solution, then there exists $t_0 \in [0, 1]$ and $1 \leq i_0 \leq N$ s.t $q(t_0) = q_{i_0}(t_0)$. Since $q_{i_0}(t + 1/2) = R(2)q_{i_0}(t)$, one has $q(t_0 + k/2) = q_{i_0}(t_0 + k/2), \forall 0 \leq k \leq 2$. So, by (1) of Lemma 11, we get

$$\frac{1}{N} \int_0^1 \left[\frac{1}{2} |\dot{q} - \dot{q}_{i_0}|^2 + \frac{N}{|q - q_{i_0}|} \right] dt$$

$$= \frac{2}{N} \int_0^{1/2} \left[\frac{1}{2} |\dot{q} - \dot{q}_{i_0}|^2 + \frac{N}{|q - q_{i_0}|} \right] dt \tag{24}$$

$$\geq \frac{3}{2} (4\pi)^{2/3} N^{-1/3}.$$

For noncollision pair q, q_i $(i \neq i_0)$, we claim $\int_0^1 (q(t) - q_i(t)) dt = 0$. In fact, it follows from (1) that $q_{i+1}(t) = q_i(t + 1/N)$, which implies

$$q_i(t) = q_1\left(t + \frac{i-1}{N}\right). \tag{25}$$

Therefore, by (4), we have

$$\begin{aligned}
\int_0^1 & q_1(t) \, dt \\
&= \int_0^{1/N} q_1(t) \, dt + \cdots + \int_{(N-1)/N}^1 q_1(t) \, dt \\
&= \int_0^{1/N} q_1(t) \, dt + \cdots + \int_0^{1/N} q_1\left(t + \frac{N-1}{N}\right) dt \\
&= \int_0^{1/N} \sum_{i=1}^N q_i(t) \, dt \\
&= 0.
\end{aligned} \tag{26}$$

Since $q_1(t+1) = q_1(t)$, by (25) and (26), we obtain

$$\int_0^1 q_i(t) \, dt = \int_0^1 q_1\left(t + \frac{i-1}{N}\right) dt = \int_0^1 q_1(t) \, dt = 0. \tag{27}$$

Combined with (17) and (27), we have $\int_0^1 (q(t) - q_i(t)) dt = 0$. Hence, by (2) of Lemma 11, we can get

$$\begin{aligned}
\frac{1}{N} \int_0^1 \sum_{i \neq i_0} & \left[\frac{1}{2} |\dot{q} - \dot{q}_i|^2 + \frac{N}{|q - q_i|} \right] dt \\
&\geq \left(\frac{N-1}{N} \right) \frac{3}{2} (2\pi)^{2/3} N^{2/3}.
\end{aligned} \tag{28}$$

For the other term of f, using the expression for the orbits $q_1, \ldots, q_N$ as in (1), one has

$$-\frac{1}{2N} \int_0^1 \sum_{i=1}^N |\dot{q}_i|^2 dt = -\frac{1}{2}(2\pi)^2 r^2. \tag{29}$$

Therefore, it follows from (24), (28), and (29) that

$$\begin{aligned}
\inf_{q \in \partial \Lambda_-} & f(q) \\
&\geq \frac{3}{2}(4\pi)^{2/3} N^{-1/3} + \left(\frac{N-1}{N} \right) \frac{3}{2}(2\pi)^{2/3} N^{2/3} - \frac{1}{2}(2\pi)^2 r^2 \\
&\triangleq d_0.
\end{aligned} \tag{30}$$

$\square$

Lemma 14. *For* $\partial \Lambda_+ = \{q \in W^{1,2}(R/Z, R^2) \mid q(t + 1/N) = R(N)q(t), \exists 1 \leq \tilde{i}_0 \leq N, \tilde{t}_0 \in [0, 1] \ s.t. \ q_{\tilde{i}_0}(\tilde{t}_0) = q(\tilde{t}_0)\}$, *we have*

$$\begin{aligned}
\inf_{q \in \partial \Lambda_+} & f(q) \\
&\geq \frac{3}{2}(2\pi)^{2/3} N^{1/3} + \left(\frac{N-1}{N} \right) \frac{3}{2}(2\pi)^{2/3} N^{2/3} - \frac{1}{2}(2\pi)^2 r^2 \\
&\triangleq d_1.
\end{aligned} \tag{31}$$

Proof. Similar to the proof of Lemma 13, Lemma 14 holds. $\square$

Proof of Theorem 2. In order to get Theorem 2, we are going to find a test loop $\tilde{q} \in \Lambda_-$ such that $f(\tilde{q}) = d_2$. Then the minimizer of f on $\overline{\Lambda}_-$ must be a noncollision solution if $d_2 < d_0$.

Let $a > 0$, $b > 0$, $\theta \in [0, 2\pi)$ and

$$\tilde{q} - q_N = (a \cos(-2\pi t + \theta), b \sin(-2\pi t + \theta))^T. \tag{32}$$

Hence,

$$\begin{aligned}
\tilde{q} - q_1 = \bigg(& r \cos 2\pi t - r \cos\left(2\pi t + \frac{2\pi}{N}\right) \\
& + a \cos(-2\pi t + \theta), r \sin 2\pi t \\
& - r \sin\left(2\pi t + \frac{2\pi}{N}\right) + b \sin(-2\pi t + \theta) \bigg)^T,
\end{aligned} \tag{33}$$

$$\vdots$$

$$\begin{aligned}
\tilde{q} - q_i = \bigg(& r \cos 2\pi t - r \cos\left(2\pi t + \frac{i}{N}2\pi\right) \\
& + a \cos(-2\pi t + \theta), r \sin 2\pi t \\
& - r \sin\left(2\pi t + \frac{i}{N}2\pi\right) + b \sin(-2\pi t + \theta) \bigg)^T,
\end{aligned} \tag{34}$$

$$\vdots$$

$$\begin{aligned}
\tilde{q} - q_{N-1} = \bigg(& r \cos 2\pi t - r \cos\left(2\pi t + \frac{N-1}{N}2\pi\right) \\
& + a \cos(-2\pi t + \theta), r \sin 2\pi t \\
& - r \sin\left(2\pi t + \frac{N-1}{N}2\pi\right) + b \sin(-2\pi t + \theta) \bigg)^T.
\end{aligned} \tag{35}$$

It is easy to see that $\tilde{q} \in \Lambda_-$ and

$$\left|\dot{\tilde{q}} - \dot{q}_1\right|^2 = (2\pi)^2 \left\{ \frac{a^2 + b^2}{2} - \frac{a^2 - b^2}{2} \cos(4\pi t - 2\theta) \right.$$
$$+ r^2 \left(2 - 2\cos\frac{2\pi}{N}\right) - (a+b)r$$
$$\times \left[\cos(4\pi t - \theta) - \cos\left(4\pi t + \frac{2\pi}{N} - \theta\right)\right]$$
$$\left. + (a-b)r\left[\cos\theta - \cos\left(\frac{2\pi}{N} + \theta\right)\right] \right\},$$
(36)

$$\left|\tilde{q} - q_1\right| = \left\{ \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos(4\pi t - 2\theta) \right.$$
$$+ r^2 \left(2 - 2\cos\frac{2\pi}{N}\right) + (a+b)r$$
$$\times \left[\cos(4\pi t - \theta) - \cos\left(4\pi t + \frac{2\pi}{N} - \theta\right)\right]$$
$$\left. + (a-b)r\left[\cos\theta - \cos\left(\frac{2\pi}{N} + \theta\right)\right] \right\}^{1/2},$$

$$\vdots$$
(37)

$$\left|\dot{\tilde{q}} - \dot{q}_i\right|^2 = (2\pi)^2 \left\{ \frac{a^2 + b^2}{2} - \frac{a^2 - b^2}{2} \cos(4\pi t - 2\theta) \right.$$
$$+ r^2 \left(2 - 2\cos\frac{i}{N}2\pi\right) - (a+b)r$$
$$\times \left[\cos(4\pi t - \theta) - \cos\left(4\pi t + \frac{i}{N}2\pi - \theta\right)\right]$$
$$\left. + (a-b)r\left[\cos\theta - \cos\left(\frac{i}{N}2\pi + \theta\right)\right] \right\},$$
(38)

$$\left|\tilde{q} - q_i\right| = \left\{ \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos(4\pi t - 2\theta) \right.$$
$$+ r^2 \left(2 - 2\cos\frac{i}{N}2\pi\right) + (a+b)r$$
$$\times \left[\cos(4\pi t - \theta) - \cos\left(4\pi t + \frac{i}{N}2\pi - \theta\right)\right]$$
$$\left. + (a-b)r\left[\cos\theta - \cos\left(\frac{i}{N}2\pi + \theta\right)\right] \right\}^{1/2},$$

$$\vdots$$
(39)

$$\left|\dot{\tilde{q}} - \dot{q}_N\right|^2 = (2\pi)^2 \left(\frac{a^2 + b^2}{2} - \frac{a^2 - b^2}{2} \cos(4\pi t - 2\theta) \right),$$
(40)

$$\left|\tilde{q} - q_N\right| = \sqrt{\frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos(4\pi t - 2\theta)},$$
(41)

$$\left|\dot{q}_1\right|^2 = \cdots = \left|\dot{q}_N\right|^2 = (2\pi)^2 r^2.$$
(42)

Therefore by (36)–(42), we get

$$f(\tilde{q})$$
$$= \frac{1}{N} \int_0^1 \sum_{i=1}^N \left[\frac{1}{2}\left|\dot{\tilde{q}} - \dot{q}_i\right|^2 + \frac{N}{\left|\tilde{q} - q_i\right|} \right] dt - \frac{1}{2N} \int_0^1 \sum_{i=1}^N \left|\dot{q}_i\right|^2 dt$$
$$= \left(a^2 + b^2\right)\pi^2 + (2\pi)^2 r^2$$
$$+ 2\pi^2(a-b)r\left[\cos\theta - \frac{1}{N}\sum_{i=1}^N \cos\left(\frac{i}{N}2\pi + \theta\right)\right]$$
$$+ \sum_{i=1}^N \int_0^1 \left\{ \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos(4\pi t - 2\theta) \right.$$
$$+ r^2 \left(2 - 2\cos\frac{i}{N}2\pi\right) + (a+b)r$$
$$\times \left[\cos(4\pi t - \theta) - \cos\left(4\pi t + \frac{i}{N}2\pi - \theta\right)\right]$$
$$\left. + (a-b)r\left[\cos\theta - \cos\left(\frac{i}{N}2\pi + \theta\right)\right] \right\}^{-1/2} dt$$
$$- \frac{1}{2}(2\pi)^2 r^2$$
$$= 2\pi^2 \left(\frac{a^2 + b^2}{2} + r^2 \right)$$
$$+ 2\pi^2(a-b)r\left[\cos\theta - \frac{1}{N}\sum_{i=1}^N \cos\left(\frac{i}{N}2\pi + \theta\right)\right]$$
$$+ \sum_{i=1}^N \int_0^1 \left\{ \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos(4\pi t - 2\theta) \right.$$
$$+ r^2 \left(2 - 2\cos\frac{i}{N}2\pi\right) + (a+b)r$$
$$\times \left[\cos(4\pi t - \theta) - \cos\left(4\pi t + \frac{i}{N}2\pi - \theta\right)\right]$$
$$\left. + (a-b)r\left[\cos\theta - \cos\left(\frac{i}{N}2\pi + \theta\right)\right] \right\}^{-1/2} dt$$
$$= d_2(a, b, \theta).$$
(43)

In order to estimate d_2, we have computed the numerical values of $d_2 = f(q)$ over some selected test loops. The computation of the integral that appears in (43) has been done using the function {quad} of Mathematica 7.1 with an error less than 10^{-6}. The results of the numerical explorations are given in Table 1.

TABLE 1: Parameters of test loops for Theoerm 2.

N	a	b	θ	d_0	d_2
2	0.11	0.29	$\pi/2$	9.813298	9.753899
2	0.13	0.25	π	9.813298	9.753243
2	0.13	0.27	π	9.813298	9.643849
2	0.15	0.21	π	9.813298	9.790068
2	0.15	0.23	$\pi/5$	9.813298	9.746803
2	0.17	0.19	π	9.813298	9.751237
2	0.17	0.21	$\pi/5$	9.813298	9.625193
2	0.19	0.17	π	9.813298	9.784859
2	0.19	0.19	π	9.813298	9.564603
2	0.21	0.21	π	9.813298	9.279445
2	0.23	0.15	π	9.813298	9.793205
2	0.23	0.23	$\pi/5$	9.813298	9.109854
2	0.25	0.13	$\pi/2$	9.813298	9.730925
2	0.25	0.25	$\pi/5$	9.813298	9.038706
3	0.15	0.67	$\pi/30$	11.523843	11.505860
3	0.15	0.69	$\pi/30$	11.523843	11.444212
3	0.17	0.65	$\pi/30$	11.523843	11.493238
3	0.17	0.67	$\pi/20$	11.523843	11.452135
3	0.17	0.69	$\pi/20$	11.523843	11.400124
3	0.19	0.63	$\pi/30$	11.523843	11.519350
3	0.19	0.65	$\pi/20$	11.523843	11.455969
3	0.19	0.67	$\pi/20$	11.523843	11.386608
3	0.61	0.23	π	11.523843	11.516685
3	0.63	0.19	π	11.523843	11.489791
3	0.63	0.21	π	11.523843	11.436105
3	0.65	0.17	π	11.523843	11.461786
3	0.65	0.19	π	11.523843	11.392115
3	0.65	0.21	π	11.523843	11.349366
3	0.67	0.15	π	11.523843	11.472422
3	0.67	0.17	π	11.523843	11.383978
3	0.67	0.19	π	11.523843	11.324970
3	0.69	0.13	π	11.523843	11.522980
3	0.69	0.15	π	11.523843	11.412094
3	0.69	0.17	π	11.523843	11.334189

For the parameters a, b, and θ given in Table 1, we all have $d_2 < d_0$. For $4 \leq N \leq 1000$, we do not find suitable parameters $a \in [0.01, 2r)$, $b \in [0.01, 2r)$, and $\theta = \pi, \pi/2, \pi/3, \ldots, \pi/29, \pi/30$ such that $d_2 < d_0$, where r is the radius of the first N bodies with equal masses. Hence we only consider the case $N = 2, 3$. This completes the proof of Theorem 2. □

Proof of Theorem 4. To get Theorem 4, we are going to find a test loop $\overline{q} \in \Lambda_+$ such that $f(\overline{q}) = d_3$. Then the minimizer of f on $\overline{\Lambda}_+$ must be a noncollision solution if $d_3 < d_1$.

Let $a > 0$, $\theta \in [0, 2\pi)$, and

$$\overline{q} - q_N = ae^{\sqrt{-1}2\pi t}e^{\sqrt{-1}\theta}. \tag{44}$$

Hence,

$$\overline{q} - q_1$$
$$= q_N + ae^{\sqrt{-1}2\pi t}e^{\sqrt{-1}\theta} - q_1$$
$$= re^{\sqrt{-1}2\pi t}\left(1 - e^{\sqrt{-1}(2\pi/N)}\right) + ae^{\sqrt{-1}2\pi t}e^{\sqrt{-1}\theta}, \tag{45}$$

$$\vdots$$

$$\overline{q} - q_i$$
$$= q_N + ae^{\sqrt{-1}2\pi t}e^{\sqrt{-1}\theta} - q_i$$
$$= re^{\sqrt{-1}2\pi t}\left(1 - e^{\sqrt{-1}(i/N)2\pi}\right) + ae^{\sqrt{-1}2\pi t}e^{\sqrt{-1}\theta}, \tag{46}$$

$$\vdots$$

$$\overline{q} - q_{N-1}$$
$$= q_N + ae^{\sqrt{-1}2\pi t}e^{\sqrt{-1}\theta} - q_{N-1}$$
$$= re^{\sqrt{-1}2\pi t}\left(1 - e^{\sqrt{-1}((N-1)/N)2\pi}\right) + ae^{\sqrt{-1}2\pi t}e^{\sqrt{-1}\theta}. \tag{47}$$

It is easy to see that $\overline{q} \in \Lambda_+$ and

$$\left|\dot{\overline{q}} - \dot{q}_1\right|^2 = (2\pi)^2\left\{r^2\left(2 - 2\cos\frac{2\pi}{N}\right) + a^2 \right.$$
$$\left. + 2ar\left[\cos\theta - \cos\left(\theta - \frac{2\pi}{N}\right)\right]\right\}, \tag{48}$$

$$\left|\overline{q} - q_1\right| = \left\{r^2\left(2 - 2\cos\frac{2\pi}{N}\right) + a^2 \right.$$
$$\left. + 2ar\left[\cos\theta - \cos\left(\theta - \frac{2\pi}{N}\right)\right]\right\}^{1/2}, \tag{49}$$

$$\vdots$$

$$\left|\dot{\overline{q}} - \dot{q}_i\right|^2 = (2\pi)^2\left\{r^2\left(2 - 2\cos\frac{i}{N}2\pi\right) + a^2 \right.$$
$$\left. + 2ar\left[\cos\theta - \cos\left(\theta - \frac{i}{N}2\pi\right)\right]\right\}, \tag{50}$$

$$\left|\overline{q} - q_i\right| = \left\{r^2\left(2 - 2\cos\frac{i}{N}2\pi\right) + a^2 \right.$$
$$\left. + 2ar\left[\cos\theta - \cos\left(\theta - \frac{i}{N}2\pi\right)\right]\right\}^{1/2}, \tag{51}$$

$$\vdots$$

$$\left|\dot{\overline{q}} - \dot{q}_N\right|^2 = (2\pi)^2 a^2, \qquad \left|\overline{q} - q_N\right| = a, \tag{52}$$

$$\left|\dot{q}_1\right|^2 = \cdots = \left|\dot{q}_N\right|^2 = (2\pi)^2 r^2. \tag{53}$$

Therefore by (48)–(53), we get

$$
\begin{aligned}
f\left(\overline{q}\right) \\
&= \frac{1}{N} \int_0^1 \sum_{i=1}^N \left[\frac{1}{2}\left|\dot{\overline{q}} - \dot{q}_i\right|^2 + \frac{N}{\left|\overline{q} - q_i\right|} \right] dt \\
&\quad - \frac{1}{2N} \int_0^1 \sum_{i=1}^N |\dot{q}_i|^2 dt \\
&= 2\pi^2 a^2 + (2\pi)^2 r^2 \\
&\quad + (2\pi)^2 \left[\arccos\theta - \frac{ar}{N}\sum_{i=1}^N \cos\left(\theta - \frac{i}{N}2\pi\right) \right] \\
&\quad + a^{-1} + \sum_{i=1}^{N-1} \left\{ r^2\left(2 - 2\cos\frac{i}{N}2\pi\right) + a^2 \right.\\
&\qquad\qquad \left. + 2ar\left[\cos\theta - \cos\left(\theta - \frac{i}{N}2\pi\right)\right] \right\}^{-1/2} \\
&\quad - \frac{1}{2}(2\pi)^2 r^2 \\
&= 2\pi^2\left(a^2 + r^2\right) \\
&\quad + (2\pi)^2 \left[\arccos\theta - \frac{ar}{N}\sum_{i=1}^N \cos\left(\theta - \frac{i}{N}2\pi\right) \right] \\
&\quad + a^{-1} + \sum_{i=1}^{N-1} \left\{ r^2\left(2 - 2\cos\frac{i}{N}2\pi\right) + a^2 \right.\\
&\qquad\qquad \left. + 2ar\left[\cos\theta - \cos\left(\theta - \frac{i}{N}2\pi\right)\right] \right\}^{-1/2} \\
&= d_3\left(a, \theta\right).
\end{aligned}
\tag{54}
$$

In order to estimate d_3, we have computed the numerical values of $d_3 = f(q)$ over some selected test loops. The results of the numerical explorations show that by choosing some $a \in [0.11, 12.50]$ and $\theta = \pi$, one has $d_3 < d_1$ for $2 \le N \le 32402$. From the computational data, let $\theta = \pi, \pi/2, \pi/3, \ldots, \pi/19, \pi/20$, and we find that $d_3 < d_1$ holds for $2 \le N \le 19$ with some $a \in [0.11, 0.90]$. But for $20 \le N \le 32402$, no matter how we choose the value of $a \in [0.11, 12.50]$, we obtain $d_3 < d_1$ only if $\theta = \pi$. For $N = 32403$, the computational data shows that $d_3 \gg d_1$ for all $a \in [0.11, 10.00]$ and $\theta = \pi, \pi/2, \pi/3, \ldots, \pi/99, \pi/100$. Then combined with Figure 1, it is easy to see that $a.e.$ $d_3 > d_1$ for all $a \in [0.11, 12.50]$ and $\theta = \pi, \pi/2, \pi/3, \ldots, \pi/99, \pi/100$. The same holds for $N = 32404$ from Figure 2 and the computational data. Moreover, we can get the same conclusions for $32405 \le N \le 100000$. Hence, we only consider the case $2 \le N \le 32402$. This completes the proof of Theorem 4. □
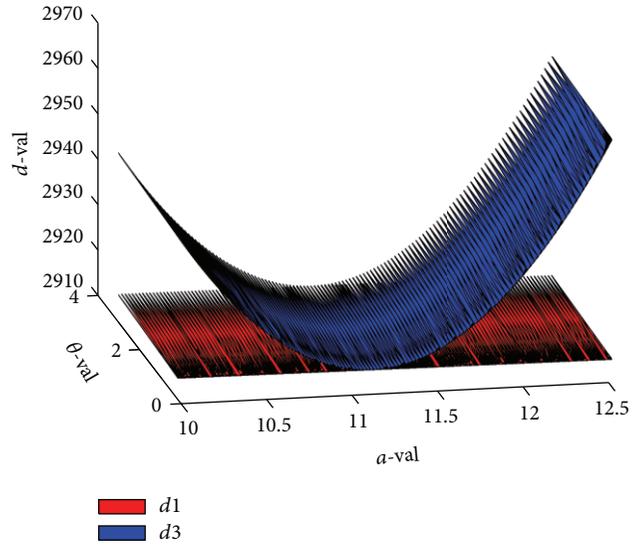
FIGURE 1: $N = 32403$.
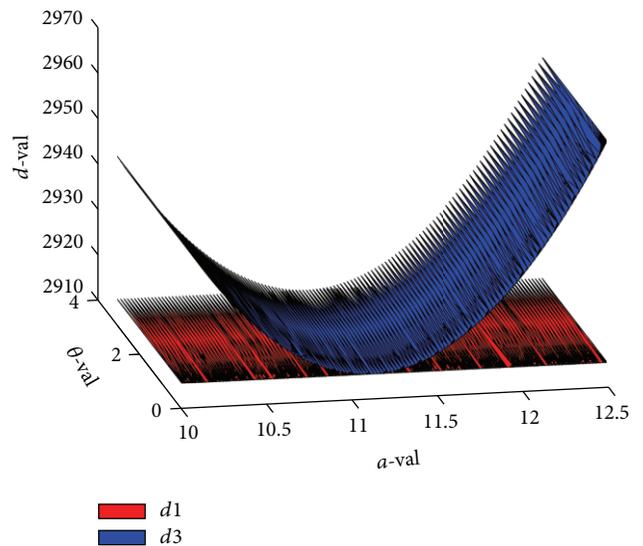
FIGURE 2: $N = 32404$.

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